Alliances in Directed Graphs

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Abstract

Alliances in undirected graphs were introduced by Hedetniemi, Hedetniemi, and Kristiansen, and generalized to k-alliances by Shafique and Dutton. We translate these definitions of alliances to directed graphs. We establish basic properties of alliances and examine bounds on the size of minimal alliances in directed graphs. In general, the bounds established for alliances in undirected graphs do not hold when alliances are considered over the larger class of directed graphs and we construct examples which break these bounds.

Hedetniemi, Hedetniemi, and Kristiansen [4] introduced several definitions of alliances in undirected graphs. This model represents groups of nations or individuals who form alliances either for mutual protection or for aggression. Shafique and Dutton [8, 9] further generalize these definitions to k-alliances. Both models reflect two way relationships, so that if one nation can threaten another, then the relationship goes both ways and if one nation can defend another the relationship is mutual. This two-way model reflects relationships that are determined, for example, by geographic proximity. In this paper we consider the possibility that some or all of these relationships may be one way: Some nations may have greater power than neighboring nations, or some individuals may have the ability to influence others in a non-reciprocal fashion. We will expand the definitions for alliances in undirected graphs to directed graphs, but require some preliminary definitions.

A directed graph D will consist of a set of vertices $V(D)$ and a set of directed arcs $A(D)$. The directed graph may contain the pair of arcs (u,v) and (v,u) but may not have loops or multi-arcs. If there is an arc from u to v then u is an *in-neighbor* of v, and v is an *out-neighbor* of u. The set of in-neighbors of v is the *open in-neighborhood* of v, denoted $I(v)$. The *closed in-neighborhood* of v is the set $I[v] = I(v) \cup \{v\}$. For

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a set of vertices S, the open in-neighborhood of S is $I(S) = \bigcup_{v \in S} N(v)$, and the closed in-neighborhood of S is $I[S] = I(S) \cup S = \bigcup_{v \in S} N[v]$. The *in-boundary* of S, $\partial_i(S)$ is $I[S] - S$. The out-neighborhoods $O(v), O(S)$, closed out-neighborhoods $O[v], O[S]$ and out-boundary $\partial_o(S)$ are defined analogously.

The in-degree of a vertex v, denoted $id(v)$, is the size of the inset of v, $|I(v)|$. The minimum in-degree in D is denoted $\delta_-(D)$ and the maximum in-degree is $\Delta_-(D)$. The out-degree of v, denoted $od(v)$ is the size of the outset of v, $|O(v)|$. Likewise $\delta_+(D)$ and $\Delta_+(D)$ are the minimum and maximum out-degrees of D, respectively.

A non-empty set of vertices $S \subseteq V$ is called a *k-defensive alliance* if and only if, for every $v \in S$, $|I(v) \cap S| \geq |I(v) \cap (V - S)| + k$. Consistent with the alliances found in [4], if $k = -1$, then S is a *defensive alliance*, and for all $v \in S$, $|I[v] \cap S| \geq |I[v] \cap (V - S)|$. If $k = 0$, then S is a *strong defensive alliance* and for all $v \in S$, $|I[v] \cap S| > |I[v] \cap (V - S)|$. Given $v \in S$, $u \in (I[v] \cap S)$ *defends* v, and $u \in (I(v) - S)$ *attacks* v. Note that we say v defends itself, which is consistent with [4]. If $k \leq 0$ then V is always a k-defensive alliance, but it is possible, if $k \geq 1$, that a directed graph has no k-defensive alliance.

A non-empty set of vertices $S \subseteq V$ is a *k-offensive alliance* if for every $v \in \partial_o(S)$, $|I(v) \cap S| \geq |I(v) \cap (V - S)| + k$. If $k = 1$, then S is an *offensive alliance,* and for all $v \in \partial_o(S)$, $|I[v] \cap S| \geq |I[v] \cap (V - S)|$. If $k = 2$, then S is a *strong offensive alliance*, and for all $v \in \partial_o(S)$, $|I[v] \cap S| > |I[v] \cap (V - S)|$. Given $v \in \partial_o(S)$, $u \in (I(v) \cap S)$ *attacks* v, and $u \in (I[v] - S)$ *defends* v. The vertex v defends itself. In any directed graph V is a k-offensive alliance for any k.

