# $(2,10)$ GWhD$(10n+1)$ - Existence Results

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**Abstract**

GWhD($v$)s, or Generalized Whist Tournament Designs on $v$ players, are a relatively new type of design. GWhD($v$)s are (near) resolvable $(v, k, k-1)$ BIBDs. For $k = et$ each block of the design is considered to be a game involving $e$ teams of $t$ players each. The design is subject to the requirements that every pair of players appears together in the same game exactly $t-1$ times as teammates and exactly $k-t$ times as opponents. These conditions are referred to as the Generalized Whist Conditions, and when met, we refer to the (N)RBIBD as a $(t, k)$ GWhD($v$). When $k = 10$, necessary conditions on $v$ are that $v \equiv 0, 1 \pmod{10}$. In this study we focus on the existence of $(2,10)$ GWhD($v$), $v \equiv 1 \pmod{10}$. It is known that a $(21,10,9)$-NRBIBD does not exist. Therefore it is impossible to have a $(2,10)$ GWhD(21). It is established here that $(2,10)$ GWhD$(10n+1)$ exist for all other $v$ with at most 42 additional possible exceptions.

# 1 Introduction

Generalized Whist Tournament Designs were first introduced in [5]. Generalized Whist Tournament Designs are defined as follows.

**Definition 1.1** *Let $e, k, t, v$ be positive integers such that $k = et$ and $v \equiv 0, 1 \pmod{k}$. Let $a$ be a positive rational number. A $(t, k)$* **Generalized Whist Tournament Design** *on $v$ players, having parameter $a$, is a $(v, k, a(k-1))$-(N)RBIBD that satisfies the conditions indicated below. Each block of the BIBD is considered to be a game in which $e$ teams of $t$ players each compete simultaneously. Players on the same team are called partners and players in the same game but not on the same team are called opponents. For each pair of players, say $\{x, y\}$, $x$ is to be a partner of $y$ exactly $a(t-1)$ times and $x$ is to be an opponent of $y$ exactly $a(k-t)$ times. Such a design is denoted by $(t, k)$ $GWhD_a(v)$. When $v \equiv 1 \pmod{k}$ consistency with the definition of a NRBIBD requires that $a$ be an integer. When $v \equiv 0 \pmod{k}$, practical reasons require that each of $a(v-1)$, $a(k-1)$, $a(t-1)$ and $a(k-t)$ be an integer. When $a = 1$, all reference to the parameter $a$ is suppressed. The teammate and opponent balance requirements are referred to as the* **Generalized Whist Conditions,** *so that if an (N)RBIBD satisfies the additional requirements of the Generalized Whist Conditions, then we will have a GWhD.*

In this paper we consider $a = 1$ and, in this case, it is customary to eliminate $a$ from the symbolism. The rounds of a $(t, k)$ GWhD($v$) are given by the (near) resolution classes of the design. The games are displayed in the form $(a_{11}, a_{12}, \ldots, a_{1t}; \ldots; a_{e1}, a_{e2}, \ldots, a_{et})$ where the semicolons separate the teams.

**Definition 1.2** *Let $v \equiv 0, 1 \pmod{k}$. A $Z$-cyclic $(t, k)$ GWhD($v$) is a generalized whist tournament design wherein players are taken as elements in $Z_{v-1} \cup \{\infty\}$ if $v = kn$ or in $Z_v$ if $v = kn + 1$. If $v = kn + 1$, the $Z$-cyclic generalized whist tournament has the property that round $i + 1$ is obtained from round $i$ by adding $1 \pmod{v}$ to every element in round $i$. If $v = kn$ then addition is modulo $v - 1$ and $\infty + 1 = \infty$.*

For $Z$-cyclic designs, the entire $(t, k)$ GWhD($v$) can be constructed from any one of its rounds. This one round is called the designs's initial round. If $v = kn$, the initial round is conventionally chosen as any round in which $\infty$ and 0 are team-mates. When $v = kn + 1$, the conventional initial round is that which omits 0.

**Example 1.3** *The initial round of a $Z$-cyclic $(2, 10)$ GWhD$(10)$:*
$(\infty, 0; 5, 4; 2, 7; 1, 8; 6, 3)$.

**Example 1.4** *The initial round of a Z-cyclic* $(2, 10)$ *GWhD*$(11)$*:*
$(1, 10; 2, 9; 3, 8; 4, 7; 5, 6)$.

**Example 1.5** *The initial round of a Z-cyclic* $(2, 10)$ *GWhD*$(20)$*:*
$(\infty, 0; 14, 17; 10, 1; 3, 11; 5, 18)$, $(13, 15; 16, 4; 2, 6; 8, 9; 7, 12)$.

**Example 1.6** *The initial round of a Z-cyclic* $(2, 10)$ *GWhD*$(91)$ *is obtained by multiplying each of the following blocks by* 1,16 *and* 74*:*
$(53, 3; 47, 7; 89, 27; 80, 66; 49, 25)$, $(39, 26; 81, 45; 44, 42; 43, 33; 75, 74)$, $(60, 37; 70, 12; 57, 11; 82, 19; 64, 58)$.

The designs in the next two examples were obtained by whistifying BIBDs found by Morales [20, 21].

**Example 1.7** *The initial round of a Z-cyclic* $(2, 10)$ *GWhD*$(40)$*:*
$(\infty, 0; 9, 1; 10, 38; 12, 30; 16, 4)$, $(7, 27; 11, 13; 14, 23; 18, 33; 25, 35)$,
$(8, 5; 19, 26; 22, 28; 29, 34; 2, 3)$, $(6, 31; 15, 32; 17, 21; 20, 36; 24, 37)$.

Our last direct construction in this section is 1-rotational, over $Z_{50} \cup \{\infty\}$. The first base block is short and generates a partial parallel class missing $\infty$; the other base blocks form a partial parallel class missing 0.

