

New 2-critical sets in the abelian 2-group

Carlo Hamalainen

Centre for Discrete Mathematics and Computing
Department of Mathematics
The University of Queensland
Queensland 4072, Australia
carloh@maths.uq.edu.au

Abstract

In this paper we determine a class of critical sets in the abelian 2-group that may be obtained from a greedy algorithm. These new critical sets are all 2-critical (each entry intersects an an intercalate, a trade of size 4) and completes in a top down manner.

1 Introduction

Critical sets are minimal defining sets in latin squares [4]. Some recent work has investigated the structure and size of critical sets in the latin square L_s derived from the abelian 2-group of order 2^s ([6], [5]). In this paper we present a new family of critical sets derived from isotopisms of L_s .

Section 2 presents background definitions. Section 3 has basic properties of greedy critical sets. Then Section 4 develops some properties of greedy critical sets in L_s , and Section 5 completes the proof of the main result, which is Theorem 5.1. The Appendices provide extra examples to aid in the understanding of the Theorem and also have more detail for the inductive hypotheses.

2 Definitions

We begin with some definitions. Let $N_n^k = \{nk, nk + 1, \dots, nk + n - 1\}$ for integers $k \geq 0$ and $n > 0$. A *latin square* L of order n is an $n \times n$ array with rows indexed by N_n^k , columns by $N_n^{k'}$, and with entries from the set $N_n^{k''}$. Further, each $e \in N_n^{k''}$ appears exactly once in each row and

exactly once in each column. This is equivalent to the usual definition where $k = k' = k'' = 0$ but allows more flexibility when discussing subsquares. A *partial latin square* is an $n \times n$ array where each entry of $N_n^{k''}$ occurs at most once in each row and at most once in each column.

A latin square L may also be represented as a set of ordered triples, where $(r, c; e) \in L$ denotes the fact that symbol e appears in the cell at row r , column c , of L . The *size* of a partial latin square P is the number of filled cells, denoted by $|P| = |\{(r, c; e) \mid (r, c; e) \in P\}|$.

A partial latin square L of order n is *isotopic* to L' (also of order n) if the rows, columns, and entries of L can be rearranged to obtain L' . Specifically, we say that L is isotopic to L' if there exist permutations α, β, γ on the row labels, column labels, and symbols (respectively) such that $L' = \{(\alpha r, \beta c; \gamma e) \mid (r, c; e) \in L\}$. We say that (α, β, γ) is an *isotopism* from L onto L' , and we write this as $L' = (\alpha, \beta; \gamma)L$. We write α instead of (α, ι, ι) when it is clear from the context that the columns and entries are left fixed.

Given a partial latin square P of order n , we define the partial latin square $P^r = \{(i, j; k + nr) \mid (i, j; k) \in P\}$. Note that if P has symbols selected from N_n^p , then P^r has symbols from N_n^{p+nr} . We use this exponent notation when recursively constructing larger partial latin squares. For example, suppose that A, B, C , and D are partial latin squares of order n . Then by

$$P = \begin{array}{|c|c|} \hline A & B \\ \hline C & D \\ \hline \end{array}$$

we mean the partial latin square P of order $2n$ where

$$P = \{(i, j; k) \mid (i, j; k) \in A\} \cup \{(i, j + n; k) \mid (i, j; k) \in B\} \\ \cup \{(i + n, j; k) \mid (i, j; k) \in C\} \cup \{(i + n, j + n; k) \mid (i, j; k) \in D\}$$

Let P and Q be partial latin squares of order n . Suppose that α, β, γ are bijections between the row, column, and symbol sets (respectively) of P and Q such that

1. $P = (\alpha, \beta, \gamma)Q$.
2. α and β are monotone.

Then P and Q are said to be *similar*, written $P \approx Q$. Informally, P and Q are similar when the rows and columns of Q can be relabelled (preserving order) to give Q' such that $P = (\iota, \iota, \gamma)Q'$.

Given a partial latin square P we can define a binary relation (P, \ll) on the elements of P as follows (see also [1]). For all $(x, y; z), (r, s; t) \in P$, $(x, y; z) \ll (r, s; t)$ if and only if

1. $x < r$, or
2. $x = r$ and $y \leq s$.

We can verify that $(x, y; z) \ll (x, y; z)$ so \ll is reflexive. If $(x, y; z) \ll (r, s; t)$ and $(r, s; t) \ll (x, y; z)$ then $x = r$ and $y = s$, so \ll is antisymmetric. Finally, suppose that $(x, y; z) \ll (r, s; t)$ and $(r, s; t) \ll (u, v; w)$. If $x < r$ then $x < u$, so $(x, y; z) \ll (u, v; w)$. On the other hand, if $x = r$ and $r < u$ then $x < u$, so $(x, y; z) \ll (u, v; w)$ again. Finally, if $x = r$ and $r = u$, then $y \leq s$ and $s \leq v$ so $y \leq v$, which implies that $(x, y; z) \ll (u, v; w)$. Hence \ll is transitive, and (P, \ll) is a weak partial order.

In fact, (P, \ll) is a total order since for any distinct $(x, y; z), (r, s; t) \in P$, either $x < r$, or $r < x$, or $r = x$ and $y \leq s$ or $s \leq y$. Given a partial latin square P we denote the *least element* of (P, \ll) by $(rl_P, cl_P; el_P)$ and the *greatest element* by $(rg_P, cg_P; eg_P)$. Since \ll is the only partial order used in this paper we simply say that $(i, j; k) \in P$ is the least (greatest) element of P .

