On Potentially ${}_kC_l$-graphic Sequences *

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Abstract: For given integers k and ℓ, $3 \le k \le \ell$, a graphic sequence $\pi = (d_1, d_2, \ldots, d_n)$ is said to be potentially ${}_kC_\ell$-graphic if there exists a realization of π containing C_r for each r, where $k \le r \le \ell$ and C_r is the cycle of length r. Luo (Ars Combinatoria 64(2002)301–318) characterized the potentially C_ℓ-graphic sequences without zero terms for $\ell = 3, 4, 5$. In this paper, we characterize the potentially ${}_kC_\ell$-graphic sequences without zero terms for $k = 3,\ 4 \le \ell \le 5$ and $k = 4,\ \ell = 5$.
Keywords: graph, degree sequence, potentially graphic sequence.
MR subject classification(2000): 05C07, 05C38.

1. Introduction

A n-term non-negative integer sequence, $\pi = (d_1, d_2, \ldots, d_n)$, is said to be *graphic* if it is the degree sequence of a simple graph G on n vertices and such a graph G is referred to as a *realization* of π. For a given non-negative integer sequence $\pi = (d_1, d_2, \ldots, d_n)$, define $\sigma(\pi) = d_1 + d_2 + \cdots + d_n$. The set of all n-term non-increasing graphic sequences without zero terms is denoted by GS_n. For a given graph H, a sequence $\pi \in GS_n$ is said to

*Supported by National Natural Science Foundation of China (Grant No. 10401010) and China Scholarship Council.
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be *potentially H-graphic* if there exists a realization of π containing H as a subgraph. Moreover, for given integers k and ℓ, $3 \le k \le \ell$, a sequence $\pi \in GS_n$ is said to be *potentially* ${}_kC_\ell$*-graphic* if there exists a realization of π containing C_r as a subgraph for each r, $k \le r \le \ell$.

In [2], Gould, Jacobson and Lehel considered the potentially H-graphic sequences. They considered the following problem: determine the smallest even integer $\sigma(H,n)$ such that every n-term graphic sequence π with $\sigma(\pi) \ge \sigma(H,n)$ is potentially H-graphic. If $H = K_r$, the complete graph on r vertices, this problem was considered by Erdős, Jacobson and Lehel [1] where they showed that $\sigma(K_3,n) = 2n$ for $n \ge 6$ and conjectured that $\sigma(K_r,n) = (r-2)(2n-r+1)+2$ for sufficiently large n. [2,6,7,8] confirmed the conjecture for $r \ge 4$ and n sufficiently large. In [11], Rao also considered the problem of characterizing the degree sequences of graphs containing a clique of prescribed size and gave a characterization for a sequence π to be potentially K_r-graphic. Although the proof by Rao [11] remains unpublished, Kézdy and Lehel [3] have given a different proof using network flows. Li et al. [7] also obtained a sufficient condition for a graphic sequence π to be potentially K_r-graphic. For $H = K_{r,s}$, the $r \times s$ complete bipartite graph, Gould et al. [2] determined $\sigma(K_{2,2},n)$. Yin and Li [12,13] obtained some sufficient conditions for a graphic sequence π to be potentially $K_{r,s}$-graphic and determined $\sigma(K_{r,r},n)$ for $r \ge 3$ and n sufficiently large.

Recently, Yin, Li and Chen [14] further considered the following problem: for given integers k and ℓ, $3 \le k \le \ell$, determine the smallest even integer $\sigma({}_kC_\ell,n)$ such that each n-term graphic sequence π with $\sigma(\pi) \ge \sigma({}_kC_\ell,n)$ is potentially ${}_kC_\ell$-graphic. [2] determined $\sigma(C_4,n)$. [4,5,9] determined $\sigma({}_3C_\ell,n)$ for $\ell \ge 4$ and n sufficiently large. [14] completely determined $\sigma({}_kC_\ell,n)$ for $\ell \ge 7$ and $3 \le k \le \ell$. Motivated by the above problem, Luo [10] considered the problem: characterize $\pi \in GS_n$ such that π is potentially C_k-graphic, and obtained the following results.