**Example 1.8** *A* $(2, 10)$ *GWhD*$(51)$ *over* $Z_{50} \cup \{\infty\}$*:*
$(0, 25; 5, 30; 10, 35; 15, 40; 20, 45)$, $(\infty, 15; 5, 38; 7, 27; 9, 24; 26, 48)$,
$(1, 3; 21, 25; 30, 37; 33, 45; 40, 46)$, $(10, 13; 14, 35; 17, 36; 20, 47; 41, 49)$,
$(2, 11; 6, 42; 12, 28; 19, 43; 39, 44)$, $(4, 22; 8, 18; 16, 29; 23, 34; 31, 32)$.

In Examples 1.3–1.8 one can utilize symmetric differences to verify that the generalized whist conditions are satisfied [6].

Existence results have been established for a few specific cases of Generalized Whist Tournament Designs. In particular, the cases $(t, k) = (2, 6)$, $(3, 6)$ and $(4, 8)$, for both $v = kn$ and $v = kn + 1$, have been studied [1, 2, 4, 3, 12, 13]. Existence of $(t, 12)$ GWhD$(12n + 1)$, $t = 2, 3, 4$, was investigated in [11]. In this paper we investigate the existence of $(2, 10)$ GWhD$(10n + 1)$.

# 2 Preliminaries

In this section we provide some theorems that are of fundamental importance for our existence arguments.

**Theorem 2.1** [1] *If $v = kn + 1$ is an odd prime power then there exists a cyclic (directed) $(t, k)$ GWhD$(v)$ for every $t$ such that $t|k$.*

The following theorem is a generalization of a theorem due to Baker [9]:

**Theorem 2.2** *Let $s, u \equiv 1 \pmod{k}$ be such that $(t, k)$ GWhD($s$) and $(t, k)$ GWhD($u$) both exist. If there exist $k-1$ MOLS of order $s$ then there exists a $(t, k)$ GWhD($su$).*

**Theorem 2.3** [5, 15] *Suppose there exists an RTD($km+1, kn+1$) that is given by a $km+1$ by $kn+1$ difference matrix over an Abelian group, $G$, of order $kn+1$ and a $(kn+1, k, k-1)$ NRBIBD that is generated by a difference family over $G$. Suppose also, $0 \le w \le n$, and a $(km, k, k-1)$ RBIBD, a $(km+1, k, k-1)$ NRBIBD plus a $(kw+1, k, k-1)$ NRBIBD all exist. Then a $(km(kn+1)+kw+1, k, k-1)$ NRBIBD exists. Furthermore, if the RBIBD and all the input NRBIBDs are $(t, k)$ GWhDs, then the resulting NRBIBD is a $(t, k)$ GWhD.*

An application of Theorem 2.1 combined with results of Furino et al. (see Corollary 2.5.4 in [15]) indicate that if $kn+1$ is a prime power for which $kn+1 > km+1$ then the requirements on $kn+1$ in Theorem 2.3 will be satisfied if we take $G = GF(kn+1)$.

Materials related to frames, for the most part, appear in the research literature. An excellent text resource is the book by Furino, Miao and Yin [15]. We provide the following material on frames which is sufficient for our present purposes.

**Definition 2.4** *A frame is a group divisible design $GDD_\lambda(X, \mathcal{G}, \mathcal{B})$, such that*

1. *the size of each block is the same, say $k$,*
2. *the block set can be partitioned into a family $\mathcal{R}$ of partial parallel classes, and*
3. *each $R_i \in \mathcal{R}$ can be associated with a group $G_j \in \mathcal{G}$ so that $R_i$ contains every point of $X \setminus G_j$ exactly once.*

We use the notation $(k, \lambda)$ Frame of type $g_1, g_2, g_3, \ldots$ to describe a frame with groups of size $g_1, g_2, \ldots$, ($g_i = |G_i|$), blocks of size $k$ and index $\lambda$. To say that the frame has index $\lambda$ means that any two elements from different groups occur together in $\lambda$ blocks. If $m$ of the groups of the frame are of the same size, say $g_1 = g_2 = \cdots = g_m$ then we say we have a $(k, \lambda)$ Frame of type $g_1^m, g_{m+1}, \ldots$. A frame is said to be a $(t, k)$ GWhFrame of type $g_1, g_2, \ldots$ if the blocks of the frame satisfy the generalized whist conditions.

**Example 2.5** *The blocks of a $(2, 10)$ GWhFrame of type $10^{13}$ are obtained by multiplying the following blocks by $81^i$ for $0 \le i \le 2$:* (11, 45; 82, 10; 112, 80; 122, 55; 64, 126), (87, 51; 63, 47; 68, 14; 59, 96; 119, 97), (20, 61; 75, 40; 18, 38; 128, 125; 69, 54), (127, 46; 92, 41; 123, 3; 24, 43; 9, 116). *The groups are* $\{13n : 0 \le n \le 9\} + 0, 1, \ldots, 12$.

**Example 2.6** *The blocks of a* $(2,10)$ *GWhFrame of type* $10^{29}$ *are obtained by multiplying the following blocks by* $81^i$ *for* $0 \leq i \leq 6$*:* (141, 155; 132, 20; 182, 200; 202, 175; 224, 56), (67, 251; 163, 17; 124, 198; 189, 206; 209, 197), (150, 221; 75, 280; 178, 148; 288, 115; 159, 204), (47, 146; 222, 231; 183, 53; 64, 133; 49, 216). *The groups are* $\{29n : 0 \leq n \leq 9\} + 0, 1, \ldots, 28$.

**Example 2.7** *The blocks of a* $(2,10)$ *GWhFrame of type* $10^{37}$ *are obtained by multiplying the following blocks by* $81^i$*, for* $0 \leq i \leq 8$*:* (21, 75; 92, 10; 152, 210; 182, 85; 254, 36), (17, 231; 183, 7; 84, 198; 209, 216; 359, 217), (180, 311; 205, 60; 308, 298; 358, 315; 289, 184), (287, 116; 282, 331; 293, 233; 124, 283; 239, 306). *The groups are* $\{37n : 0 \leq n \leq 9\} + 0, 1, \ldots, 36$.