In the case of k-defensive alliances, if every vertex is in S or $I[S]$, then S is a global alliance. A k-offensive alliance is global if every vertex is in S or $O[S]$. If $k = 0$, then S is an offensive alliance if and only if for every vertex v, $v \in S$ or at least $1/2$ of $I(v)$ is in S. This makes a global 0-offensive alliance a $1/2$-dominating set, as defined on undirected graphs by Dunbar, Hoffman, Laskar, and Markus [2] and generalized to directed graphs by Langley, Merz, Stewart, and Ward [5].

If D is a symmetric digraph, then the above definitions for alliances are consistent with the definitions for alliances on undirected graphs as seen in [4, 8]. Both papers consider bounds on alliance sizes. The definitions for bounds for directed graphs follow.

An alliance, S, of any kind is called *critical* if no proper subset of S is an alliance of the same kind.

The *lower k-defensive alliance number*, $a_k(D)$ is the minimum size of a k-defensive alliance in D. If $k = -1$, we will write $a(D)$. If $k = 0$, we will write $\hat{a}(D)$.

The *upper k-defensive alliance number*, $A_k(D)$ is the maximum size of a critical defensive alliance in D. If $k = -1$, we will write $A(D)$. If $k = 0$, we will write $\hat{A}(D)$.

The *lower k-offensive alliance number*, $a_k^o(D)$ is the minimum size of a k-offensive alliance in D. If $k = 1$, we will write $a^o(D)$. If $k = 2$, we will write $\hat{a}^o(D)$.

The *upper k-offensive alliance number*, $A_k^o(D)$ is the maximum size of a critical defensive alliance in D. If $k = 1$, we will write $A^o(D)$. If $k = 2$, we will write $\hat{A}^o(D)$.

Observe, as in undirected graphs, if S is a k-defensive alliance then S is a $(k-1)$-defensive alliance, and if S is a k-offensive alliance then S is a $(k-1)$-offensive alliance, so if $k < l$, $a_k(D) \leq a_l(D)$ provided each number is defined and $a_k^o(D) \leq a_l^o(D)$. Likewise, $A_k(D) \leq A_l(D)$ and $A_k^o(D) \leq A_l^o(D)$.

Next we look at examples of classes of directed graphs in defensive and offensive alliances, as well as possible bounds for the alliance numbers.

1. Defensive Alliances

Since the definitions of defensive alliances on directed graphs are consistent with the definitions of defensive alliances on graphs, many of the properties for defensive alliances in [4] still hold for the corresponding symmetric digraphs. We will not list constructions for classes of such symmetric graphs here.

Observation 1.1. If S is a critical k-defensive alliance of a directed graph D, then the induced subgraph of D restricted to S is weakly connected.

In any symmetric digraph a weakly connected subgraph is necessarily strongly connected. For general digraphs other possibilities may hold.

Observation 1.2. In a directed graph D, any vertex of in-degree zero is a k-defensive alliance for any $k \leq 0$. In general, any set S such that $I(S) - S = \emptyset$ is a k-defensive alliance if $k \leq 0$.

Observation 1.3. Suppose D is weakly connected with vertex set $V' \subset V$ such that $\partial_i(V') = \emptyset$ and let D' be the induced subgraph of D on V'. If S is a k-defensive alliance of D', then S is a k-defensive alliance of D. If S is critical in D', then S is critical in D.

Observation 1.4. If S is a k-defensive alliance and $v \in S$ is a vertex such that $|I(v)| = t$, then $|S| \geq (t+k)/2 + 1$.

Proof. This is an immediate consequence of the definition. If S is a k-defensive alliance, then $|I(v) \cap S| \geq |I(v) \cap (V - S)| + k$. Since $|I(v)| = t$, $|I(v) \cap S| + |I(v) \cap (V - S)| = t$, so $|I(v) \cap S| \geq t - |I(v) \cap S| + k$ and $|I(v) \cap S| \geq (t+k)/2$. Since S includes $I(v) \cap S$ and v as well, the inequality follows. □

Corollary 1.5. For any directed graph D, $a(D) \geq \lceil(\delta_-(D)+1)/2\rceil$ and $\hat{a}(D) \geq \lceil \delta_-(D)/2+1 \rceil$.