It is convenient to refer to the set of entries occurring in a particular row or column of a partial latin square P . For each row i of P , define $\mathcal{R}_P^i = \{k \mid \text{there exists } j \text{ such that } (i, j; k) \in P\}$. Also, for each column j of P , we define $\mathcal{C}_P^j = \{k \mid \text{there exists } i \text{ such that } (i, j; k) \in P\}$. The *shape* of a partial latin square P is the set of filled cells, defined by $S_P = \{(i, j) \mid (i, j; k) \in P\}$.

For some partial latin square P we use the following notation to specify a subsquare:

$$\mathcal{Q}_{i,j}^k(P) = \{(x, y; z) \in P \mid i \leq x < i + k, j \leq y < j + k\}$$

We also use this notation for defining subsquares in a partial latin square. For example, $\mathcal{Q}_{i,j}^k(P) = L$ places the order k latin square L into P starting with the top-left corner at cell (i, j) .

Let P be a partial latin square of order n contained in the latin square L . Without loss of generality, suppose that the rows and columns are indexed by $N_n = N_n^0$, and that each entry is from N_n . Let $R \subseteq N_n$, $C \subseteq N_n$, and $S = R \times C$. For each $(r, c) \in S$, define

$$S_{r,c} = \begin{cases} \emptyset, & \text{if } (r, c; e) \in P \text{ for some } e \in N_n \\ N \setminus (\mathcal{R}_P^r \cup \mathcal{C}_P^c), & \text{otherwise.} \end{cases}$$

Then the *array of alternatives* of S with respect to P and L is given by $\mathcal{A}(P, S, L) = \{(r, c; S_{r,c}) \mid (r, c) \in S\}$. For clarity we write $\mathcal{A}(P, S, L)_{r,c}$ for $S_{r,c}$.

We say that $\mathcal{A}(P, S, L)$ is similar to $\mathcal{A}(P', S', L')$ if there are relabellings of the row names, column names and symbols so that the table for $\mathcal{A}(P, S, L)$ is equal to the relabelled table for $\mathcal{A}(P', S', L')$.

A partial latin square T forms a *latin trade* in a latin square of order n if there exists a partial latin square T' , the *disjoint mate*, such that:

1. T and T' are of the same order.
2. $\{(i, j) \mid (i, j; k) \in T \text{ for some symbol } k\}$
 $= \{(i, j) \mid (i, j; k') \in T' \text{ for some symbol } k'\}$
3. For each $(i, j; k) \in T$ and $(i, j; k') \in T'$, $k \neq k'$.
4. For each $i \in N$, $\mathcal{R}_T^i = \mathcal{R}_{T'}^i$, and $\mathcal{C}_T^i = \mathcal{C}_{T'}^i$.

Informally, Condition 2 says that T and T' have the same shape, Condition 3 says that they are disjoint, and Condition 4 says that T and T' are *row balanced* and *column balanced*.

Let L and L' be two disjoint latin square of the same order. Let $T = L \setminus L'$ and $T' = L' \setminus L$. Then T and T' form a latin trade. We assume that all latin trades are nonempty. A partial latin square P is *uniquely completable* if there is just one latin square L of the same order as P such that $P \subseteq L$.

A partial latin square P of order n is *strongly completable* if it is uniquely completable to L , there is a sequence of partial latin squares $P_0 = P \subset P_1 \subset P_2 \subset \dots \subset P_m = L$ where $m = n^2 - |P|$, and for each P_k there exists r, c such that $|\mathcal{A}(N_n \times N_n, P_k)_{r,c}| = 1$.

A partial latin square $C \subseteq L$ is a *critical set* if

1. C has unique completion to L , and
2. no proper subset of C satisfies 1.

A *strong critical set* is a critical set that has strong completion. We say that a (uniquely completable) partial latin square *extends top down* if, given that rows $0, 1, \dots, i$ are filled in, then row $i + 1$ can be shown to have unique extension. If all rows can be extended in this manner then the critical set has *unique completion top down*.

Lemma 2.1. *Let P be a critical set in the latin square L and T a latin trade in L . Then $P \cap T \neq \emptyset$.*

Lemma 2.2. *Let L be a latin square and $C \subseteq L$ a critical set. For each $x \in C$ there exists a latin trade $T \subseteq L$ such that $C \cap T = \{x\}$.*

The latin trade containing the least number of entries is a 2×2 sub-square, known as an *intercalate*. Let C be a critical set in L , $c \in C$, and $I \subseteq L$ an intercalate such that $C \cap I = \{c\}$. Then c is said to be *2-essential*. If all $c \in C$ are 2-essential then C is *2-critical*.

Algorithm A

Input: Partial latin square P of order n with unique completion, and bijection $f : \{1, \dots, |P|\} \rightarrow \mathcal{S}_P$.

```
 $P_0 \leftarrow P$ 
for  $i = 1, \dots, |P|$ 
  let  $x, y, z$  be integers such that  $(x, y; z) \in P_{i-1}$  and  $f(i) = (x, y)$ 
  if  $P_{i-1} \setminus \{(x, y; z)\}$  has unique completion then
     $P_i \leftarrow P_{i-1} \setminus \{(x, y; z)\}$ 
  else
     $P_i \leftarrow P_{i-1}$ 
return  $P_{|P|}$ 
```

3 Greedy Critical Sets

Algorithm A was first presented in [1]. Given a partial latin square P with unique completion, and a bijection on its cells, the algorithm produces a critical set.