**Example 2.8** *The blocks of a* $(2,10)$ *GWhFrame of type* $10^{53}$ *are obtained by multiplying the following blocks by* $81^i$*, for* $0 \leq i \leq 12$*:* (321, 355; 112, 20; 142, 420; 182, 445; 414, 246), (57, 311; 233, 47; 154, 218; 219, 386; 319, 497), (340, 351; 295, 520; 438, 328; 508, 415; 339, 334), (167, 276; 522, 441; 433, 213; 34, 243; 509, 496). *The groups are* $\{53n : 0 \leq n \leq 9\} + 0, 1, \ldots, 52$.

Theorems 2.9–2.18 are special cases of "abstract" theorems found in [15]. Proofs are provided here because our applications require that the blocks of the frames satisfy the generalized whist conditions. Thus we need to demonstrate that this is indeed the case.

**Theorem 2.9** *A* $(t,k)$ $GWhD(kn+1)$ *is equivalent to a* $(t,k)$ *GWhFrame of type* $(1^{kn+1})$.

*Proof:* Let $x$ be any element in the design and let $\{x\}$ to be the corresponding group of the target frame. The partial parallel class that misses $\{x\}$ is the round of the $(t,k)$ $GWhD(kn+1)$ that omits $x$. ∎

**Theorem 2.10** *Suppose we have a "master" K-GDD with g groups and a group type vector of* $(|G_j| : j = 1, \ldots, g)$*, and a weighting that assigns a positive weight of* $w(x)$ *to each point* $x$*. Let* $W(B_i)$ *be the weight vector of the i-th block. If, for every block* $B_i$*, we have an ingredient* $(t,k)$ *GWhFrame with a group type vector of* $W(B_i)$*, then there exists a* $(t,k)$ *GWhFrame with a group size vector of* $(\sum_{x \in G_j} w(x) : j = 1, \ldots, g)$.

*Proof:* Considering the ingredient and resultant designs merely as frames, the result is a variant of Wilson's fundamental construction, known as the GDD construction for frames [15, Corollary 2.4.3]. In that construction, every block, $B$, of the master GDD, say $(x_1, x_2, \ldots, x_{|B|})$, is replaced by an ingredient GDD, formed on the point set $\{x \times w(x) : x \in B\}$, having a group vector $(w(x_1), w(x_2), \ldots, w(x_{|B|}))$, where points with the same first element are in the same group of the ingredient design. Clearly, any pair of points in different master groups, $x, y$ say, occur together in exactly one block of the master design, and then our construction produces $k-1$ blocks containing the pair $(x,i), (y,j)$ in the ingredient design on this master block, and of these $k-1$ blocks, the pair will be teammates in $t-1$ of these blocks. ∎

**Corollary 2.11** *If $k+1$ is a prime power, then there exists a $(t,k)$ GWh-Frame of type $k^{k+2}$.*

*Proof:* Take AG$(2, k+1)$, and remove a point and its blocks. The deleted blocks define the groups of a $\{k+1\}$-GDD of type $k^{k+2}$. Since $k+1$ is a prime power, a $(t,k)$ GWh$(k+1)$ (i.e., a $(t,k)$ GWhFrame of type $1^{k+1}$) exists, and we can apply Theorem 2.10, giving all points a weight of 1. ∎

**Theorem 2.12** *Suppose we have a $(t,k)$ GWhFrame with a group size vector of $(|G_1|, |G_2|, \ldots, |G_g|)$, and an RTD$(k,n)$. Then there exists a $(t,k)$ GWhFrame with a group size vector of $(n|G_1|, n|G_2|, \ldots, n|G_g|)$.*

*Proof:* Considering both the $(t,k)$ GWhFrame designs merely as frames, the result is a variant of Wilson's fundamental construction, known as inflation by an RTD [15, Corollary 2.4.6]. In that construction, every block, $B$, of the input frame, say $(x_1, x_2, \ldots, x_{|B|})$, is replaced by $n^2$ blocks of an RTD formed on the point set $\{x \times I_n : x \in B\}$. The team (alt. opponent) property is inherited from the input frame, so if any pair of points in different input groups, $x, y$ say, occur together as teammates (alt. opponents) in some block of the input design, and then our construction produces one block in the output frame containing the pair $(x,i), (y,j)$ as teammates (alt. opponents). Since the input frame has the pair $x, y$ in $t-1$ blocks as teammates, and $k-t$ blocks as opponents, that property is inherited by the pair $(x,i), (y,j)$. ∎

**Theorem 2.13** *Suppose we have a $(t,k)$ GWhFrame with a group size vector of $(h|G_1|, h|G_2|, \ldots, h|G_g|)$, and we have a $(t,k)$ GWhFrame with a group type of $h^{|G_i|}w^1$ with $w \geq 0$ for $2 \leq i \leq g$. Then there exists a $(t,k)$ GWhFrame with group types of $h^{T-|G_1|}(h|G_1|+w)^1$, where $T = \sum_{i=1}^{i=g} |G_i|$. If a $(t,k)$ GWhFrame with a group type of $h^{|G_1|}w^1$ also exists, then there exists a $(t,k)$ GWhFrame with group type of $h^T w^1$.*

*Proof:* Considering both the $(t,k)$ GWhFrame designs merely as frames, the result is a standard one, sometimes known as breaking up groups [15, Corollary 2.4.7], or else is known as filling the groups. Clearly, no two points in different groups of the frame to be filled occur in any block of the filling designs, so their generalized whist properties stay intact. Points within the same group of the frame to be filled will either be in identical groups of the filling design, or will get their generalized whist properties from the filling design. A new point in the group of size $w$ will get its generalized whist properties (with respect to some other point of the frame to be filled) from the unique filling design that contains that other point (unless that point lies in the first group of size $h|G_1|$ and the group of size $h|G_1|+w$ does not need to be filled). Pairs of new points are in the same group of size $w$ or $h|G_1|+w$. ∎