Theorem 1.1. [10] Let $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $n \ge 3$. Then π is potentially C_3-graphic if and only if $d_3 \ge 2$ except for two cases: $\pi = (2^4)$ and $\pi = (2^5)$, where the symbol x^y in a sequence stands for y consecutive terms, each equal to x.

Theorem 1.2. [10] Let $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$. Then π is potentially C_4-graphic if and only if all the following conditions must be satisfied:

(1) $d_4 \ge 2$;

(2) $d_1 = n-1$ implies $d_2 \ge 3$;

(3) If $n = 5, 6$, then $\pi \ne (2^n)$.

Theorem 1.3. [10] Let $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$. Then π is potentially C_5-graphic if and only if π satisfies all the following conditions:

(1) $d_5 \ge 2$ and $\pi \ne (2^n)$ for n=6,7;

(2) For $i = 1, 2$, $d_1 = n - i$ implies $d_{4-i} \ge 3$;

(3) If $\pi = (d_1, d_2, 2^k, 1^{n-k-2})$, then $d_1 + d_2 \le n + k - 2$.

Motivated by the above theorems, we consider the problem: for given integers k and ℓ, $3 \leq k \leq \ell$, characterize $\pi \in GS_n$ such that π is potentially ${}_kC_\ell$-graphic. In this paper, we give the characterizations for a sequence $\pi \in GS_n$ to be potentially ${}_kC_\ell$-graphic for $k = 3$ and $\ell = 4$, $k = 3$ and $\ell = 5$, and $k = 4$ and $\ell = 5$.

2. Preliminaries

In order to prove our main results, we need the following results.

Theorem 2.1. [2] If $\pi = (d_1, d_2, \ldots, d_n)$ is a graphic sequence with a realization G containing H as a subgraph, then there exists a realization G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π.

For a non-negative integer sequence $\pi = (d_1, d_2, \ldots, d_n)$, let $d'_1 \geq d'_2 \geq \cdots \geq d'_{n-1}$ be the rearrangement of $d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$ in non-increasing order. Then $\pi' = (d'_1, d'_2, \ldots, d'_{n-1})$ is called the *residual sequence* obtained by laying off d_1 from π. It is easy to see that if π' is graphic then so is π, since a realization G of π can be obtained from a realization G' of π' by adding a new vertex v_1 of degree d_1 to G' and joining it to the vertices whose degrees are reduced by one in going from π to π'.

Theorem 2.2. [9] If $\pi = (d_1, d_2, \ldots, d_n)$ is a non-increasing non-negative integer sequence with $d_1 \leq n - 1$, $d_1 - d_n \leq 1$ and $\sigma(\pi)$ is even, then π is graphic.

Lemma 2.1. Let $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ be a potentially C_4-graphic sequence. If $d_3 \geq 3$, or $d_1 \geq 4$ and $d_2 \geq 3$, then π is potentially ${}_3C_4$-graphic.

Proof. By Theorem 2.1, we may let G be a realization of π with $C_4 = x_1x_2x_3x_4x_1 \subseteq G$ and $\{d_G(x_1), d_G(x_2), d_G(x_3), d_G(x_4)\} = \{d_1, d_2, d_3, d_4\}$, where $d_G(x)$ denotes the degree of the vertex x in G. Since x_1x_3 or $x_2x_4 \in E(G)$ implies that π is potentially ${}_3C_4$-graphic, we assume that x_1x_3 and $x_2x_4 \notin E(G)$, and first prove the following two claims:

Claim 1. If there exist $x, y \in V(C_4)$, $x \neq y$ and $z \in V(G) \setminus V(C_4)$ such that $xz, yz \in E(G)$, then π is potentially ${}_3C_4$-graphic.