Lemma 3.1 (Lemma 2.1, [1]). *Let P be a partial latin square that uniquely completes to L . Then for every bijection f over $\{1, \dots, |P|\}$, Algorithm A returns a critical set.*

Proof. Algorithm A works on a sequence of partial latin squares, $P_0 = P \supseteq P_1 \supseteq \dots \supseteq P_m$ where $m = |P|$. The initial partial latin square $P_0 = P$ has unique completion, and the if statement ensures that each P_i , for $i > 0$, has unique completion. Hence P_m has unique completion.

To see that P_m is minimal, suppose otherwise. Then there is an $x \in P_m$ such that $P_m \setminus \{x\}$ has unique completion. Also, let k be the integer such that $f(k) = x$. Then P_k is the partial latin square where x is inspected (and not removed) by Algorithm A. Since $P_m \setminus \{x\}$ has unique completion, we can add entries to $P_m \setminus \{x\}$ until we have precisely $P_k \setminus \{x\}$. This has unique completion, yet Algorithm A apparently did not remove x , a contradiction. Hence P_m is minimal and so P_m is a critical set. \square

Since a latin square trivially has unique completion, we get:

Corollary 3.2. *If the input to Algorithm A is a latin square of order n then the output is a critical set for any bijective function f .*

We refer to Algorithm A as the *generalised greedy critical set* algorithm, and abbreviate this to $ggcs(L, f)$ for given latin square L and map f .

Lemma 3.3. *These two sets are equal:*

$$\{ggs(L, f) \mid f \text{ is a bijection on } \{1, \dots, n^2\}\}$$

and

$$\{C \mid C \subseteq L \text{ and } C \text{ is a critical set of } L\}$$

for some latin square L of order n .

Let $f_0 : \{1, \dots, n^2\} \rightarrow \mathcal{S}_L$ be the bijection defined by

$$f_0(i) = \left(\left\lfloor \frac{i-1}{n} \right\rfloor, n-i \pmod{n} \right)$$

for $1 \leq i \leq n^2$ and L of order n . Then f_0 orders the cells of L from right to left along each row and from the bottom row to the top row. We abbreviate $ggs(L, f_0)$ to $gcs(L)$ and call this the *greedy critical set of L* .

We now characterise greedy critical sets in terms of the partial order \ll . Let L be a latin square, and $\mathcal{I} = \{I \mid I \subset L \text{ and } I \text{ is a latin trade}\}$. Each $I \in \mathcal{I}$ is a partial latin square implying I has a least element and a greatest element.

Lemma 3.4 (Lemma 2.4, [1]). *Let C be a critical set in L . Then $C = gcs(L)$ if and only if for all $(x, y, z) \in C$ there exists an $I \in \mathcal{I}$ such that $I \cap C = \{(x, y, z)\}$ and $(x, y, z) = (rl_I, cl_I; el_I)$.*

Proof. (if) Algorithm A with input L and map f_0 computes on a sequence of partial latin squares $P_1 = L \supseteq P_2 \supseteq \dots \supseteq P_m = C$. Suppose that $f_0(k) = (x, y)$ and that $P_{k-1} \setminus \{(x, y, z)\}$ completes to $L_0 = L, L_1, \dots, L_s$ for some $s \geq 1$. Then there is an L_i , $i \geq 1$, such that $P_{k-1} \cap (L \setminus L_i) = \{(x, y, z)\}$. In other words, $T = L \setminus L_i$ is a latin trade in L . The definition of f_0 implies that for any $(r, c; e) \in T$ then either $r > x$, or, if $r = x$ then $c \geq y$. Hence (x, y, z) is the least element of T .

(only if) Assume that for all $(x, y, z) \in C$ there exists an $I \in \mathcal{I}$ such that $I \cap C = \{(x, y, z)\}$ and $(x, y, z) = (lr_I, lc_I; le_I)$, but C is not the greedy critical set $gcs(L)$. Let $D = L \cap ((C \setminus gcs(L)) \cup (gcs(L) \setminus C))$, that is, the intersection with the symmetric difference.

The set D is a partial latin square and has a greatest element $(gr_D, gc_D; ge_D)$ since $D \neq \emptyset$. Thus for all $(a, b, c) \in L$ such that $a > gr_D$, or $a = gr_D$ and $b > gc_D$, $(a, b, c) \in C$ if and only if $(a, b, c) \in gcs(L)$. The reason is that (a, b, c) is not in D , so $(a, b, c) \notin ((C \setminus gcs(L)) \cup (gcs(L) \setminus C))$.

By the definition of D there are two possibilities:

1. $(gr_D, gc_D; ge_D) \in C$, and $(gr_D, gc_D; ge_D) \notin gcs(L)$. Since $(gr_D, gc_D; ge_D)$ is in C , there exists an $I \in \mathcal{I}$ such that $I \cap C = \{(gr_D, gc_D; ge_D)\}$

and $(gr_D, gc_D; ge_D) = (lr_I, lc_I; le_I)$. But for all $(a, b; c) \in L$ such that $a > gr_D$, or $a = gr_D$ and $b > gc_D$, $(a, b; c) \in C$ if and only if $(a, b; c) \in gcs(L)$, so $I \cap gcs(L) = \emptyset$, which is a contradiction.