Corollary 1.6. For any directed graph D, $a_k(D) = 1$ if and only if D contains a vertex v with $id(v) \leq -k$.

For undirected graphs, it is conjectured in [4] that $a(G) \leq \lceil \frac{n}{2} \rceil$ and $\hat{a}(G) \leq \lceil \frac{n}{2} \rceil + 1$. The second bound was proven in [7]. Neither bound holds for directed graphs in general.

Define D on $3r$ vertices as follows: Partition the vertices into r directed 3-cycles: $C_1, C_2, C_3, \ldots, C_r$, then place arcs from each vertex of C_1 to each vertex of C_2 and so on, with C_r directed toward C_1. Each vertex will have in-degree 4, so if $v \in S$, then at least two in-neighbors of v are in S as well. Consequently, at least one vertex in each component must be in S. If v is any vertex in C_i such that $I(v) \cap C_i = \emptyset$, then at least two vertices of C_{i-1} must be in S. So, if C_i has exactly one vertex in S, then all 3 of C_{i+1} must be in S. If x is the number of C_i with exactly one vertex, then $|S| \geq x + 3x + 2(r-2x) = 2r$.

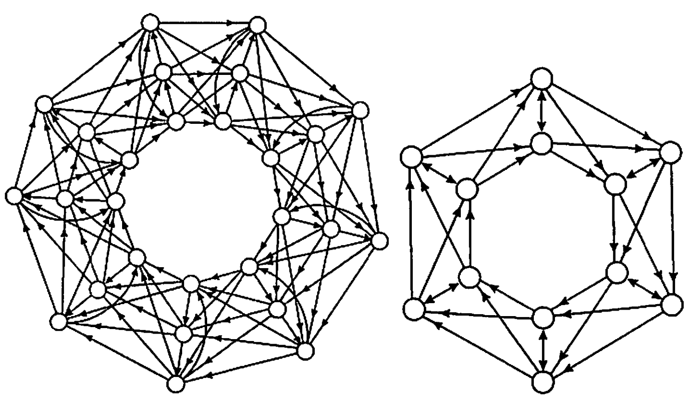

Figure 1: Graphs with relatively large $a(D)$ and $\hat{a}(D)$.

Conjecture 1. If D is a directed graph on n vertices, $a(D) \leq \frac{2n}{3}$.

What about $\hat{a}(D)$? On a directed cycle $\hat{a}(D) = n$. If we require vertices have out-degree at least two, then construct D on $2r$ vertices as follows: Partition the vertices into r pairs C_i, where C_i consists of a pair of mutually adjacent vertices. Place arcs from each vertex of C_1 to each vertex of C_2 and so on, with C_r directed toward C_1. Since each vertex has inset of size 3, if x is in S then so are at least two in-neighbors of x. If $x \in C_i \cap S$, but $y \in C_i \cap \overline{S}$, then both vertices of C_{i+1} must be in S. Consequently, $|S| \geq 3r/2$.

Conjecture 2. If D is a directed graph on n vertices and $\delta_-(D) \geq 2$, $\hat{a}(D) \leq \lceil \frac{3n}{4} \rceil$.

If D is an acyclic digraph, any k-defensive alliance must have a vertex v with $I(v) \cap S = \emptyset$. If $k \leq 0$, such a vertex is necessarily a k-defensive alliance on its own, but if $k > 0$, no vertex in S may have only itself as a defender. Consequently we have the following:

Theorem 1.7. Let D be acyclic digraph.

1. If $k \leq 0$, then $A_k(D) = a_k(D) = 1$.

2. If $k > 0$, then D has no k-defensive alliance.

We say D is *m-regular* if all vertices have the same in-degree and out-degree, both equal to m. If a regular graph is weakly connected it is also strongly connected. A component of D is a maximal strongly connected induced subgraph of D, and is necessarily regular.

