Some remarks on lower bounds on the p-domination number in trees

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Abstract

Let G be a simple graph, and let p be a positive integer. A subset $D \subseteq V(G)$ is a *p-dominating set* of the graph G, if every vertex $v \in V(G) - D$ is adjacent with at least p vertices of D. The *p-domination number* $\gamma_p(G)$ is the minimum cardinality among the p-dominating sets of G. Note that the 1-domination number $\gamma_1(G)$ is the usual *domination number* $\gamma(G)$. The *covering number* of a graph G is denoted by $\beta(G)$. If T is a tree of order $n(T)$, then Fink and Jacobson [1] proved in 1985 that

$$\gamma_p(T) \geq \frac{(p-1)n(T)+1}{p}$$

The special case $p = 2$ of this inequality easily leads to

$$\gamma_2(T) \geq \beta(T) + 1 \geq \gamma(T) + 1$$

for every non-trivial tree T. Inspired by the article of Fink and Jacobson [1], we characterize in this paper the family of trees T with $\gamma_p(T) = \lceil((p-1)n(T)+1)/p\rceil$ as well as all non-trivial trees T with $\gamma_2(T) = \gamma(T) + 1$ and $\gamma_2(T) = \beta(T) + 1$.

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1. Terminology

We consider finite, undirected, and simple graphs G with vertex set $V(G)$ and edge set $E(G)$. The number of vertices $|V(G)|$ of a graph G is called the *order* of G and is denoted by $n = n(G)$.

The *open neighborhood* $N(v) = N_G(v)$ of a vertex v consists of the vertices adjacent to v and $d(v) = d_G(v) = |N(v)|$ is the *degree* of v. The *closed neighborhood* of a vertex v is defined by $N[v] = N_G[v] = N(v) \cup \{v\}$. By $\delta = \delta(G)$ and $\Delta = \Delta(G)$, we denote the *minimum degree* and the *maximum degree* of the graph G, respectively. A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*. An edge incident with a leaf is called a *pendant edge*. Let $L(G)$ be the set of leaves of a graph G. For a subset $S \subseteq V(G)$, we define $N(S) = N_G(S) = \bigcup_{v \in S} N(v)$, $N[S] = N_G[S] = N(S) \cup S$, and $G[S]$ is the subgraph induced by S.

We write K_n for the *complete graph* of order n, and $K_{p,q}$ for the the *complete bipartite graph* with bipartition X, Y such that $|X| = p$ and $|Y| = q$. A bipartite graph is called *p-semiregular*, if its vertex set can be partioned in such a way that every vertex in one partite set has degree p.

The *subdivision graph* $S(G)$ of a graph G is that graph obtained from G by replacing each edge uv of G by a vertex w and edges uw and vw. In the case that G is the trivial graph, we define $S(G) = G$. Let SS_t be the subdivision graph of the star $K_{1,t}$. A tree is a *double star* if it contains exactly two vertices of degree at least two. A double star with respectively s and t leaves attached at each support vertex is denoted by $S_{s,t}$. A *subdivided double star* $SS_{s,t}$ is obtained from a double star $S_{s,t}$ by subdividing each edge by exactly one vertex. The *corona graph* $G \circ K_1$ of a graph G is the graph constructed from a copy of G, where for each vertex $v \in V(G)$, a new vertex v' and a pendant edge vv' are added.

A vertex and an edge are said to *cover* each other if they are incident. A *vertex cover* in a graph G is a set of vertices that covers all edges of G. The minimum cardinality of a vertex cover in a graph G is called the *covering number* of G and is denoted by $\beta(G) = \beta$.

Let p be a positive integer. A subset $D \subseteq V(G)$ is a *p-dominating set* of the graph G, if $|N_G(v) \cap D| \geq p$ for every $v \in V(G) - D$. The *p-domination number* $\gamma_p(G)$ is the minimum cardinality among the p-dominating sets of G. Note that the 1-domination number $\gamma_1(G)$ is the usual *domination number* $\gamma(G)$. A p-dominating set of minimum cardinality of a graph G is called a $\gamma_p(G)$-set.

In [1], [2], Fink and Jacobson introduced the concept of p-domination. For a comprehensive treatment of domination in graphs, see the monographs by Haynes, Hedetniemi, and Slater [4], [5].

2. Preliminary results

The following well-known results play an important role in our investigations.