2. $(gr_D, gc_D; ge_D) \in gcs(L)$, and $(gr_D, gc_D; ge_D) \notin C$. Let k be the integer such that $f_0(k) = (gr_D, gc_D)$. Then at step k Algorithm A removes $(gr_D, gc_D; ge_D)$ and $P_{k-1} \setminus \{(gr_D, gc_D; ge_D)\}$ is found to have at least two completions, say L and L' . So $T = L \setminus L'$ is a latin trade and the least element of T is $(gr_D, gc_D; ge_D)$. Once again, this implies that $T \cap C = \emptyset$, which is a contradiction.

Hence $D = \emptyset$, which contradicts our original assumption that C was different to the greedy critical set. \square

Corollary 3.5. *Let L be a latin square of order n and $G = gcs(L)$. If $(i, j; k) \in G$ then $i \neq n - 1$ and $j \neq n - 1$.*

4 Greedy Critical Sets in the Abelian 2-Group

We define L_s to be the latin square corresponding to the abelian 2-group of order $n = 2^s$ and the partial latin square $P_s \subset L_s$ as in [2]. That is,

$$P_1 = \begin{array}{|c|c|} \hline 0 & \\ \hline & \\ \hline \end{array} \quad L_1 = \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array}$$

and for $s \geq 2$,

$$L_s = L_1 \times L_{s-1} = \{(x, y; z), (x, y + n/2; z + n/2), (x + n/2, y; z + n/2), (x + n/2, y + n/2; z) \mid (x, y; z) \in L_{s-1}\}, \text{ and}$$

$$P_s = P_1 \otimes P_{s-1} = \{(x, y; z), (u, v + n/2; w + n/2), (u + n/2, v; w + n/2), (u + n/2, v + n/2; w) \mid (u, v; w) \in P_{s-1} \text{ and } (x, y; z) \in L_{s-1}\}.$$

For example, L_3 and P_3 are:

0	1	2	3	4	5	6	7
1	0	3	2	5	4	7	6
2	3	0	1	6	7	4	5
3	2	1	0	7	6	5	4
4	5	6	7	0	1	2	3
5	4	7	6	1	0	3	2
6	7	4	5	2	3	0	1
7	6	5	4	3	2	1	0

0	1	2	3	4	5	6	
1	0	3	2	5	4		
2	3	0	1	6		4	
3	2	1	0				
4	5	6	7	0	1	2	
5	4			1	0		
6		4		2		0	

In general, we may take a latin square L of order $n/2$ and form the order n latin square $L_1 \times L$ by defining:

$$L_1 \times L = \{(x, y, z), (x, y + n/2, z + n/2), (x + n/2, y, z + n/2), \\ (x + n/2, y + n/2, z) \mid (x, y, z) \in L\}$$

The next Lemma is similar to the doubling construction of [3] which gives 2-critical sets.

Lemma 4.1. *Let M be a latin square of order n such that $\text{gcs}(M)$ is 2-critical. Then $\text{gcs}(L_1 \times M)$ is 2-critical.*

Proof. Define P , a partial latin square of order $2n$, by

$$P = \begin{array}{|c|c|} \hline M & \text{gcs}^1(M) \\ \hline \text{gcs}^1(M) & \text{gcs}(M) \\ \hline \end{array}$$

Note that $P \subset L_1 \times M$. Choose some $(i, j, k) \in \mathcal{Q}_{0,0}^n(P)$. Then (i, j, k) will be 2-essential in one of two ways:

1. If $(i, j + n, k') \notin P$ then the set of cells

$$I = \{(i, j, k), (i, j + n, k'), (i + n, j, k'), (i + n, j + n, k)\}$$

is an intercalate in $L_1 \times M$ such that $P \cap I = \{(i, j, k)\}$.

2. Otherwise, $(i, j + n, k') \in P$. Since $(i, j + n, k') \in \text{gcs}^1(M)$ which is 2-critical, for some integers $0 < |a|, |b| < n$ there exists an intercalate

$$I = \{(i, j + n, k'), (i + a, j + n, l), (i, j + n + b, l), (i + a, j + n + b, k')\}$$

for which $I \cap \text{gcs}^1(M) = \{(i, j + n, k')\}$. Hence there is an intercalate

$$I' = \{(i, j, k), (i, j + n + b, l), (i + n + a, j, l), (i + n + a, j + n + b, k)\}$$

such that $I' \cap P = \{(i, j, k)\}$ implying that (i, j, k) is 2-essential.

Hence each $(i, j, k) \in P$ is the least element of an intercalate so P is 2-critical by Lemma 3.4. \square

Corollary 4.2. *For all $s \geq 1$, $\text{gcs}(L_s) = P_s$ and P_s is 2-critical.*

Lemma 4.3. *Let α be a row isotopism of L_s defined below:*

$$\alpha(i) = \begin{cases} 4k_1 + 1, & i = 4k_1 + 2 \\ 4k_1 + 2, & i = 4k_1 + 1 \\ \vdots & \\ 4k_p + 1, & i = 4k_p + 2 \\ 4k_p + 2, & i = 4k_p + 1 \\ i, & \text{otherwise} \end{cases}$$

where

$$0 \leq p < 2^{s-2}, k_i \neq k_j \text{ for } i \neq j, \text{ and } 0 \leq 4k_i < 2^s \quad (1)$$

Then $\text{gcs}(\alpha L_s)$ is 2-critical.

Proof. We proceed by induction. There are two base cases to check. First, define H_2 by the bracketed entries in the following square and \hat{H}_2 to be the completion (as shown) of H_2 .