**Corollary 2.14** *Suppose there exists a $(t,k)$ GWhFrame of type $v^p$ where $v \equiv 0 \pmod{k}$. If there exists a $(t,k)$ GWhD($v+1$), then there exists a $(t,k)$ GWhD($pv+1$).*

Theorem 2.2 can now be interpreted as starting with a $(t,k)$ GWhFrame with a group type of $1^u$ (noting Theorem 2.9), inflating by an RTD$(k,s)$ (using Theorem 2.12) to get a $(t,k)$ GWhFrame with a group type of $s^u$, then filling the groups with $(t,k)$ GWhFrames with a group type of $1^s$ to get a $(t,k)$ GWhFrame with a group type of $1^{su}$ by Theorem 2.13, then using Theorem 2.9 again to treat this design as a $(t,k)$ GWhD($su$).

**Lemma 2.15** *Let $p \equiv 1 \pmod{k}$ be a prime power. If there exists a Z-cyclic $(t,k)$ GWhD($v$) defined on $Z_{v-1} \cup \{\infty\}$ then there exists a $(t,k)$ GWhFrame of type $v^p$.*

*Proof:* Take $X = \{Z_{v-1} \cup \{\infty\}\} \times \mathrm{GF}(p)$. Let $x$ denote a primitive element for GF($p$). Set $a = (p-1)/k$, $k = et$ and let

$$(a_{i1}^{(1)}, a_{i2}^{(1)}, \ldots, a_{it}^{(1)}; \ldots; a_{i1}^{(e)}, \ldots, a_{it}^{(e)}), \quad i = 1, \ldots, v/k$$

denote the games in the initial round of the $Z$-cyclic $(t,k)$ GWhD($v$) defined on $Z_{v-1} \cup \{\infty\}$. Without loss of generality we assume that $a_{11}^{(1)} = \infty$ and $a_{12}^{(1)} = 0$.

In [1], the initial round tables of a $(t,k)$ GWhD($p$) defined on GF($p$) are given by $(team_1^b; team_2^b; \ldots; team_e^b)$, $\quad b = 0, 1, \ldots, a-1$, where

$$team_1^b = x^b, x^{ea+b}, x^{2ea+b}, \ldots, x^{(t-1)ea+b},$$
$$team_2^b = x^{a+b}, x^{(e+1)a+b}, x^{(2e+1)a+b}, \ldots, x^{((t-1)e+1)a+b},$$
$$\vdots$$
$$team_e^b = x^{(e-1)a+b}, x^{(2e-1)a+b}, \ldots, x^{(te-1)a+b}.$$

Take as groups (for our target frame) the sets $G_i = \{Z_{v-1} \cup \{\infty\}\} \times \{i\}$, $i \in$GF($p$). Let

$$S_0 = \{(a_{i1}^1, x^s), (a_{i2}^1, x^{s+a}), \ldots, (a_{it}^1, x^{s+(t-1)a}); (a_{i1}^2, x^{s+ta}), \ldots,$$
$$(a_{it}^2, x^{s+(2t-1)a}); \ldots, (a_{i1}^e, x^{s+(e-1)ta}), \ldots, (a_{it}^e, x^{s+(k-1)a})\},$$
$$i = 1, 2, \ldots, v/k; \; s = 0, 1, \ldots, a-1.$$

Note that $S_0$ consists of $av/k$ blocks (games) and hence $av$ elements. Set $S_y = S_0 \cdot (1, x^{ay})$, $y = 0, 1, \ldots, k-1$. If $S$ denotes the collection of all blocks contained in $S_y$ then it is easy to see from the description that $S$ is a partial parallel class that misses $G_0$. Furthermore $S + (g, 0)$, $g \in Z_{v-1}$ is also a partial parallel class that misses $G_0$. Thus we have $v-1$ partial parallel classes that miss $G_0$.

Let $0 \le m \le a-1$ and $z \in Z_{v-1} \cup \{\infty\}$. Consider the collection, $T$, of all blocks $T_{m,z}$ with

$$T_{m,z} = \{(z, x^{m+b}), (z, x^{m+ea+b}), \ldots, (z, x^{m+(t-1)ea+b}); \ldots;$$
$$(z, x^{m+(e-1)a+b}), (z, x^{m+(2e-1)a+b}), \ldots, (z, x^{m+(te-1)a+b})\},$$
$$b = 0, 1, \ldots, a-1.$$

Then $T$ is a $v$-th partial parallel class that misses $G_0$. Now, if $C$ is any partial parallel class that misses $G_0$ then $C + (0, x^y)$ for a fixed $y \in \{0, 1, \ldots, p-2\}$ is a partial parallel class that misses $G_{x^y}$. Applying this latter operation to each of the $v$ partial parallel classes that miss $G_0$ and for each $y \in \{0, 1, \ldots, p-2\}$ produces the desired frame. That the blocks satisfy the generalized whist conditions is discerned from the $(t, k)$ GWhD($v$) structure in the $S$ sets, the $(t, k)$ GWhD($p$) structure in the $T$ sets, the utilization of the rounds of the $(t, k)$ GWhD($v$) in the operation $S + (g, 0)$ and the utilization of the rounds of the $(t, k)$ GWhD($p$) in the operation $C + (0, x^y)$. ∎

In Lemma 2.15 it is clear that we could relax the $Z$-cyclicity assumption on the GWhD($v$), noting the first members in the ordered pairs in $S_0$ follow the structure of the first round of the $(t, k)$ GWhD($v$), and in place of the operation $S + (g, 0)$ one could use each of the rounds of the $(t, k)$ GWhD($v$). However, we can also dispense with the primality assumption and the $Z$-cyclicity assumption on the GWhD($p$) (which will be satisfied if $p$ is a prime anyway). The key observation is that, in the proof of Lemma 2.15, we inflated the GWhD($v$) with a partial difference matrix that had no zeros in it. If $p \equiv 1 \pmod k$ is a prime power, then we have a $(k+1)$ by $p$ DM, and we could assume that the first row and column were all 0, and delete these.