Theorem 2.1 (Ore [6] 1962) If G is a graph without isolated vertices, then

$$\gamma(G) \leq \frac{n(G)}{2}.$$

Theorem 2.2 (Fink, Jacobson [1] 1985) Let $p \geq 1$ be an integer. If T is a tree, then

$$\gamma_p(T) \geq \frac{(p-1)n(T)+1}{p} \tag{1}$$

and $\gamma_p(T) = ((p-1)n(T)+1)/p$ if and only if T is a p-semiregular tree or $n(T) = 1$.

Corollary 2.3 (Fink, Jacobson [1] 1985) If T is a tree, then

$$\gamma_2(T) \geq \frac{n(T)+1}{2}$$

and $\gamma_2(T) = (n(T)+1)/2$ if and only if T is the subdivision graph of another tree.

Theorem 2.4 (Payan, Xuong [7] 1982, Fink Jacobson, Kinch, Roberts [3] 1985) For a graph G with even order n and no isolated vertices, $\gamma(G) = n/2$ if and only if the components of G consist of the cycle C_4 or the corona graph $H \circ K_1$ for any connected graph H.

Proofs of Theorem 2.1 as well as of Theorem 2.4 can also be found in the book of Volkmann [9], pp. 223-224. In 1998, Randerath and Volkmann [8] and independently, in 2000, Xu, Cockayne, Haynes, Hedetniemi and Zhou [10] (cf. also [4], pp. 42-48) characterized the odd order graphs G for which $\gamma(G) = \lfloor n/2 \rfloor$. In the next theorem we only note the part of this characterization which we will use in the next section

Theorem 2.5 (Randerath, Volkmann [8] 1998, Xu, Cockayne, Haynes, Hedetniemi, Zhou [10] 2000) Let T be a non-trivial tree of odd order with $\gamma(T) = \lfloor n/2 \rfloor$. Then the following cases are possible:
(1) $|N_T(L(T))| = |L(T)| - 1$ and $T - N_T[L(T)] = \emptyset$.
(2) $|N_T(L(T))| = |L(T)|$ and $T - N_T[L(T)]$ is an isolated vertex.
(3) $|N_T(L(T))| = |L(T)|$ and $T - N_T[L(T)]$ is a star of order three such that the center of the star has degree two in T.

3. Main results

Inspired by Theorem 2.2, we will characterize the family of trees T which satisfies the identity $\gamma_p(T) = \lceil ((p-1)n(T)+1)/p \rceil$.

Theorem 3.1 If T is a tree of order $n = n(T)$, then

$$\gamma_p(T) = \left\lceil \frac{(p-1)n+1}{p} \right\rceil \tag{2}$$

if and only if

(i) $n = pt + 1$ for an integer $t \geq 0$ and T is a p-semiregular tree or $n(T) = 1$ or

(ii) $n = pt + r$ for integers $t \geq 0$ and $2 \leq r \leq p$ and T consists of r trees $T_1, T_2, \ldots, T_r$ which satisfy the conditions in (i) and $r-1$ further edges such that the trees $T_1, T_2, \ldots, T_r$ together with these $r-1$ edges result in a tree.

Proof. Condition (i) is a part of Theorem 2.2, and the identity $\gamma_p(T) = ((p-1)n(T)+1)/p$ yields $n = pt+1$ for an integer $t \geq 0$.

Therefore we assume next that $n = pt + r$ for any integers $t \geq 0$ and $2 \leq r \leq p$.

Assume that T satisfies the conditions given in (ii). It follows from inequality (1) and Condition (i) that

$$\begin{aligned}
\left\lceil \frac{(p-1)n+1}{p} \right\rceil &\leq \gamma_p(T) \\
&\leq \gamma_p(T_1) + \gamma_p(T_2) + \ldots \gamma_p(T_r) \\
&= \frac{(p-1)n(T_1)+1}{p} + \frac{(p-1)n(T_2)+1}{p} + \ldots \\
&\quad + \frac{(p-1)n(T_r)+1}{p} \\
&= \frac{(p-1)n+r}{p} = \left\lceil \frac{(p-1)n+1}{p} \right\rceil
\end{aligned}$$

and hence (2) is proved.