(0)	(1)	(2)	3
(2)	3	(0)	1
(1)	(0)	3	2
3	2	1	0

We note that H_2 is isotopic to P_2 and so H_2 is a critical set. Further, each entry of H_2 is the least element of some intercalate contained in \hat{H}_2 . For the second base case we need to check a square of order 8. First we construct a general critical set G_s of order $n = 2^s$ for $s \geq 3$ which will be shown to be equivalent to $\text{gcs}(\alpha L_s)$ for α satisfying (1).

Let i, j be integers such that $i, j \equiv 0 \pmod{4}$ and $0 \leq i, j < 2^s$. We define each subsquare $\mathcal{Q}_{i,j}^4(G_s)$ as follows:

- If $\alpha(i+1) = i+1$ then set $\mathcal{Q}_{i,j}^4(G_s) = \mathcal{Q}_{i,j}^4(P_s)$.
- Otherwise, $\alpha(i+1) = i+2, \alpha(i+2) = i+1$. Let l be the integer such that $\mathcal{Q}_{i,j}^4(L_s) = L_2^l$. If $\mathcal{Q}_{i,j}^4(P_s)$ is similar to L_2 then set $\mathcal{Q}_{i,j}^4(G_s) = \hat{H}_2^l$ otherwise set $\mathcal{Q}_{i,j}^4(G_s) = H_2^l$.

Since H_2 is isotopic to P_2 it follows that G_s is isotopic to P_s , so G_s is a critical set. To finish the second base case, we observe that each entry of G_3 is the least element of some intercalate contained in αL_3 .

Next, fix the integer $s > 3$. We can partition G_s into 4×4 subsquares $\mathcal{Q}_{i,j}^4(G_s)$ where $i, j \equiv 0 \pmod{4}$. See Figure 1 for a diagram of the situation. There are two cases for each subsquare:

1. If $\mathcal{Q}_{i,j}^4(G_s)$ is similar to P_2 or H_2 then each $(x, y, z) \in \mathcal{Q}_{i,j}^4(G_s)$ is the least element of an intercalate.
2. Otherwise, $\mathcal{Q}_{i,j}^4(G_s)$ is isotopic to $\mathcal{Q}_{i,j}^4(P_s) = L_2$. Since G_s is isotopic to P_s , the definition of P_s implies that $(i/4, j/4; k) \in P_{s-2}$ for some $k \in N_{n/4}$. Then we know that there is an intercalate

$$I = \{(i/4, j/4; k), (i/4 + a, j/4; k'), \\ (i/4, j/4 + b; k'), (i/4 + a, j/4 + b; k)\}$$

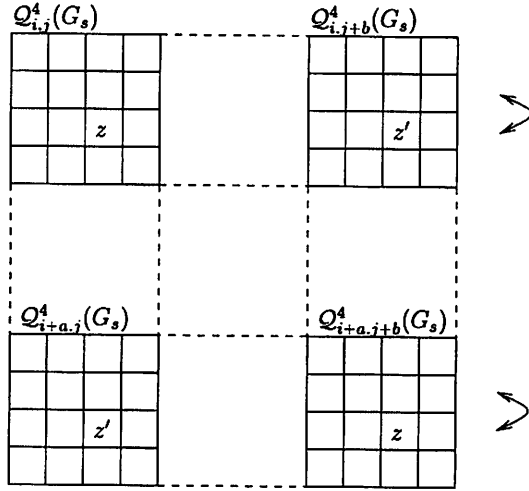


Figure 1: Inner swap. Shows four subsquares of G_s which form an 8×8 subsquare isomorphic to P_3 . An entry $(x, y; z) \in Q_{i,j}^4(G_s)$ is 2-essential due to the intercalate that sits in the square (indicated by z and z'), regardless of possible row swaps (indicated by arrows).

in L_{s-2} such that $I \cap P_{s-2} = \{(i/4, j/4; k)\}$ and $a, b > 4$. Due to this intercalate I and the definition of P_s we now see that the subsquare

$$R = Q_{i,j}^4(G_s) \cup Q_{i,j+b+n/4}^4(G_s) \cup Q_{i+a+n/4,j}^4(G_s) \cup Q_{i+a+n/4,j+b+n/4}^4(G_s)$$

is similar to P_3 . We verified earlier that each $(x, y; z) \in Q_{i,j}^4(G_s)$ is the least element of an intercalate and is 2-essential.

By Lemma 3.4 we have $G_s = \text{gcs}(\alpha L_s)$ and that $\text{gcs}(\alpha L_s)$ is 2-critical. \square

5 The Main Result

Theorem 5.1. *Let $\alpha_{k,k'}$ be a row isotopism on a latin square of order $n = 2^s$, defined by*

$$\alpha_{k,k'}(i) = \begin{cases} k', & i = k \\ k, & i = k' \\ i, & \text{otherwise.} \end{cases}$$

where

$$|k - k'| < 3 \text{ and } j \leq k < k' < j + 4 \text{ for some } j \equiv 0 \pmod{4} \quad (2)$$

Then $\text{gcs}(\alpha_{k,k'}L_s)$ is 2-critical, strong, and completes top down to $\alpha_{k,k'}L_s$.

The proof of Theorem 5.1 is based on induction. The case $s = 2$ is treated separately in Section 5.1. The remaining sections contain the inductive proof, beginning with the base case of $s = 3$.

5.1 Case $s = 2$

The six possible $\text{gcs}(\alpha_{k,k'}L_2)$ are shown below. Each critical set is 2-critical, strong, and completes top down.