**Theorem 2.16** *Let $p \equiv 1 \pmod k$. Suppose a TD with a parallel class, i.e., a TD*$^*(k+1, p)$*, and a $(t, k)$ GWhD($v$) exist. If there exists a $(t, k)$ GWhD($v$), $v \equiv 0 \pmod k$, then there exists a $(t, k)$ GWhFrame of type $v^p$.*

*Proof:* We contruct the frame on $I_v \times I_p$ and, without loss of generality, we can assume the parallel class of our TD$^*(k+1, p)$ is the block set $I_{k+1} \times \{i\}$ for $i = 0, 1, \ldots, p-1$. We delete these blocks, and one group of the TD. Note that each deleted point defines a holey parallel class that includes one deleted block as the hole. This modified TD is an example of a double GDD, which has two sets of groups, one (the original) of type $p^k$, and one (introduced by the deletion of the parallel class) of type $k^p$. A group from the first system has one point in common with every group of the second system. DGDDs with this property are often called modified GDDs (MGDDs) and were introduced by Assaf [8]. We will follow the construction we used in Theorem 2.12 and, for each block, $B$ of the $(t,k)$ GWhD($v$), form a design on $B \times I_p$ using this modified RTD to form the new blocks, with the team assignments inherited from $B$. Since we are inflating the GWhD with a deficient RTD, it's clear we will get some sort of deficient GDD of type $p^v$. We fill these groups of this design with $v$ copies of a $(t,k)$ GWhD($p$). Now let's examine the deficiency. If $B = (b_1, b_2, \ldots, b_k)$ is a block of the GWhD($v$), then we never form the blocks $\{(b_i, j) : i = 1, 2, \ldots, k\}$ for any $j$ (and if we did, they would be the deficit in the GDD of type $p^v$). With the new groups of the target GDD as $G_j = I_v \times \{j\}$, we see that we never produce blocks with groupmates in the new system, and counting pairs, these are the only missing pairs in the design we produce, so we actually have (after the filling $(t,k)$ GWhD($p$)s are adjoined) a GDD of type $v^p$. It is easy to see that the $(t,k)$ generalized whist conditions hold, so we now look at the resolvability.

Take the $v/k$ blocks of a parallel class of the GWhD($v$), and inflate them using the holey parallel class of the MGDD that misses $I_k \times \{j\}$, to produce a holey parallel class missing $G_j$ having $v(p-1)/k$ blocks. Repeat for the other parallel classes of the GWhD($v$), and we have $(v-1)$ parallel classes missing $G_j$. The last parallel class missing $G_j$ is formed from the holey parallel classes of the filling designs on $\{i\} \times I_p$. Take the parallel class that misses $(i,j)$ from the $i$-th filler, and collect the $v$ sets of $(p-1)/k$ blocks to give the final class missing $G_j$. ∎

**Corollary 2.17** *Let $k+1$ be a prime power. If there is a $(t,k)$ GWhD($k$), then a $(t,k)$ GWhFrame of type $k^{k+1}$ exists.*

**Theorem 2.18** [5] *Let $0 \leq w_i \leq u$ for $1 \leq i \leq n$ and let $W = \sum w_i$, and suppose that:*

1. *a TD($k+1+n, u$) exists;*

2. *a $(t,k)$ GWhFrame of type $k^m$ exists for $k+1 \leq m \leq k+1+n$;*

3. *a $(t,k)$ GWhD($kv+1$) exists for $v = u$ and $v = w_i$ for $0 \leq w_i \leq u$;*

*Then a $(t,k)$ GWhD($(k+1)ku + kW + 1$) exists.*

*Proof:* Truncate the $i$-th group of the TD to size $w_i$. This gives a $\{k+1, k+2, \ldots, k+1+n\}$ GDD of type $u^{k+1}w_1^1 w_2^n \cdots w_n^1$. Now give all points a weight of $k$ in Theorem 2.10, using the hypothesized frames as ingredients. Finally, use Theorem 2.13, and fill the groups with the GWhDs, using an extra point. ∎

# 3 Golay type frames

We do give some examples of $(2,10)$ GWhFrames, and also give $(2,10)$ GWhD$(v)$s for $v = 481, 551$ here, as well as examples of frames with other block sizes. We have expanded the section beyond the $(2,10)$ GWhFrames as we think further results might be available in this area, even if we cannot yet provide them.

**Example 3.1** *The blocks of a* $(2,10)$ *GWhFrame of type* $2^{11}$ *are given by developing the following blocks over* $Z_{22}$*:*

$$(4,6;3,16;13,18;12,8;20,21) \quad (12,6;4,7;3,10;5,13;9,19)$$

*Each base block,* $B$*, generates* 11 *holey parallel classes of the form* $\{B+i, B+(11+i)\}$ *missing the group* $\{i, 11+i\}$ *for* $0 \leq i \leq 10$.

**Example 3.2** *The blocks of a* $(2,10)$ *GWhFrame of type* $3^{11}$ *are given by developing the following blocks over* $Z_{33}$*:*

$$(23,13;3,28;26,30;27,32;20,18) \quad (1,18;14,32;4,17;5,2;9,30)$$
$$(1,28;14,7;4,13;5,19;20,21)$$

*Each base block,* $B$*, generates* 11 *holey parallel classes of the form* $\{B+i, B+(11+i), B+(22+i)\}$ *missing the group* $\{i, 11+i, 22+i\}$ *for* $0 \leq i \leq 10$.