Conversely, assume that the identity (2) is valid. This implies that $\gamma_p(T) = (p-1)t + r$. If D is a $\gamma_p(T)$-set and $S = V(T) - D$, then $|D| = (p-1)t + r$ and $|S| = t$. In view of the definition of D, each vertex in S is adjacent with p or more vertices of D. Now let E_p be an edge set of $E(T)$ consisting of $p|S| = pt$ edges such that each vertex of S is incident with exactly p edges leading from S to D. Then the subgraph induced by the edges of E_p is a p-semiregular forest T'. Because of $|E(T)| = pt + r - 1$ and $|E(T')| = pt$, there are $r-1$ further edges in T. If we delete in T all edges of the edge set $E(T) - E_p$, then we obtain a spanning forest T'' of T, consisting of r trees $T_1, T_2, \ldots, T_r$, which are p-semiregular or isolated vertices. Since D is a $\gamma_p(T)$-set, we deduce that $D \cap V(T_i)$ is a minimum p-dominating set of T_i for $i = 1, 2, \ldots, r$. Let now $n(T_i) = pt_i + k_i$ with integers $t_i \geq 0$ and $1 \leq k_i \leq p$ for $i = 1, 2, \ldots, r$. Next we will show that $n(T_i) = pt_i + 1$ for $i = 1, 2, \ldots, r$. If not, then $k_1 + k_2 + \ldots + k_r \geq r + 1$. Since

$$n = pt + r = p(t_1 + t_2 + \ldots + t_r) + (k_1 + k_2 + \ldots k_r),$$

it follows that $t_1 + t_2 + \ldots + t_r \leq t - 1$. Applying inequality (1), we arrive at the following contradiction:

$$\begin{aligned}
\gamma_p(T) &= (p-1)t + r = \gamma_p(T_1) + \gamma_p(T_2) + \ldots \gamma_p(T_r) \\
&\geq \left\lceil \frac{(p-1)n(T_1)+1}{p} \right\rceil + \left\lceil \frac{(p-1)n(T_2)+1}{p} \right\rceil + \ldots \\
&\quad + \left\lceil \frac{(p-1)n(T_r)+1}{p} \right\rceil \\
&= ((p-1)t_1 + k_1) + ((p-1)t_2 + k_2) + \ldots + ((p-1)t_r + k_r) \\
&= (pt_1 + k_1) + (pt_2 + k_2) + \ldots + (pt_r + k_r) - (t_1 + t_2 + \ldots + t_r) \\
&= n - (t_1 + t_2 + \ldots + t_r) \\
&\geq pt + r - (t-1) \\
&= (p-1)t + r + 1 \\
&> |D|
\end{aligned}$$

However, if $n(T_i) = pt_i + 1$ for $i = 1, 2, \ldots, r$, then (1) leads to

$$\begin{aligned}
\gamma_p(T) &= (p-1)t + r = \gamma_p(T_1) + \gamma_p(T_2) + \ldots \gamma_p(T_r) \\
&\geq \frac{(p-1)n(T_1)+1}{p} + \frac{(p-1)n(T_2)+1}{p} + \ldots + \frac{(p-1)n(T_r)+1}{p} \\
&= \frac{(p-1)n + r}{p} \\
&= \frac{(p-1)(pt+r) + r}{p} \\
&= (p-1)t + r.
\end{aligned}$$

This implies $\gamma_p(T_i) = ((p-1)n(T_i)+1)/p$ for $i = 1, 2, \ldots, r$. Hence, according to (i), we deduce that the trees $T_1, T_2, \ldots, T_r$ are p-semiregular trees or isolated vertices and the proof is complete. □.

According to Theorem 3.1 and Corollary 2.3, we obtain the next result.

Corollary 3.2 If T is a tree of order $n = n(T)$, then

$$\gamma_2(T) = \left\lceil \frac{n+1}{2} \right\rceil$$

if and only if

(i) n is odd and T is a the subdivision graph of another tree or

(ii) n is even and T consists of two subdivision trees $S(T_1)$ and $S(T_2)$ and a further edge, connecting $S(T_1)$ with $S(T_2)$.

Theorem 3.3 (Fink, Jacobson [1] 1985) If $p \geq 2$ is an integer and G is a graph with $p \leq \Delta(G)$, then

$$\gamma_p(G) \geq \gamma(G) + p - 2.$$

If $p \geq 3$, then Theorem 3.3 shows that $\gamma_p(G) \geq \gamma(G) + 1$. This is not true in general for $p = 2$, as for example the complete bipartite $K_{2,q}$ with $q \geq 2$ demonstrate. However, for non-trivial trees, we will show in the following proposition that this bound is true.

Proposition 3.4 If T is a non-trivial tree, then

$$\gamma_2(T) \geq \beta(T) + 1 \geq \gamma(T) + 1.$$

Proof. Since T is a bipartite graph, we observe that $\beta(T) \leq n(T)/2$. Hence it follows from Corollary 2.3 that

$$\gamma_2(T) \geq \frac{n(T)+1}{2} \geq \beta(T) + \frac{1}{2}.$$

Because of the well-known fact that $\beta(T) \geq \gamma(T)$, this yields the desired inequality chain. □

Next we will characterize the family of trees T with $\gamma_2(T) = \gamma(T) + 1$ as well as all trees T with $\gamma_2(T) = \beta(T) + 1$.