Theorem 1.8. Let D be an m-regular directed graph. A set S is a k-defensive alliance in D if and only if the in-degree of each vertex in D restricted to S is at least $(m+k)/2$.

Proof. This follows from from the definition of k-defensive alliances. Suppose S is a defensive alliance. If $v \in S$, $|I(v) \cap S| \geq |I(v) \cap (V-S)| + k$. Since $|I(v)| = m$, $|I(v) \cap S| + |I(v) \cap (V-S)| = m$. So, $|I(v) \cap S| \geq m - |I(v) \cap S| + k$ and $|I(v) \cap S| \geq (m+k)/2$. On the other hand, suppose every vertex in D restricted to S has degree at least $(m+k)/2$. Then $2|I(v) \cap S| \geq (m+k)$. So, $|I(v) \cap S| \geq m - |I(v) \cap S| + k = |I(v) \cap (V-S)| + k$. ☐

Corollary 1.9. If $k \leq -m$, $a_k(D) = A_k(D) = 1$.

Proof. If $k \leq -m$ then $(m+k)/2 \leq 0$. Any set of vertices form a k-defensive alliance in D, so the only critical defensive alliances are single vertices. ☐

Corollary 1.10. If $k = m-1$ or $k = m$ then $a_k(D)$ is the size of the smallest component of D and $A_k(D)$ is the size of the largest component of D.

Proof. If $k = m$, then $(m+k)/2 = m$. If $k = m-1$, then $(m+k)/2 = m - 1/2$. However, the in-degree of any vertex is an integer. So, if $k = m-1$, then every vertex has in-degree m. In either case, the only subgraph of an m-regular graph D that is itself m-regular is a connected component of D. ☐

Corollary 1.11. If $k \geq m+1$, then D has no non-empty alliances.

Proof. If $k \geq m+1$, then every vertex in a defensive alliance must have in-degree at least $(m+m+1)/2$, which is greater than m. □

If D is 1-regular then D consists of the disjoint union of directed cycles. It follows from the preceding theorem and corollaries that any vertex of D is a defensive alliance, so $a_k(D) = A_k(D) = 1$ if $k \leq -1$. On the other hand if a vertex is in a strong alliance or $k = 1$, its in-neighbor must also be in the alliance. So, $a_k(D)$ is the size of the smallest directed cyclic component of D and $A_k(D)$ is the size of the largest directed cyclic component of D if $k = 0$ or $k = 1$.

Note, as a consequence, any directed cycle has $a(D) = 1$ and $\hat{a}(D) = n$. So, the difference between $a(D)$ and $\hat{a}(D)$ may be arbitrary large.

If D is 2-regular, then any alliance is also a strong alliance. From the preceding corollaries, $a_k(D) = A_k(D) = 1$ if $k \leq -2$, $a(D) = \hat{a}(D)$ equals the size of the smallest cycle in D, and $A(D) = \hat{a}(D) =$ the size of the largest chordless cycle in D. Also, if $k = 1$ or 2, $a_k(D)$ equals the size of the smallest connected component of D, and if $k = 1$ or 2 $A_k(D)$, equals the size of the largest connected component of D.

If D is 3-regular, $(3-1)/2 = 1$ so any cycle is a defensive alliance. Consequently $a(D)$ is the size of the smallest cycle and $A(D)$ is the size of the largest chordless cycle in D.

2. Offensive Alliances

Many of the properties of offensive alliances in directed graphs either parallel properties of defensive alliances or those of offensive alliances in undirected graphs as shown by Favaron, Fricke, Goddard, Hedetniemi, Hedetniemi, Kristiansen, Laskar, Skaggs, [3].

Observation 2.1. Any vertex of out-degree zero is a k-offensive alliance. If $O(S) - S$ is empty then S is a k-offensive alliance.

This is a trivial consequence of the definition of offensive alliances. Unlike the corresponding property of defensive alliances, an offensive alliance contained within such a set isn't necessarily an offensive alliance for D.

Observation 2.2. Let S be a k-offensive alliance of D. If $y \in O(S) - S$, and $|I(y)| = t$, then there are at least $(t+k)/2$ vertices in S.