**Example 3.3** *The blocks of a* $(2,10)$ *GWhFrame of type* $4^{11}$ *are given by developing the following blocks over* $Z_{44}$*:*

$$(23,24;3,28;4,30;27,32;20,18) \quad (1,29;3,32;15,6;38,2;31,19)$$
$$(1,28;14,18;37,13;38,8;20,10) \quad (1,8;3,24;4,10;16,29;42,39)$$

*Each base block,* $B$*, generates* 11 *holey parallel classes of the form* $\{B+i, B+(11+i), B+(22+i), B+(33+i)\}$ *missing the group* $\{i, 11+i, 22+i, 33+i\}$ *for* $0 \leq i \leq 10$.

**Lemma 3.4** *A* $(2,10)$ *GWhD*$(v)$ *exists for* $v = 481,\ 551$.

*Proof:* For 481, take an AG$(2,11)$ and delete a point and its blocks to get a 11-GDD of type $10^{12}$. Give all points of this design a weight of 4 in Theorem 2.10, and use the $(2,10)$ GWhFrame of type $4^{11}$ of Example 3.3 as the ingredient design to get a $(2,10)$ GWhFrame of type $40^{12}$. Similarly for 551, take a TD$(11,25)$, and give all its points a weight of 2 using the $(2,10)$ GWhFrame of type $2^{11}$ from Example 3.1 to get a $(2,10)$ GWhFrame of type $50^{11}$. Applying Corollary 2.14 to these frames of types $40^{12}$ and $50^{11}$ now gives the required designs. ∎

Although Lemma 3.4 gives the only needed designs that we can use the frames of Examples 3.1–3.3 to construct, we could inflate the $(2, 10)$ GWh-Frame of type $2^{11}$ with an an RTD$(10, 80)$ and, if we had a $(2, 10)$ GWhD$(v)$ for $v = 161$, we could then fill the groups for a $(2, 10)$ GWhD$(10n + 1)$ for $n = 176$.

Example 3.1 was first constructed as a frame, and then team assignments were made in our frame. We think the construction of the frames merits further comment, so let us ignore the team assignments in Examples 3.1–3.3 and rewrite the base blocks over $G \times Z_{11}$ where $G$ is some group (here $G = Z_2$ or $Z_3$ or $Z_4$), and since each base block has no elements that are congruent to $0 \pmod{11}$, we can omit column 0.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $Z_2 \times Z_{11}$ | | | | | | | | | | |
| $B_1$ | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| $B_2$ | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 1 | 0 |
| $Z_3 \times Z_{11}$ | | | | | | | | | | |
| $B_1$ | 2 | 1 | 0 | 2 | 0 | 1 | 0 | 0 | 2 | 2 |
| $B_2$ | 1 | 2 | 2 | 1 | 2 | 2 | 0 | 0 | 0 | 2 |
| $B_3$ | 1 | 1 | 2 | 1 | 2 | 1 | 1 | 1 | 2 | 0 |
| $Z_4 \times Z_{11}$ | | | | | | | | | | |
| $B_1$ | 3 | 0 | 3 | 0 | 3 | 0 | 2 | 2 | 0 | 0 |
| $B_2$ | 1 | 2 | 3 | 3 | 2 | 2 | 1 | 3 | 3 | 0 |
| $B_3$ | 1 | 1 | 2 | 1 | 2 | 0 | 2 | 0 | 0 | 2 |
| $B_4$ | 1 | 0 | 3 | 0 | 0 | 3 | 1 | 0 | 2 | 2 |

Now let us define the aperiodic autocorrelations at lag $t$ for the $|G|$ by $k$ array A.

$$\phi_t^L = \bigcup_{i=1}^{i=|G|} \bigcup_{j=1}^{j=k-t} a_{i,j}^{-1} \bullet a_{i,j+t}$$

$$\phi_t^R = \bigcup_{i=1}^{i=|G|} \bigcup_{j=1}^{j=k-t} a_{i,j+t}^{-1} \bullet a_{i,j}$$

where $a \bullet b$ is the group operation, and $\cup$ represents the multiset union operator, so frequencies are preserved. We also denote the multiset where each element of $G$ occurs exactly $m$ times as $m \cdot G$.

In order for the array to form a set of base blocks, we need

$$\phi_t^L \cup \phi_{k+1-t}^R = (k-1) \cdot G$$

for $1 \leq t \leq k$, and we have the convention that $\phi_k^L = \phi_k^R = \emptyset$.

When $G$ is $Z_2$, (or $Z_2 \times Z_2$ or any number of direct products of $Z_2$) every element $x$ of $G$ satisfies $x^{-1} = x$, so in this case we always have $\phi_t^L = \phi_t^R$.

For the above design over $Z_2$, the pair of base blocks $B_1, B_2$ satisfy not just the required condition for the above designs, (i.e., $\phi_t^L \cup \phi_{k+1-t}^R = (k-1) \cdot G$) but also $\phi_t^L = (k-t) \cdot G$ (for $1 \le t \le k$. This extra condition means the given pair of base blocks is a Golay complementary pair [17], and for such a pair $(B_1, B_2)$ we can double the length of the blocks $B_1$, $B_2$ to obtain another Golay complementary pair, and hence also a $(20, 19)$ frame of type $2^{21}$. (If $||$ is the concatenation operator, the doubled pair is $(B_1||B_2,\ B_1||(B_2+1))$.).

