Sarvate-Beam Triple Systems for $v \equiv 2 \pmod 3$

R.G. Stanton
Department of Computer Science
University of Manitoba
Winnipeg, MB, Canada R3T 2N2
stanton@cc.umanitoba.ca

Abstract

A Sarvate-Beam type of triple system is defined in the case $v \equiv 2 \pmod 3$ and an enumeration is given of such systems for $v = 5$.

1 Introduction

A Sarvate-Beam Triple System for $v \equiv 0$ or $1 \pmod 3$ is a set of triples such that the $v(v-1)/2$ pairs all have distinct frequencies, ranging from 1 to $v(v-1)/2$ (see [1]). A Restricted SB Triple System is one in which only triples of the form $(1xx)$ or $(2xx)$ are used (see [2]). In this note, we allow v to be congruent to $2 \pmod 3$; this necessitates taking a pair (12) and then having $v(v-1)/2 - 1$ triples, with the frequencies of triples ranging from 2 to $v(v-1)/2$.

Since we have assigned a pair (12) to occur as a single block, the triples for $v = 5$ are (134), (135), (145), (234), (235), (245), (345). The frequencies of the 9 pairs, in some order, are the integers from 2 to 10. This gives $2 + 3 + \ldots + 10 = 54$ pairs, and so we require 18 triples. We assign the frequencies in order to the seven triples listed above as a_1, b_1, c_1, a_2, b_2, c_2, d.

We immediately have (using F for frequency):

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$$\begin{array}{ll} F(12) = 1 & \\ F(13) = a_1 + a_2 & F(23) = b_1 + b_2 \\ F(14) = a_1 + a_3 & F(24) = b_1 + b_3 \\ F(15) = a_2 + a_3 & F(25) = b_2 + b_3 \end{array}$$

$$\begin{aligned} F(34) &= a_1 + b_1 + d \\ F(35) &= a_2 + b_2 + d \\ F(45) &= a_3 + b_3 + d \end{aligned}$$

It follows at once that

$$a_1 + a_2 + a_3 + b_1 + b_2 + b_3 + d = 18.$$

Also, all the a_i are distinct, and all the b_i are distinct.
The solutions with $d = 0$ give the Restricted SB Triple Systems.

Note: We have defined this type of SB Triple System by assigning label 1 to the single pair (12) and labels $2, 3, \ldots, 10$, to the other pairs $13, 14, \ldots, 45$, occuring in the 18 triples of the system. It would be possible to assign labels 4 or 7 or 10 to (12), along with 17, 16, 15, triples respectively, but such assignment appears much less natural. For example, if (12) is assigned label 10, then a restricted SB Triple System could be obtained using $A = (0, 1, 3)$, $B = (2, 4, 5)$, corresponding to the 15 triples $(135) + 3(145) + 2(234) + 4(235) + 5(245)$. Here $F(12) = 10$, $F(13) = 1$, $F(14) = 3$, $F(15) = 4$, $F(23) = 6$, $F(24) = 7$, $F(45) = 9$, $F(34) = 2$, $F(35) = 5$, $F(45) = 8$. Obviously, this is less natural than assigning label 1 to (12). Furthermore, the assignment of label 1 to (12) makes the number of triples obey the same formula as for $v \equiv 0$ or 1 (mod 3), namely $\lfloor N(v) \rfloor$, where

$$N(v) = \frac{(v^2 - v)(v^2 - v + 2)}{24}.$$

For $v \equiv 0$ or 1 (mod 3), the quantity $N(v)$ is itself an integer. For $v \equiv 2$ (mod 3), the floor of $N(v)$ is given by

$$\frac{(v^2 - v + 4)(v^2 - v - 2)}{24}.$$

We note that, in this case, $N(v) - \lfloor N(v) \rfloor = \frac{1}{3}$.

2 The Restricted Systems

We are seeking 2 vectors $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ such that the 9 quantities $a_1 + a_2$, $a_2 + a_3$, $a_3 + a_1$, $b_1 + b_2$, $b_2 + b_3$, $b_3 + b_1$, $a_1 + b_1$, $a_2 + b_3$, are the 9 integers $2, 3, \ldots, 10$, in some order. We have $\sum a_i + \sum b_i = 18$, and we agree to convection $\sum a_i \leq \sum b_i$. Clearly, $\sum a_i \geq 0 + 2 + 3 = 5$. So there are 5 cases, as $\sum a_i$ ranges from 5 to 9.

Case 1. $\sum a_i = 5$.

Then $a_1 = 0$, $a_2 = 2$, $a_3 = 3$. Clearly, $\sum b_i = 13$, and so the smallest possible b_i is 3. This forces the other two values of b_i to be 4 and 6. These values work if we put $A = (0, 2, 3)$, $B = (4, 6, 3)$. This solution gives the SB system

$$2(135) + 3(145) + 4(234) + 6(235) + 3(245),$$

with frequencies $F(12) = 1$, $F(13) = 2$, $F(14) = 3$, $F(15) = 5$, $F(23) = 10$, $F(24) = 7$, $F(25) = 9$, $F(34) = 4$, $F(35) = 8$, $F(45) = 6$.