If $xy \in E(C_4)$, then G clearly contains C_3 as a subgraph. If $xy \notin (C_4)$, without loss of generality, we may assume that $x = x_1$ and $y = x_3$, then $G' = G - \{x_1x_2, x_3x_4\} + \{x_1x_3, x_2x_4\}$ is a realization of π and contains C_3 and C_4 as subgraphs.

Claim 2. If there exist $x, y \in V(C_4)$, $x \neq y$ and $x', y' \in V(G) \setminus V(C_4)$, $x' \neq y'$ such that $xx', yy' \in E(G)$ and $x'y' \notin E(G)$, then π is potentially ${}_3C_4$-graphic.

If $xy \notin E(G)$, without loss of generality, we may assume that $x = x_1$

and $y = x_3$, then $G' = G - \{xx', yy'\} + \{x_1x_3, x'y'\}$ is a realization of π and contains C_3 and C_4 as subgraphs. If $xy \in E(G)$, without loss of generality, we may assume that $x = x_1$ and $y = x_2$, then $G' = G - \{xx', yy', x_3x_4\} + \{x'y', x_1x_3, x_2x_4\}$ is a realization of π and contains C_3 and C_4 as subgraphs.

Let $A_i = N(x_i) \setminus \{x_{i-1}, x_{i+1}\}$ for $1 \le i \le 4$, where $x_0 = x_4$, $x_5 = x_1$ and $N(x)$ denotes the neighbor set of the vertex x in G. Then $|A_i| = d_G(x_i) - 2$ for $1 \le i \le 4$, and by Claims 1 and 2, for any i and j, $1 \le i < j \le 4$, we may assume that $A_i \cap A_j = \emptyset$ and $xy \in E(G)$ for any $x \in A_i$ and $y \in A_j$. Hence it is enough to prove that $d_4 \ge 3$, since $d_4 \ge 3$ implies that $|A_i| \ge 1$ for $1 \le i \le 4$ and G clearly contains C_3 as a subgraph. Now it easily follows from $d_3 \ge 3$, or $d_1 \ge 4$ and $d_2 \ge 3$ that $\cup_{i=1}^4 |A_i| \ge 3$ and there exists a vertex $u \in \cup_{i=1}^4 A_i$ such that $d_G(u) \ge 3$. Thus, we have $d_4 \ge d_G(u) \ge 3$. □

Lemma 2.2. Let $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ be a potentially C_5-graphic sequence. If $d_3 \ge 3$, or $d_1 \ge 4$ and $d_2 \ge 3$, then π is potentially ${}_3C_5$-graphic.

Proof. By Theorem 2.1, we may assume that π has a realization G with $C_5 = x_1x_2x_3x_4x_5x_1 \subseteq G$ and $\{d_G(x_1), d_G(x_2), d_G(x_3), d_G(x_4), d_G(x_5)\} = \{d_1, d_2, d_3, d_4, d_5\}$. If there exists an i, $1 \le i \le 5$ such that $x_ix_{i+2} \in E(G)$, where $x_6 = x_1$ and $x_7 = x_2$, then π clearly is potentially ${}_3C_5$-graphic. So we further assume that $x_ix_{i+2} \notin E(G)$ for $1 \le i \le 5$. First we prove the following

Claim. If there exist $x, y \in V(C_5)$, $x \ne y$ and $x', y' \in V(G) \setminus V(C_5)$, $x' \ne y'$ such that $xx', yy' \in E(G)$ and $x'y' \notin E(G)$, then π is potentially ${}_3C_5$-graphic.