We note that this doubling process makes use of the fact that the array $D_1$ below is a $(2,2,1)$ difference matrix. If the above design over $Z_3$ also satisfied $\phi_t^L = (k-t) \cdot G$ (for $1 \le t \le k$ then by similarly noting that the array $D_2$ below is a $(3,3,1)$ difference matrix, we could triple the block sizes to obtain a $(30, 29)$ frame of type $3^{31}$. The tripled base blocks would be $(B_1||B_2, |B_3)$, $(B_1||(B_2+1)||(B_3+2)$ and $(B_1||(B_2+2)||(B_3+1)$. Unfortunately, there is no set of 3 base blocks for a $3^{11}$ frame satisfying $\phi_t^L = (k-t) \cdot G$ for $1 \le t \le 10$; (for the $3^{11}$ design above, this condition fails for $t = 4$ and 7).

$$D_1 : \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad D_2 : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

More generally if $G$ is the addive component of GF$(g)$, then there exists a $(G, g, 1)$ difference matrix $D$, so if $B_1, B_2 \dots, B_g$ are our base blocks (for a $(k, k-1)$ frame of type $g^{k+1}$ over $G$ satisfying the extra condition $\phi_t^L = (k-t) \cdot G$ for $1 \le t \le k$, then the base blocks for a $(gk, gk-1)$ frame of type $g^{gk-1}$ can be taken as

$$B_1 + D_{i,1},\ B_2 + D_{i,2},\ B_3 + D_{i,3}, \dots, B_g + D_{i,g} \text{ for } 1 \le i \le g.$$

For our other two cyclic groups, $\phi_t^R$ can be computed from $\phi_t^L$ by replacing $a$ by $-a$ for every element in the multiset. It would be interesting to know what Golay type $|G|$-complementary sequences there were over groups other than $Z_2$, and if there were any analogue for them of the Golay doubling construction. There are some other known sequences that yield $(k, k-1)$ Frames of type $|G|^{k+1}$.

In the following examples, the first two are Golay pairs; the last is taken from Lu and Zhu [22, Lemma 2.1]; some are taken from Geramita and Seberry [16, Lemma 4.144]. The $GF(4) \times Z_5$ has the property that $\phi_t^L = \phi_t^R = (k-t) \cdot G$ for $1 \le t \le k$, and the $Z_3 \times Z_5$ and $Z_3 \times Z_{17}$ arrays have the property that $\phi_t^L = \phi_t^R$ plus the property that the arrays are cyclically developed over $Z_3$ (i.e., $B_i = B_{i-1} + 1$ for $i = 2, 3$). This means that the index can be reduced by deleting the last two base blocks. We highlight this in Theorem 3.5.

| $Z_2 \times Z_3$ | |
|---|---|
| $B_1$ | 11 |
| $B_2$ | 10 |
| $Z_2 \times Z_6$ | |
| $B_1$ | 11101 |
| $B_2$ | 10011 |
| $Z_3 \times Z_5$ | |
| $B_1$ | 2112 |
| $B_2$ | 0220 |
| $B_3$ | 1001 |
| $GF(4) \times Z_5$ | |
| $B_1$ | 10,00,01,00 |
| $B_2$ | 01,11,00,01 |
| $B_3$ | 00,11,01,01 |
| $B_4$ | 11,00,00,00 |

| $Z_2 \times Z_{27}$ | |
|---|---|
| $B_1$ | 11100111010000010110010000 |
| $B_2$ | 11100111010010101001101111 |
| $Z_2 \times Z_{14}$ | |
| $B_1$ | 1111001110101 |
| $B_2$ | 0111101100100 |
| $Z_3 \times Z_{17}$ | |
| $B_1$ | 0010122002210100 |
| $B_2$ | 1121200110021211 |
| $B_3$ | 2202011221102022 |

**Theorem 3.5** *A* $(16, 5)$ *frame of type* $3^{17}$ *exists.*

There is another consequence illustrated by the Lu and Zhu example. This is not a new result, and amounts to taking the trivial $(p, p-1)$ frame of type $1^{p+1}$ given by the base block $Z_{p+1} \setminus \{0\}$, and inflating with an RTD$(p, p)$, but perhaps viewing this construction in our array sequence setting might be suggestive.

**Theorem 3.6** *If* $p$ *is a prime power, then a* $(p, p-1)$ *Frame of type* $p^{p+1}$ *exists.*

*Proof:* For our GF$(p)$ by $Z_{p+1}$ array, place the transpose of a difference matrix over GF$(p)$ in columns 1–$p$ of the array. ∎

# 4 Existence results

Our objective here is to establish that Table 1 contains all our possible exceptions.

There is one known non-existence result. Greig et al. [18, 19] showed that no $(21, 10, 9)$ NRBIBD exists, so certainly no $(2, 10)$ GWhD$(21)$ exists.

**Lemma 4.1** $(2, 10)$ *GWhD*$(10n + 1)$ *exist for each* $n \leq 142$ *that does not appear in Table 1.*

*Proof:* There are $142 - 31 - 1 = 110$ cases to verify. Verification for these cases is provided as follows:

Solutions for $n = 5$ and $9$ can be found in Examples 1.8 and 1.6, while $n = 11$ is given by Lemma 2.15.

Table 1: Table of $n$ for which no $(2,10)$ GWhD$(10n+1)$ is known

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| (2) | 14 | 16 | 17 | 20 | 22 | 23 | 26 | 30 | 35 |
| 38 | 39 | 47 | 50 | 51 | 58 | 59 | 68 | 73 | 79 |
| 80 | 83 | 90 | 92 | 93 | 95 | 98 | 112 | 113 | 114 |
| 116 | 119 | 170 | 171 | 173 | 175 | 176 | 177 | 178 | 179 |
| 205 | 207 | 268 | | | | | | | |

There are 61 values of $n < 143$ for which $10n + 1$ is a prime or prime power.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 4 | 6 | 7 | 8 | 10 | 12 | 13 | 15 |
| 18 | 19 | 21 | 24 | 25 | 27 | 28 | 31 | 33 | 36 |
| 40 | 42 | 43 | 46 | 49 | 52 | 54 | 57 | 60 | 63 |
| 64 | 66 | 69 | 70 | 75 | 76 | 81 | 82 | 84 | 88 |
| 91 | 94 | 96 | 97 | 99 | 102 | 103 | 105 | 106 | 109 |
| 115 | 117 | 118 | 120 | 123 | 129 | 130 | 132 | 133 | 136 |
| 138 | | | | | | | | | |

The following 39 values of $n$ are covered by Theorem 2.3 with $km = 10$ and $kn + 1 \in \{31, 41, 61, 71, 81, 101, 121, 131\}$:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 32 | 34 | 41 | 44 | 45 | 61 | 62 | 65 | 67 | 71 |
| 72 | 74 | 77 | 78 | 85 | 86 | 87 | 89 | 101 | 104 |
| 107 | 108 | 110 | 111 | 121 | 122 | 124 | 125 | 126 | 127 |
| 128 | 131 | 134 | 135 | 137 | 139 | 140 | 141 | 142 | |

An application of Corollary 2.14 with the $(2,10)$ GWhFrames of types $10^{29}$, $10^{37}$ and $10^{53}$ given in Examples 2.6–2.8 provides $(2,10)$ GWhD$(10n+1)$ for $n = 29$, 37 and 53.