Theorem 3.5 A non-trivial tree T satisfies

$$\gamma_2(T) = \gamma(T) + 1 \tag{3}$$

if and only if T is a subdivided star SS_t or a subdivided star SS_t minus a leaf or a subdivided double star $SS_{s,t}$.

Proof. It is a simple matter to verify that (3) is valid for a subdivided star SS_t, a subdivided star SS_t minus a leaf, and a subdivided double star $SS_{s,t}$.

Conversely, assume that $\gamma_2(T) = \gamma(T) + 1$ for a non-trivial tree T. It follows from Theorem 2.1 that $\gamma(T) \leq n(T)/2$.

Case 1. Assume that $n = n(T)$ is even. If $\gamma(T) < n(T)/2$, then Corollary 2.3 yields

$$\gamma_2(T) \geq \frac{n+2}{2} = \frac{n-2}{2} + 2 \geq \gamma(T) + 2.$$

In the remaining case that $\gamma(T) = n/2$, we deduce from Theorem 2.4 that $T = T' \circ K_1$ for any tree T'. It is straightforward to verify that T is a subdivided star SS_t minus a leaf.

Case 2. Assume that $n = n(T)$ is odd. If $\gamma(T) < n(T)/2$, then Corollary 2.3 leads to

$$\gamma_2(T) \geq \frac{n+1}{2} = \frac{n-3}{2} + 2 \geq \gamma(T) + 2.$$

Therefore let now $\gamma(T) = (n-1)/2$. In view of Theorem 2.5, we have to investigate three cases.

Subcase 2.1. Assume that $|N_T(L(T))| = |L(T)| - 1$ and furthermore that $T - N_T[L(T)] = \emptyset$. This leads to $T = SS_1$, because otherwise, we arrive at the contradiction $\gamma_2(G) \geq (n+3)/2 = \gamma(T) + 2$.

Subcase 2.2. Assume that $|N_T(L(T))| = |L(T)|$ and $T - N_T[L(T)]$ is an isolated vertex. This easily shows that $T = SS_t$.

Subcase 2.3. Assume that $|N_T(L(T))| = |L(T)|$ and $T - N_T[L(T)]$ is a star of order three such that the center of the star has degree two in T. It is a simple matter to obtain $T = SS_{s,t}$. □

Theorem 3.6 A non-trivial tree T of order $n = n(T)$ satisfies

$$\gamma_2(T) = \beta(T) + 1 \tag{4}$$

if and only if

a) n is odd and T is the subdivision graph of another non-trivial tree or

b) n is even and T consists of two subdivision trees $S(T_1)$ and $S(T_2)$ and an edge e between $S(T_1)$ and $S(T_2)$ such that e is incident with one vertex of T_1 and one of T_2.

Proof. It is an easy exercise to verify that (4) is valid for the collection of trees given in a) and b).

Conversely, assume that $\gamma_2(T) = \beta(T) + 1$ for a non-trivial tree T. It follows from Corollary 2.3 that $\gamma_2(T) \geq (n+1)/2$.

Case 1. Assume that $n \geq 3$ is odd. If $\gamma_2(T) \geq (n+3)/2$, then

$$\gamma_2(T) \geq \frac{n}{2} + \frac{3}{2} \geq \beta(T) + \frac{3}{2}$$

and thus $\gamma_2(T) \geq \beta(T) + 2$. This is a contradiction to the hypothesis $\gamma_2(T) = \beta(T) + 1$. In the remaining case that $\gamma(T) = (n+1)/2$, we deduce from Corollary 3.2 that T is the subdivision graph of another non-trivial tree.

Case 2. Assume that $n \geq 2$ is even. If $\gamma_2(T) \geq (n+4)/2$, then we arrive at the contradiction

$$\gamma_2(T) \geq \frac{n}{2} + 2 \geq \beta(T) + 2.$$

In the remaining case that $\gamma(T) = (n+2)/2$, we deduce from Corollary 3.2 that T consists of two subdivision trees $S(T_1)$ and $S(T_2)$ and a further edge e, connecting $S(T_1)$ with $S(T_2)$. In the case that e is incident with a vertex from $V(S(T_i)) - V(T_i)$ for $i = 1$ or $i = 2$, we observe that $\beta(T) = (n-2)/2$ and $\gamma_2(T) = (n+2)/2$ and hence $\gamma_2(T) = \beta(T) + 2$. Hence e is incident with one vertex of T_1 and one of T_2. □

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