The previous two observations may be combined to provide bounds on the size of offensive alliances in directed graphs. Let U be the smallest set of vertices with $O(U) = \emptyset$.

Observation 2.3. For any strongly connected directed graph D, $a_k^o(D) \geq \min\{|U|, \lceil(\delta^-(D)+k)/2\rceil\}$.

Favaron et al.[3] observe that bounds on $\hat{a}^o(G)$ are asymptotic to 1/2 over classes of graphs of increasing minimum degree. This is a direct consequence of a theroem in Füredi and Mubayi [6]. Although this theorem refers to undirected graphs, the probabilistic argument contained within the proof applies equally well to directed graphs.

Theorem 2.4. For directed graphs with order n and minimum in-degree δ_-, $a^o(D) \leq \hat{a}^o(D) \leq n(1/2 + o(\delta_-))$.

Also Favaron et al.[3] find bounds for the size of offensive alliances in undirected graphs. They observe that if $n \geq 2$, $a^o(G) \leq 2n/3$, and if $n \geq 3$, $\hat{a}^o \leq 5n/6$. For graphs of minimum degree 2, $\hat{a}^o \leq 3n/4$. Like the bounds on defensive alliances, these bounds also do not suffice for directed graphs in general.

To examine these bounds, we begin with a specific construction. Create a regular directed graph D_r on $n \geq r+1$ vertices as follows: Label the vertices $x_0, \ldots, x_{n-1}$. Place an arc from x_i to $x_{i+1 \ (mod\ n)}$, $x_{i+2 \ (mod\ n)}$, $\ldots, x_{i+r \ (mod\ n)}$.

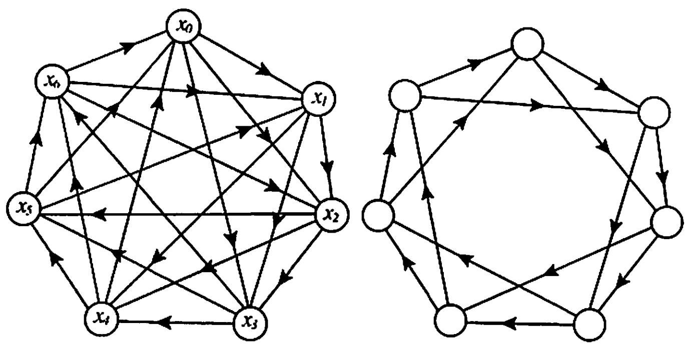

Figure 2: Construction of D_3 and D_2 on 7 vertices

Theorem 2.5. Let r be an integer greater than 1.

1. If r is even, then $a^o(D_r) = \hat{a}^o(D_r) = \left\lceil n\frac{r+2}{2(r+1)} \right\rceil$.

2. If r is odd, then $a^o(D_r) = \left\lceil \frac{n}{2} \right\rceil$.

3. If r is odd, then $\hat{a}^o(D_r) = \left\lceil n\frac{r+3}{2(r+1)} \right\rceil$.

Proof. First we show any offensive or strong offensive alliance in D_r is universal. If x_i is a vertex in the boundary of S, then there must be at least $\lceil \frac{r+1}{2} \rceil \geq 2$ vertices of $I(x_i)$ in S if S is an offensive alliance and

$\lceil \frac{r}{2}+1 \rceil \geq 2$ vertices of $I(x_i)$ in S if S is a strong offensive alliance. Suppose x_i is neither in S nor in $\partial_o(S)$. Then, since $x_{i-1(mod\ n)}$ is not in S, and $|In(x_i) \cap In(x_{i-1(mod\ n)})| = 1$, $x_{i-1\ (mod\ n)}$ cannot be in the out-boundary of S either. It follows that S must be empty, which is a contradiction. Therefore, every vertex in D_r is either in S or in the boundary of S.