If $xy \notin E(G)$, without loss of generality, we may assume that $x = x_1$ and $y = x_3$, then clearly $G' = G - \{xx', yy'\} + \{x_1x_3, x'y'\}$ is a realization of π and contains C_3, C_4 and C_5 as subgraphs. If $xy \in E(G)$, without loss of generality, we may assume that $x = x_1$ and $y = x_2$, then $G' = G - \{x_3x_4, xx', yy'\} + \{x_1x_3, x_2x_4, x'y'\}$ is a realization of π and contains C_3, C_4 and C_5 as subgraphs.

Let $A_i = N(x_i) \setminus \{x_{i-1}, x_{i+1}\}$ for $1 \le i \le 5$, where $x_0 = x_5$ and $x_6 = x_1$. Thus, by the Claim, we may assume that for any i and j, $1 \le i < j \le 5$, $xy \in E(G)$ for any $x \in A_i$, $y \in A_j$ and $x \ne y$. So it is easy to get from $d_3 \ge 3$, or $d_1 \ge 4$ and $d_2 \ge 3$ that there exists a vertex $u \in \cup_{i=1}^5 A_i$ such that $d_G(u) \ge 3$. Now by $d_5 \ge d_G(u) \ge 3$, i.e., $|A_i| \ge 1$ for $1 \le i \le 5$, it is clear that G contains C_3 and C_4 as subgraphs. □

3. Main Results

Theorem 3.1. Let $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ be a potentially C_4-graphic sequence. Then π is potentially ${}_3C_4$-graphic if and only if π satisfies one of the following conditions:

(1) $d_2 \geq 3$ and $\pi \neq (3^2, 2^4)$;

(2) $\pi = (d_1, 2^k, 1^{n-k-1})$ with $2 \leq d_1 \leq 3$ and $k \geq 6$, and $\pi \neq (2^8)$ and (2^9);

(3) $\pi = (d_1, 2^k, 1^{n-k-1})$ with $4 \leq d_1 \leq n-2$ and $k \geq 5$, and $\pi \neq (4, 2^6)$ and $(4, 2^7)$.

Proof. First assume that π is potentially ${}_3C_4$-graphic. If $d_2 \geq 3$, then $\pi \neq (3^2, 2^4)$, since it is easy to see that $(3^2, 2^4)$ is not potentially ${}_3C_4$-graphic. Now assume that $d_2 = 2$. Then $\pi = (d_1, 2^k, 1^{n-k-1})$, where $k \geq 5$. If $2 \leq d_1 \leq 3$, then $k \geq 6$ and $\pi \neq (2^8)$and (2^9), since $k \leq 5$ implies that π is not potentially ${}_3C_4$-graphic, and (2^8) and (2^9) are not potentially ${}_3C_4$-graphic. If $d_1 \geq 4$, then by Theorem 1.2, $d_1 \leq n-2$, and it follows from π to be potentially ${}_3C_4$-graphic that $k \geq 5$, and $\pi \neq (4, 2^6)$ and $(4, 2^7)$.

To prove the sufficiency, we first assume that π satisfies (1). By Lemma 2.1, we may assume that $\pi = (3^2, 2^k, 1^{n-k-2})$, where $k \geq 2$. If $4 \leq n \leq 6$, then π is one of the following sequences:

$$(3^2, 2^2), \quad (3^2, 2^3), \quad (3^2, 2^2, 1^2).$$

It is easy to see that all of them are potentially ${}_3C_4$-graphic. If $n \geq 7$, let $\pi_* = (2^{k-2}, 1^{n-k-2})$, then by $\sigma(\pi_*) = \sigma(\pi) - 10$ is even and Theorem 2.2, π_* is graphic. Let G^* be a realization of π_*. Then $(K_4 - e) \cup G^*$ is a realization of π and contains C_3 and C_4 as subgraphs, where $K_4 - e$ is the graph obtained from K_4 by removing one edge.