The remaining cases are for $n = 48$, 55, 56 and 100. $n = 48$ and 55 are given by Lemma 3.4, while $n = 56$ and 100 can be obtained by applying Theorem 2.2 with $s = 11$ and $u = 51$ or 91. ∎

**Theorem 4.2** *Let $0 \le w_i \le u$ for $i = 1, 2$, and suppose that a TD$(13, u)$ exists, and that a $(2,10)$ GWhD$(kv + 1)$ exists for $v = u, w_1, w_2$. Then a $(2,10)$ GWhD$(110u + 10w_1 + 10w_2 + 1)$ exists.*

*Proof:* This is an application of Theorem 2.18 with $n = 2$. The ingredient designs we need are $(2,10)$ GWhFrame of type $10^m$ for $m = 11$, 12 and 13. These are given by Corollary 2.17, Corollary 2.11 and Example 2.5. ∎

**Lemma 4.3** $(2,10)$ *GWhD*$(10n+1)$ *exist for* $143 \le n \le 274$, *except possibly for the following* 11 *values of* $n$.

170 171 173 175 176 177 178 179 205 207 268

*Proof:* There are 132 values to be considered; 65 of them ($143 \le n \le 169$ and $209 \le n \le 247$, $n \ne 244$) can be handled by Theorem 4.2 with $w = 13$ or $w = 19$. 22 more cases ($n =$ 172, 174, 180, 181, 183, 186, 187, 190, 193, 195, 201, 208, 244, 252, 253, 255, 259, 262, 267, 271, 273, 274) can be handled by Theorem 2.1 since $10n+1$ is a prime power. The remaining $132-65-22-11 = 34$ cases, specifically when $n \in$ {182, 184, 185, 188, 189, 191, 192, 194, 196, 197, 198, 199, 200, 202, 203, 204, 206, 248, 249, 250, 251, 254, 256, 257, 258, 260, 261, 263, 264, 265, 266, 269, 270, 272} are obtained by Theorem 2.3 with $km = 10$ and $kn+1 \in$ {181, 191, 241, 251}. ∎

**Lemma 4.4** *There exists a* $(2,10)$ *GWhD*$(v)$ *for* $v = 10n+1$ *and* $275 \le n \le 2001$.

*Proof:* These designs can all be obtained by Theorem 4.2 with $w \in$ {25, 27, 29, 32, 37, 43, 49, 53, 61, 67, 71, 81, 89, 97, 109, 121, 131, 139, 151, 163, 169}. More specifically, the following ranges for $v$ are covered: $[110w+1, 110w+401]$ for $w = 25, 27, 29$; $[110w+1, 110w+641]$ for $w = 32$; $[110w+1, 110w+741]$ for $w =$ 37, 43, 49; $[110w+1, 110w+991]$ for $w =$ 53, 61, 67, and $[110w+1, 110w+1421]$ for $w =$ 71, 81, 89, 97, 109, 121, 131, 139, 151, 163, 169. ∎

It is now straighforward to show that a $(2,10)$ GWhD$(v)$ exists for all $v \equiv 1 \pmod{10}$, $v \ge 19911$. It is not hard to check that any number in the range $0 \le W \le 197$ can be written as $W = w_1 + w_2$ with $0 \le w_1 \le w_2 \le 181$ such that neither $w_1$ or $w_2$ is 2, nor appears in Table 1. If a $(2,10)$ GWhD$(10u+1)$ and a TD$(13,u)$ exist and $u \ge 181$, then Theorem 4.2 gives a $(2,10)$ GWhD$(v)$ for all $v \equiv 1 \pmod{10}$, $110u+1 \le v \le 110u+1971$. Further, in any set of 9 consecutive odd integers, at most three values are divisible by 3, two by 5, two by 7 and one by 11, so there is always at least one value, $u$, in this range such that a TD$(13,u)$ exists. Hence any $v \equiv 1 \pmod{10}$, $v \ge 19911 = 110 \cdot 181 + 1$ can be written as $110u + x$ where $x \equiv 1 \pmod{10}$, $0 < x = 10W + 1 < 110 \cdot 18 = 1980$, $u \ge 181$, a TD$(13,u)$ exists, and a $(2,10)$ GWhD$(10u+1)$ exists also. (Since 205, 207, 268 are divisible by 5, 3 or 2, they would not be a value of $u$ picked by the above process.) Thus Theorem 4.2 gives a $(2,10)$ GWhD$(v)$ for all $v \equiv 1 \pmod{10}$, $v \ge 19911$.

Combining all the results of this section we have:

**Theorem 4.5** *A* $(2,10)$ *GWhD*$(10n+1)$ *exists for all* $n$, *except for* $n = 2$, *and possibly for the following* 42 *values of* $n$:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 14 | 16 | 17 | 20 | 22 | 23 | 26 | 30 | 35 | 38 |
| 39 | 47 | 50 | 51 | 58 | 59 | 68 | 73 | 79 | 80 |
| 83 | 90 | 92 | 93 | 95 | 98 | 112 | 113 | 114 | 116 |
| 119 | 170 | 171 | 173 | 175 | 176 | 177 | 178 | 179 | 205 |
| 207 | 268 | | | | | | | | |

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