Consider a set of $r+1$ consecutive vertices, $x_i, x_{i+1\ (mod\ n)}, x_{i+2\ (mod\ n)}, \ldots, x_{i+r\ (mod\ n)}$, in order. Let $x_{i+r-k\ (mod\ n)}, 0 \leq k \leq r$ be the last vertex in the list which is not in S. If no such vertex exists, then all $r+1$ vertices are in S. If S is an offensive alliance, at least $\lceil \frac{r+1}{2} \rceil - k$ predecessors of $x_{i+r-k\ (mod\ n)}$ in the set are in S, and the k successors of $x_{i+r-k\ (mod\ n)}$ are in S, so if S is an offensive alliance, in any $r+1$ consecutive vertices at least $\lceil \frac{r+1}{2} \rceil$ vertices are in S. Similar arguments show that at least $\lceil \frac{r}{2}+1 \rceil$ vertices out of any consecutive $r+1$ are in S if S is a strong offensive alliance.

Case 1: Since r is even, $\lceil \frac{r+1}{2} \rceil = \lceil \frac{r}{2}+1 \rceil = \frac{r+2}{2}$. If we look at the collection of sets of $r+1$ consecutive vertices, each vertex appears in $r+1$ sets. Averaging the number of vertices in S over the n sets, we have at least $\frac{n\frac{r+2}{2}}{r+1} = n\frac{r+2}{2(r+1)}$ vertices in S.

Case 2: Since r is odd, then $\lceil \frac{r+1}{2} \rceil = \frac{r+1}{2}$, so $\frac{n\lceil \frac{r+1}{2} \rceil}{r+1} = \frac{n}{2}$.

Case 3: Since S is a strong offensive alliance,$|S| \geq \frac{n\lceil \frac{r}{2}+1 \rceil}{r+1}$. Since r is odd, $\frac{n\lceil \frac{r}{2}+1 \rceil}{r+1} = n\frac{(r+3)}{2(r+1)}$.

Construct S to meet the bounds as follows:

Case 1: Place x_i into S if $i \equiv 2j+1\ (mod\ r+1)$, for $j = 0, \ldots, r/2$. That is, vertices with indices $0, 1, 3, 5, \ldots, r-1\ (mod\ r+1)$. This will create a minimum size offensive or strong offensive alliance.

Case 2: Place x_i into S if and only if i is even.

Case 3: The construction depends upon whether $(r+1)/2$ is even or odd. If $(r+1)/2$ is odd, then place x_i into S if $i \equiv 2j+1\ (mod\ (r+1)/2)$, for $j = 0, \ldots, (r-1)/4$. That is, vertices with indices $0, 1, 3, 5, \ldots, (r+1)/2-2, (r+1)/2, (r+1)/2+1, (r+1)/2+3, \ldots, r-1\ (mod\ r+1)$. If $(r+1)/2$ is even, then place x_i into S if $i \equiv (r-1)j/2\ (mod\ r+1)$, for $j = 0, \ldots, (r+1)/2$. That is, vertices with indices $0, 1, 3, 5, \ldots, (r-1)/2-2, (r-1)/2, (r-1)/2+1, (r-1)/2+3, \ldots, r-1\ (mod\ r+1)$.

□

Conjecture 3. If D is a directed graph, $a^o(D) \leq \lceil \frac{2}{3}n \rceil$.

This bound is met by the graphs constructed above when $r = 2$ and $n \geq 3$. If $n = 5$, then $a^o(D) = 4 = (4/5)n$ which is the worst case.

Conjecture 4. If D is a directed graph with $\delta_-(D) \geq 2$, $\hat{a}^o(D) \leq \lceil \frac{3}{4}n \rceil$.

This bound is met exactly by the graphs constructed above when $r = 3$,

$n \geq 4$. If $n = 7$, then $a^o(D) = 6 = (6/7)n$. As we will see now, for digraphs of low in-degree, $\hat{a}^o(D)$ may equal n.

We finish this section with some observations regarding m-regular directed graphs. Let D be an m-regular directed graph.

Theorem 2.6. If $k \leq -m + 2$, $a_k^o(D) = A_k^o(D) = 1$.