Now we assume that π satisfies (2). If $7 \leq n \leq 9$, then π is one of the following:

$$(2^7), \quad (3, 2^6, 1), \quad (2^7, 1^2), \quad (3, 2^7, 1).$$

It is easy to see that the above four sequences are potentially ${}_3C_4$-graphic. If $n \geq 10$, let $\pi_* = (d_1 - 2, 2^{k-6}, 1^{n-k-1})$, then the residual sequence π_*' obtained by laying off $d_1 - 2$ from π_* clearly satisfies the hypotheses of Theorem 2.2, and so π_*' is graphic and thus so is π_*. Let G^* be a realization of π_*, and $x \in V(G^*)$ with $d_{G^*}(x) = d_1 - 2$. Denote

$$G = (C_4 \cup G^*) \cup \{x_1, x_2\} \cup \{xx_1, x_1x_2, x_2x\},$$

i.e., G is the graph obtained from $C_4 \cup G^*$ by adding new vertices x_1, x_2 and new edges xx_1, x_1x_2, x_2x to $C_4 \cup G^*$. Clearly, G is a realization of π and contains C_3 and C_4 as subgraphs.

Finally, we assume that π satisfies (3). If $6 \leq n \leq 8$, then π is one of the following:

$$(4, 2^5), \quad (5, 2^5, 1), \quad (4, 2^5, 1^2), \quad (5, 2^6, 1), \quad (6, 2^5, 1^2), \quad (6, 2^7).$$

It is easy to see that they are all potentially ${}_3C_4$-graphic. If $n \geq 9$, let $\pi_* = (d_1 - 4, 2^{k-5}, 1^{n-k-1})$, then the residual sequence π_*' obtained by laying off $d_1 - 4$ from π_* satisfies the hypotheses of Theorem 2.2, and so π_*' is graphic and thus so is π_*. Assume that G^* is a realization of π_*, and $x \in V(G^*)$ with $d_{G^*}(x) = d_1 - 4$. Let

$$G = G^* \cup \{x_1, x_2, x_3, x_4, x_5\} \cup \{xx_1, x_1x_2, x_2x, xx_3, x_3x_4, x_4x_5, x_5x\}.$$

Then G is a realization of π and contains C_3 and C_4 as subgraphs. □

Theorem 3.2. Let $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ be a potentially C_5-graphic sequence. Then π is potentially ${}_3C_5$-graphic if and only if π satisfies one of the following conditions:

(1) $d_2 \geq 3$, and $\pi \neq (3^2, 2^4)$ and $(3^2, 2^5)$;

(2) $\pi = (d_1, 2^k, 1^{n-k-1})$ with $2 \leq d_1 \leq 3$ and $k \geq 11$, and $\pi \neq (2^{13})$ and (2^{14});

(3) $\pi = (d_1, 2^k, 1^{n-k-1})$ with $4 \leq d_1 \leq 5$ and $k \geq 10$, and $\pi \neq (4, 2^{11})$ and $(4, 2^{12})$;

(4) $\pi = (d_1, 2^k, 1^{n-k-1})$ with $6 \leq d_1 \leq n-4$ and $k \geq 9$, and $\pi \neq (6, 2^{10})$ and $(6, 2^{11})$.

Proof. The necessity is obvious. In order to prove the sufficiency, we consider the following four cases:

Case 1. π satisfies (1). By Lemma 2.2, we may assume that $\pi = (3^2, 2^k, 1^{n-k-2})$, where $k \geq 3$. If $5 \leq n \leq 7$, then $\pi = (3^2, 2^3)$ or $(3^2, 2^3, 1^2)$. It is easy to see that $(3^2, 2^3)$ and $(3^2, 2^3, 1^2)$ are potentially ${}_3C_5$-graphic. If $n \geq 8$, let $\pi_* = (2^{k-3}, 1^{n-k-2})$, then by Theorem 2.2, π_* is graphic, and hence $\pi = (3^2, 2^3, 2^{k-3}, 1^{n-k-2})$ clearly is potentially ${}_3C_5$-graphic.