$\text{gcs}(\alpha_{0,1}L_2) =$	<table style="border-collapse: collapse; margin: auto;"> <tr><td style="padding: 2px 10px;">1</td><td style="padding: 2px 10px;">(0)</td><td style="padding: 2px 10px;">(3)</td><td style="padding: 2px 10px;">2</td></tr> <tr><td style="padding: 2px 10px;">(0)</td><td style="padding: 2px 10px;">(1)</td><td style="padding: 2px 10px;">2</td><td style="padding: 2px 10px;">3</td></tr> <tr><td style="padding: 2px 10px;">(2)</td><td style="padding: 2px 10px;">3</td><td style="padding: 2px 10px;">(0)</td><td style="padding: 2px 10px;">1</td></tr> <tr><td style="padding: 2px 10px;">3</td><td style="padding: 2px 10px;">2</td><td style="padding: 2px 10px;">1</td><td style="padding: 2px 10px;">0</td></tr> </table>	1	(0)	(3)	2	(0)	(1)	2	3	(2)	3	(0)	1	3	2	1	0	$\text{gcs}(\alpha_{0,2}L_2) =$	<table style="border-collapse: collapse; margin: auto;"> <tr><td style="padding: 2px 10px;">(2)</td><td style="padding: 2px 10px;">3</td><td style="padding: 2px 10px;">(0)</td><td style="padding: 2px 10px;">1</td></tr> <tr><td style="padding: 2px 10px;">1</td><td style="padding: 2px 10px;">(0)</td><td style="padding: 2px 10px;">(3)</td><td style="padding: 2px 10px;">2</td></tr> <tr><td style="padding: 2px 10px;">(0)</td><td style="padding: 2px 10px;">(1)</td><td style="padding: 2px 10px;">2</td><td style="padding: 2px 10px;">3</td></tr> <tr><td style="padding: 2px 10px;">3</td><td style="padding: 2px 10px;">2</td><td style="padding: 2px 10px;">1</td><td style="padding: 2px 10px;">0</td></tr> </table>	(2)	3	(0)	1	1	(0)	(3)	2	(0)	(1)	2	3	3	2	1	0
1	(0)	(3)	2																																
(0)	(1)	2	3																																
(2)	3	(0)	1																																
3	2	1	0																																
(2)	3	(0)	1																																
1	(0)	(3)	2																																
(0)	(1)	2	3																																
3	2	1	0																																
$\text{gcs}(\alpha_{0,3}L_2) =$	<table style="border-collapse: collapse; margin: auto;"> <tr><td style="padding: 2px 10px;">(3)</td><td style="padding: 2px 10px;">(2)</td><td style="padding: 2px 10px;">(1)</td><td style="padding: 2px 10px;">0</td></tr> <tr><td style="padding: 2px 10px;">(1)</td><td style="padding: 2px 10px;">0</td><td style="padding: 2px 10px;">(3)</td><td style="padding: 2px 10px;">2</td></tr> <tr><td style="padding: 2px 10px;">(2)</td><td style="padding: 2px 10px;">(3)</td><td style="padding: 2px 10px;">0</td><td style="padding: 2px 10px;">1</td></tr> <tr><td style="padding: 2px 10px;">0</td><td style="padding: 2px 10px;">1</td><td style="padding: 2px 10px;">2</td><td style="padding: 2px 10px;">3</td></tr> </table>	(3)	(2)	(1)	0	(1)	0	(3)	2	(2)	(3)	0	1	0	1	2	3	$\text{gcs}(\alpha_{1,2}L_2) =$	<table style="border-collapse: collapse; margin: auto;"> <tr><td style="padding: 2px 10px;">(0)</td><td style="padding: 2px 10px;">(1)</td><td style="padding: 2px 10px;">(2)</td><td style="padding: 2px 10px;">3</td></tr> <tr><td style="padding: 2px 10px;">(2)</td><td style="padding: 2px 10px;">3</td><td style="padding: 2px 10px;">(0)</td><td style="padding: 2px 10px;">1</td></tr> <tr><td style="padding: 2px 10px;">(1)</td><td style="padding: 2px 10px;">(0)</td><td style="padding: 2px 10px;">3</td><td style="padding: 2px 10px;">2</td></tr> <tr><td style="padding: 2px 10px;">3</td><td style="padding: 2px 10px;">2</td><td style="padding: 2px 10px;">1</td><td style="padding: 2px 10px;">0</td></tr> </table>	(0)	(1)	(2)	3	(2)	3	(0)	1	(1)	(0)	3	2	3	2	1	0
(3)	(2)	(1)	0																																
(1)	0	(3)	2																																
(2)	(3)	0	1																																
0	1	2	3																																
(0)	(1)	(2)	3																																
(2)	3	(0)	1																																
(1)	(0)	3	2																																
3	2	1	0																																
$\text{gcs}(\alpha_{1,3}L_2) =$	<table style="border-collapse: collapse; margin: auto;"> <tr><td style="padding: 2px 10px;">(0)</td><td style="padding: 2px 10px;">1</td><td style="padding: 2px 10px;">(2)</td><td style="padding: 2px 10px;">3</td></tr> <tr><td style="padding: 2px 10px;">3</td><td style="padding: 2px 10px;">(2)</td><td style="padding: 2px 10px;">(1)</td><td style="padding: 2px 10px;">0</td></tr> <tr><td style="padding: 2px 10px;">(2)</td><td style="padding: 2px 10px;">(3)</td><td style="padding: 2px 10px;">0</td><td style="padding: 2px 10px;">1</td></tr> <tr><td style="padding: 2px 10px;">1</td><td style="padding: 2px 10px;">0</td><td style="padding: 2px 10px;">3</td><td style="padding: 2px 10px;">2</td></tr> </table>	(0)	1	(2)	3	3	(2)	(1)	0	(2)	(3)	0	1	1	0	3	2	$\text{gcs}(\alpha_{2,3}L_2) =$	<table style="border-collapse: collapse; margin: auto;"> <tr><td style="padding: 2px 10px;">0</td><td style="padding: 2px 10px;">(1)</td><td style="padding: 2px 10px;">(2)</td><td style="padding: 2px 10px;">3</td></tr> <tr><td style="padding: 2px 10px;">(1)</td><td style="padding: 2px 10px;">(0)</td><td style="padding: 2px 10px;">3</td><td style="padding: 2px 10px;">2</td></tr> <tr><td style="padding: 2px 10px;">(3)</td><td style="padding: 2px 10px;">2</td><td style="padding: 2px 10px;">(1)</td><td style="padding: 2px 10px;">0</td></tr> <tr><td style="padding: 2px 10px;">2</td><td style="padding: 2px 10px;">3</td><td style="padding: 2px 10px;">0</td><td style="padding: 2px 10px;">1</td></tr> </table>	0	(1)	(2)	3	(1)	(0)	3	2	(3)	2	(1)	0	2	3	0	1
(0)	1	(2)	3																																
3	(2)	(1)	0																																
(2)	(3)	0	1																																
1	0	3	2																																
0	(1)	(2)	3																																
(1)	(0)	3	2																																
(3)	2	(1)	0																																
2	3	0	1																																