Proof. Note that for any $v \in \partial_o(S)$, $|I(v) \cap S| \geq 1$ and $|I(v) \cap (V - S)| \leq m - 1$. So $|I(v) \cap (V - S)| + k \leq (m - 1) - m + 2 = 1 \leq |I(v) \cap S|$. Consequently any non-empty set of vertices S is a k-offensive alliance, so the only critical alliances consist of a single vertex. □

Theorem 2.7. If $k > m$ then $a_k^o(D)$ is the size of the smallest component in D and $A_k^o(D)$ is the size of the largest component in D.

Proof. This is an obvious consequence of the definition of alliances. Since D is regular of degree m, no vertex may have $m + 1$ in-neighbors in any set S. In order for a set S to be a k-offensive alliance its outset must be empty. □

Corollary 2.8. For all directed cycles D on 2 or more vertices, $\hat{a}^o(D) = n$.

As with defensive alliances, in a directed cycle any single vertex is an offensive alliance so the bound between offensive alliances and strong offensive alliances may be made arbitrarily large.

3. Alternative Offensive Alliances

One consideration for a successful offensive alliance is whether it can conquer all of its neighbors, whichever the direction of influence. Such a construction would add the requirement that any offensive alliance have at least one arc toward all of its in-neighbors.

Define a *complete k-offensive alliance* as a set S which is a k-offensive alliance with the property that $\partial_i(S) \subseteq \partial_o(S)$.

Any global offensive alliance is complete. Any offensive alliance in an undirected graph is also a complete offensive alliance so this definition provides an alternative generalization to the work of [4, 3, 8]. Let $a_k^c(D)$ be the size of the smallest critical complete k-offensive alliance of D and $A_k^c(D)$ be the size of the largest critical complete k-offensive alliance of D. Clearly $a_k^o(D) \leq a_k^c(D)$ and $A_k^o(D) \leq A_k^c(D)$. As with earlier definitions, when $k = 1$, write $a^c(D)$ and when $k = 2$, write $\hat{a}^c$ or $\hat{A}^c$ and omit the subscript.

Theorems 2.4, 2.5, 2.7 apply to complete offensive alliances.

Observation 3.1. In the directed cycle $a^c(D) = A^c(D) = n - 1$.

Observation 3.2. For any directed graph D, $a_k^c(D) \geq \lceil (\delta^-(D) + k)/2 \rceil$.

References

[1] J.E. Dunbar, S.T. Hedetniemi, M.A. Henning, and P.J. Slater. Signed domination in graphs. *Graph Theory, Combinatorics and Algorithms,* Y.Alavi, and A. Schwenk, Eds., 311-321, Wiley, 1995

[2] J.E. Dunbar, D.G. Hoffman, R.C. Laskar, and L.R. Markus, "α-Domination." *Discrete Mathematics,* 211 (2000) 11-26.

[3] O. Favaron, G. Fricke, W. Goddard, S. Hedetniemi, S. Hedetniemi, P. Kristiansen, R. Laskar, D Skaggs. Offensive Alliances in Graphs, *Proceedings of the 17th Intrnl. Symposium on Computer and Information Sciences,* (2002), 298-302

[4] P. Kristiansen, S.M. Hedetniemi, S.T. Hedetniemi, Alliances in Graphs, *Proceedings of the 17th Intrnl. Symposium on Computer and Information Sciences,* (2002), 308-312.

[5] L.J. Langley, S.K.Merz, D.Stewart, C.Ward. α-Domination in tournaments. *Congressus Numerantium,* 157, (2002), 213-218.

[6] Z. Füredi, D. Mubayi. Signed domination in regular graphs and set-systems, *J. Combin. Theory Ser. B,* 76 (1999), 223-239

[7] K.H. Shafique and R.D. Dutton, On Satisfactory Partitioning of Graphs, *Congressus Numerantium,* 154,(2002) 183-194

[8] K.H. Shafique and R.D. Dutton, Maximum alliance-free and minimum alliance-cover sets, *Congressus Numerantium,* 162 (2003), 139-146.

[9] K.H. Shafique and R.D. Dutton, A Tight Bound on the Cardinalities of Maximum Alliance-Free and Minimum-Cover Sets, To appear: *Journal of Combinatorial Mathematics and Combinatorial Computing.*