Case 2. $\sum a_1 = 6$, $\sum b_i = 12$.

If $A = (0, 2, 4)$, then B must contain either $\{2, 3, 7\}$ or the set $\{3, 4, 5\}$. In both cases, we can not obtain the label 10.

If $A = (1, 2, 3)$, then B must contain $\{3, 4, 5\}$. Again, we can not obtain the label 2. So there is no solution in Case 2.

Case 3. $\sum a_i = 7$, $\sum b_1 = 11$.

If $A = (0, 2, 5)$, then B must contain $\{1, 3, 7\}$ or $\{1, 2, 8\}$. We get a solution with $B = (3, 7, 1)$ and with $B = (8, 2, 1)$.

If $A = (0, 3, 4)$, then B must contain $\{2, 3, 6\}$ since $\{1, 4, 6\}$ would give a repeated label 7. A solution is found as $B = (2, 3, 6)$.

Finally, if $A = (1, 2, 4)$, then B must contain $\{1, 3, 7\}$, since $\{1, 4, 6\}$ would have 2 elements in common with A. But label 9 can only be achieved by $7 + 2$, and label 2 can only be achieved by $1 + 1$. So we must have $B = (1, 7, 3)$.

So there are 4 solutions in Case 3.

Case 4. $\sum a_i = 8$, $\sum b_i = 10$.

Since $\sum a_i = 8$, the smallest a_i must be either 0 or 1. So the possible a_i sets are either $\{0, 2, 6\}$, $\{0, 3, 5\}$, $\{1, 2, 5\}$, $\{1, 3, 4\}$.

The possible b_i sets are $\{0, 2, 8\}$, $\{0, 3, 7\}$, $\{0, 4, 6\}$, $\{1, 2, 7\}$, $\{1, 3, 6\}$, $\{1, 4, 5\}$, $\{2, 3, 5\}$.

$B = (0, 2, 8)$ can only occur with $\{1, 2, 5\}$ or $\{1, 3, 4\}$. Both sets work with $B = (0, 2, 8)$, $A = (5, 2, 1)$ or $A = (3, 4, 1)$.

$B = (0, 3, 7)$ can only occur with $\{0, 2, 6\}$. But this set can not be arranged as an A.

$B = (0, 4, 6)$ can only occur with $\{0, 3, 5\}$. But this set can not be arranged as an A.

$B = (1, 2, 7)$ can only occur with $\{1, 3, 4\}$. This can be achieved with $A = (1, 4, 3)$.

Neither $B = (1, 3, 6)$, nor $B = (1, 4, 5)$ can occur with any possible A.

So there are 3 solutions in Case 4.

Case 5. $\sum a_i = \sum b_i = 9$.

The smallest element in A or B must be 0 or 1 (the set $\{2, 3, 4\}$ would require a partner with 2 digits below 2 or 2 digits above 4, both of which are impossible).

So A and B must contain 7 and 3 respectively; or 3 and 6; or 5 and 4.

The only possible triples are $\{0, 2, 7\}$, $\{0, 3, 6\}$, $\{0, 4, 5\}$, $\{1, 2, 6\}$, $\{1, 3, 5\}$.

Note that the elements in A and B must be disjoint in this case, since (x, α_1, α_2) and (x, β_1, β_2) imply $\alpha_1 + \alpha_2 = 9 - x = \beta_1 + \beta_2$.

So $(0, 3, 6)$ can not pair with any triple.

If we take $A = (0, 2, 7)$, then we can $B = (5, 1, 3)$.

If $A = (0, 4, 5)$, then we can take B as $(2, 6, 1)$.

So there are two solutions in Case 5, and we have

Lemma 1. For $v = 5$, there are 10 Restricted SB Triple Systems.

3 The Unrestricted SB Systems

We have $\sum a_i + \sum b_i + d = 18$. But

$$\sum a_i + \sum b_i + 3d = F(34) + F(35) + F(45) \leq 27.$$

It follows that $2d \leq 9$, $d \leq 4$. So, for $d \neq 0$, we have 4 cases.

Case 1. $d = 4$, $\sum a_i + \sum b_i = 14$.

If $\sum a_i = 5$, $\sum b_i = 9$, then $A = (0, 2, 3)$. B must come from $\{1, 2, 6\}$ or $\{1, 3, 5\}$. The first set is impossible, and the second requires the set $\{b_i + d, b_2 + d, b_3 + d\}$ to be 7, 9, 10. This is achieved by $B = (5, 1, 3)$. So there is a solution in this case.

If $\sum a_i = 6$, $\sum b_i = 8$, then $A = (0, 2, 4)$ or $(1, 2, 3)$. For $A = (0, 2, 4)$, B must come from $\{0, 3, 5\}$, since both $\{1, 2, 5\}$ and $\{1, 3, 4\}$ are impossible; but $\{0, 3, 5\}$ does not work. For $A = \{1, 2, 3\}$, B must come from $\{0, 2, 6\}$; but this likewise is impossible.