5.2 Base Cases for $s = 3$

Let $s = 3$ and $k, k' \in N_8$ such that (2) is satisfied. The base case for $s = 3$ and $\alpha_{k,k'}$ is divided into two parts: $k, k' \geq 4$ and $k, k' < 4$. For each (k, k') we see that the associated greedy critical set $\text{gcs}(\alpha_{k,k'}L_3)$ is 2-critical, strong, and completes top down.

Let Γ be the set of $(k, k') \in N_8 \times N_8$ satisfying (2) where $k, k' \geq 4$:

$$\Gamma = \{(4, 5), (4, 6), (5, 6), (5, 7), (6, 7)\} \quad (3)$$

Suppose $(k, k') = (4, 5)$. The partial latin square $\text{gcs}(\alpha_{4,5}L_3)$ is shown below as the entries in brackets. Also, we take this opportunity to define

the partial latin square $E(4, 5)_2$.

$$\text{gcs}(\alpha_{4,5}L_3) = \begin{array}{|c|c|c|c||c|c|c|c|} \hline (0) & (1) & 2 & (3) & (4) & (5) & (6) & 7 \\ \hline (1) & (0) & (3) & (2) & (5) & (4) & 7 & 6 \\ \hline (2) & (3) & (0) & (1) & (6) & 7 & (4) & 5 \\ \hline (3) & (2) & (1) & (0) & 7 & 6 & 5 & 4 \\ \hline 5 & (4) & (7) & 6 & 1 & (0) & (3) & 2 \\ \hline (4) & (5) & 6 & 7 & (0) & (1) & 2 & 3 \\ \hline (6) & 7 & (4) & 5 & (2) & 3 & (0) & 1 \\ \hline 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ \hline \end{array} \\ = \begin{array}{|c|c||c|} \hline E(4, 5)_2 & & \bullet \\ \hline \bullet & & \bullet \\ \hline \end{array}$$

The other $\text{gcs}(\alpha_{k,k'}L_3)$ and $E(k, k')_2$ for $(k, k') \in \Gamma$ are shown in Appendix B.

Otherwise, $k, k' < 4$. Let $\Lambda = \{(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)\}$. Each case defines a partial latin square $A(k, k')_2$. For example,

$$\text{gcs}(\alpha_{0,1}L_3) = \begin{array}{|c|c|c|c||c|c|c|c|} \hline 1 & (0) & (3) & (2) & 5 & (4) & (7) & 6 \\ \hline (0) & (1) & (2) & (3) & (4) & (5) & 6 & 7 \\ \hline (2) & (3) & (0) & (1) & (6) & 7 & (4) & 5 \\ \hline (3) & (2) & (1) & (0) & 7 & 6 & 5 & 4 \\ \hline (4) & (5) & (6) & 7 & (0) & (1) & (2) & 3 \\ \hline (5) & (4) & 7 & 6 & (1) & (0) & 3 & 2 \\ \hline (6) & 7 & (4) & 5 & (2) & 3 & (0) & 1 \\ \hline 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ \hline \end{array} \\ = \begin{array}{|c|c||c|} \hline A(0, 1)_2 & & \bullet \\ \hline \bullet & & \bullet \\ \hline \end{array}$$

The other $A(k, k')_2$ are shown in Appendix C.

5.3 The Final Construction

In this subsection we will define partial latin squares $E(k, k')_s$, $A(k, k')_s$, and $G(k, k', s)$. The squares $E(k, k')_s$ and $A(k, k')_s$ are used in recursively defining $G(k, k', s)$. In Section 5.4 we will show that $G(k, k', s) = \text{gcs}(\alpha_{k,k'}L_s)$.