Case 2. π satisfies (2). If $12 \leq n \leq 14$, then π is one of the following:

$$(2^{12}),\quad (3, 2^{11}, 1),\quad (3, 2^{12}, 1),\quad (2^{12}, 1^2).$$

It is easy to see that the above four sequences are potentially ${}_3C_5$-graphic. If $n \geq 15$, let $\pi_* = (d_1 - 2, 2^{k-11}, 1^{n-k-1})$, then the residual sequence π_*' obtained by laying off $d_1 - 2$ from π_* satisfies the hypotheses of Theorem 2.2, and hence π_*' and π_* are graphic. Let G^* be a realization of π_*, and $x \in V(G^*)$ with $d_{G^*}(x) = d_1 - 2$. Then $(C_5 \cup C_4 \cup G^*) \cup \{x_1, x_2\} \cup \{xx_1, x_1x_2, x_2x\}$ is a realization of π and contains C_3, C_4 and C_5 as subgraphs.

Case 3. π satisfies (3). If $11 \leq n \leq 13$, then π is one of the following:

$$(4, 2^{10}),\quad (5, 2^{10}, 1),\quad (4, 2^{10}, 1^2),\quad (5, 2^{11}, 1).$$

It is easy to see that all of them are potentially ${}_3C_5$-graphic. If $n \geq 14$, let $\pi_* = (d_1 - 4, 2^{k-10}, 1^{n-k-1})$, then by Theorem 2.2, the residual sequence

${\pi_*}'$ obtained by laying off $d_1 - 4$ from π_* is graphic, and hence so is π_*. Denote

$$G = (C_5 \cup G^*) \cup \{x_1, x_2, x_3, x_4, x_5\} \cup \{xx_1, x_1x_2, x_2x, xx_3, x_3x_4, x_4x_5, x_5x\},$$

where G^* is a realization of π_* and $x \in V(G^*)$ with $d_{G^*}(x) = d_1 - 4$. Clearly, G is a realization of π and contains C_3, C_4 and C_5 as subgraphs.

Case 4. π satisfies (4). If $10 \leq n \leq 12$, then π is one of the following:

$$(6, 2^9),\quad (7, 2^9, 1),\quad (6, 2^9, 1^2),\quad (7, 2^{10}, 1),\quad (8, 2^{11}),\quad (8, 2^9, 1^2).$$

It is easy to see that the above six sequences are potentially ${}_3C_5$-graphic. If $n \geq 13$, then the residual sequence ${\pi_*}'$ obtained by laying off $d_1 - 6$ from $\pi_* = (d_1 - 6, 2^{k-9}, 1^{n-k-1})$ satisfies the hypotheses of Theorem 2.2. Hence π_* is graphic. Let G^* be a realization of π_* and $x \in V(G^*)$ with $d_{G^*}(x) = d_1 - 6$, and denote

$$\begin{aligned} G &= G^* \cup \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\} \\ &\quad \cup \{xx_1, x_1x_2, x_2x, xx_3, x_3x_4, x_4x_5, x_5x, xx_6, x_6x_7, x_7x_8, x_8x_9, x_9x\}. \end{aligned}$$

Then G is a realization of π and contains C_3, C_4 and C_5 as subgraphs. □

The proof of the following Theorem 3.3 is very similar to that of Theorem 3.2, we omit it here.

Theorem 3.3. Let $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ be a potentially C_5-graphic sequence. Then π is potentially ${}_4C_5$-graphic if and only if π satisfies one of the following conditions:

(1) $d_2 \geq 3$;

(2) $\pi = (d_1, 2^k, 1^{n-k-1})$ with $2 \leq d_1 \leq 3$ and $k \geq 8$, and $\pi \neq (2^{10})$ and (2^{11});

(3) $\pi = (d_1, 2^k, 1^{n-k-1})$ with $4 \leq d_1 \leq n-4$ and $k \geq 7$, and $\pi \neq (4, 2^8)$ and $(4, 2^9)$.

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