If $\sum a_i = \sum b_i = 7$, then $A = (0, 2, 5)$ or $(0, 3, 4)$ or $(1, 2, 4)$. B must come from a disjoint set of the same triples. So there is no solution, and we have

Lemma 2. If $d = 4$, there is one solution.

Case 2. $d = 3$, $\sum a_i + \sum b_i = 15$.

If $\sum a_i = 5$, $\sum b_1 = 10$, then $A = (0,2,3)$. B must come from $\{0,4,6\}$ or $\{1,3,6\}$. Neither is possible.

If $\sum a_i = 6$, $\sum b_i = 9$, then $A = (0,2,4)$ or $(1,2,3)$. For $A = (0,2,4)$, there is no possible B triple. For $A = (1,2,3)$, B must come from $\{0,2,7\}$, which is impossible.

If $\sum a_i = 7$, $\sum b_i = 8$, then $A = (0,2,5)$ or $(0,3,4)$ or $(1,2,4)$. As before, B must be a disjoint set from these same triples. So there is no solutions and we have

Lemma 3. If $d = 3$, there is no solution.

Case 3. $d = 2$, $\sum a_i + \sum b_i = 16$.

If $\sum a_i = 5$, $\sum b_i = 11$, then $A = (0,2,3)$. B must come from $\{1,3,7\}$, or $\{2,4,5\}$. Both values work with $B = (7,3,1)$ or $(5,2,4)$.

If $\sum a_i = 6$, $\sum b_i = 10$, then $A = (0,2,4)$ or $(1,2,3)$. If $A = (0,2,4)$, B must be selected from $\{0,3,7\}$, $\{1,2,7\}$, or $\{2,3,5\}$, and no selection works. If $A = (1,2,3)$, then B must be selected from $\{0,2,8\}$, $\{1,3,6\}$, or $\{1,4,5\}$. No selection works.

If $\sum a_i = 7$, $\sum b_i = 9$, then $A = (0,2,5)$ or $(0,3,4)$ or $(1,2,4)$. For $A = (0,2,5)$, B must come from $\{0,3,6\}$ or $\{1,3,5\}$. The selection $A = (0,2,5)$, $B = (6,0,3)$ works; so does $A = (0,2,5)$, $B = (6,0,3)$. If $A = (0,3,4)$, no suitable B triple exists. Finally, if $A = (1,2,4)$, B must come from $\{0,2,7\}$, and the arrangment $B = (7,0,2)$ works.

So we have

Lemma 4. If $d = 2$, there are 5 solutions.

Case 4. $d = 1$, $\sum a_i + \sum b_i = 17$.

If $\sum a_i = 5$, $\sum b_i = 12$, then $A = (0,2,3)$. B must come from $\{2,4,6\}$ or $\{3,4,5\}$. No arrangement is possible in either case.

If $\sum a_i = 6$, $\sum b_i = 11$, then $A = (0,2,4)$ or $(1,2,3)$. For $A = (0,2,4)$, B must come from $\{1,2,8\}$ or $\{1,4,6\}$ or $\{2,3,6\}$, and no arrangement is possible. If $A = (1,2,3)$, then B must come from $\{3,4,5\}$, and no arrangement works.

If $\sum a_i = 7$, $\sum b_i = 10$, then $A = (0,2,5)$ or $(0,3,4)$ or $(1,2,4)$. For $A = (0,2,5)$, B must come from $\{0,4,6\}$ or $\{1,2,7\}$, and no arrangement works. For $A = (0,3,4)$, B must come from $\{0,2,8\}$ or $\{1,4,5\}$. Here we may take $A = (0,3,4)$, $B = (8,2,0)$; or we may take $A = (0,3,4)$, $B = (1,4,5)$. Finally, if $A = (1,2,4)$, then B must come from $\{0,2,8\}$ or $\{1,3,6\}$ and no arrangement is possible. So there are 2 solutions in this subcase.

If $\sum a_i = 8$, $\sum b_i = 9$, then $A = (0,2,6)$ or $(0,3,5)$ or $(1,2,5)$ or $(1,3,4)$. For $A = (0,2,6)$, B must come from $\{0,4,5\}$ and no arrangement works. For $A = (0,3,5)$, B must come from $\{0,2,7\}$, and no arrangement works. For $A = (1,2,5)$, B must come from $\{0,4,5\}$, the arrangement

$A = (1,2,5)$, $B = (0,5,4)$ works. Finally, if $A = (1,3,4)$, then B must come from $\{0,3,6\}$ and the selection $B = (0,6,3)$ works.

So we have

Lemma 5. If $d = 1$, there are 4 solutions.

4 Conclusions

With the definitions we have given for an SB Triple System as having $F(12) = 1$, we have shown that the number of Restricted SB triple Systems for $v = 5$ is equal to 10, and the total number of SB Triple Systems for $v = 5$ is equal to 20.

References

[1] Dinesh G. Sarvate and William Beam, A New Type of Block Design, *Bulletin of the ICA* **50** (2007), 26-28.

[2] R.G. Stanton, A Note on Sarvate-Beam Triple Systems, *Bulletin of the ICA* **50** (2007), 61-66.