Recall that $A(k, k')_2$ and $E(k, k')_2$ were defined in Section 5.2. For $s \geq 4$

and $\delta = 2^{s-2}$ if $k, k' < s^{s-1}$ we define

$$A(k, k')_{s-1} = \begin{cases} \begin{array}{|c|c|} \hline A(k, k')_{s-2}^0 & A(k, k')_{s-2}^1 \\ \hline L_{s-2}^1 & L_{s-2}^0 \\ \hline \end{array} & \text{if } k, k' \in N_\delta^0 \\ \begin{array}{|c|c|} \hline L_{s-2}^0 & L_{s-2}^1 \\ \hline A(k - \delta, k' - \delta)_{s-2}^1 & A(k - \delta, k' - \delta)_{s-2}^0 \\ \hline \end{array} & \text{if } k, k' \in N_\delta^1 \end{cases} \quad (4)$$

$$E(k, k')_{s-1} = \begin{cases} \begin{array}{|c|c|} \hline L_{s-2}^0 & E(k - \delta, k' - \delta)_{s-2}^1 \\ \hline L_{s-2}^1 & L_{s-2}^0 \\ \hline \end{array} & \text{if } k, k' \in N_\delta^2 \\ \begin{array}{|c|c|} \hline E(k - 2\delta, k' - 2\delta)_{s-2}^0 & L_{s-2}^1 \\ \hline E(k - 2\delta, k' - 2\delta)_{s-2}^1 & E(k - 2\delta, k' - 2\delta)_{s-2}^0 \\ \hline \end{array} & \text{if } k, k' \in N_\delta^3 \end{cases} \quad (5)$$

The previous section, Appendix B, and Appendix C give $G(k, k', s) = \text{gcs}(\alpha_{k, k'} L_3)$. For $s \geq 4$, define

$$G(k, k', s) = \begin{cases} \begin{array}{|c|c|} \hline A(k, k')_{s-1}^0 & G(k, k', s-1) \\ \hline P_{s-1}^0 & P_{s-1}^1 \\ \hline \end{array} \\ \begin{array}{|c|c|} \hline E(k, k')_{s-1}^0 & P_{s-1}^1 \\ \hline G(k - 2\delta, k' - 2\delta, s-1)^1 & G(k - 2\delta, k' - 2\delta, s-1)^0 \\ \hline \end{array} \end{cases} \quad (6)$$

for $k, k' < n/2$ and $k, k' \geq n/2$, respectively.

The following Lemma is immediate from the definition of $E(k, k')_2$ and (5).

Lemma 5.2. *Let $i, j \equiv 0 \pmod{4}$ and $W = \mathcal{Q}_{i, j}^4(E(k, k')_s)$. Then $W \approx L_2$ or $W \approx E(l, l')_2$ for some $(l, l') \in \Gamma$.*

For each $(k, k') \in \Gamma$, define

$$U(k, k') = \begin{array}{|c|c|} \hline E(k, k')_2^0 & P_2^1 \\ \hline E(k, k')_2^1 & P_2^0 \\ \hline \end{array} \quad (7)$$

Lemma 5.3. *Let $r, c \in N_4$. Then for each $(k, k') \in \Gamma$,*

$$A(U(k, k'), N_8 \times N_8, L_3)_{r, c} \cap N_4^1 = \emptyset$$

Proof. There are four cases to inspect. First, let $U = U(4, 5) = U(4, 6)$.

$$U = \begin{array}{|c|c|c|c|} \hline (0) & (1) & 2 & (3) \\ \hline (1) & (0) & (3) & (2) \\ \hline (2) & (3) & (0) & (1) \\ \hline (3) & (2) & (1) & (0) \\ \hline (4) & (5) & 6 & (7) \\ \hline (5) & (4) & (7) & (6) \\ \hline (6) & (7) & (4) & (5) \\ \hline (7) & (6) & (5) & (4) \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline (4) & (5) & (6) & 7 \\ \hline (5) & (4) & 7 & 6 \\ \hline (6) & 7 & (4) & 5 \\ \hline 7 & 6 & 5 & 4 \\ \hline (0) & (1) & (2) & 3 \\ \hline (1) & (0) & 3 & 2 \\ \hline (2) & 3 & (0) & 1 \\ \hline 3 & 2 & 1 & 0 \\ \hline \end{array}$$

In the first row there are empty cells $(0, 2)$ and $(0, 7)$ which could be filled with a 2 or 7. However $7 \notin \mathcal{A}(U, N_8 \times N_8, L_3)_{0,2}$ since $(5, 2; 7) \in U$. So $\mathcal{A}(U, N_8 \times N_8, L_3)_{0,2} = \{2\} \subset N_4^0$. Finally, $\mathcal{A}(U, N_8 \times N_8, L_3)_{r,c} = \emptyset$ for $r, c \in N_4^0$ and $(r, c) \neq (0, 2)$. This completes the proof of this case. The other three $U(k, k')$ are displayed in Appendix D. \square

For each $(k, k') \in \Gamma$, define

$$V(k, k') = \begin{array}{|c|c|} \hline E(k, k')_2^0 & P_2^1 \\ \hline P_2^1 & E(k, k')_2^0 \\ \hline \end{array} \quad (8)$$

Lemma 5.4. *Let $r \in N_4, c \in N_4^1$. Then for each $(k, k') \in \Gamma$*

$$\mathcal{A}(V(k, k'), N_8 \times N_8, L_3)_{r,c} \cap N_4^0 = \emptyset$$

As with the previous lemma there are four cases to check with very similar reasoning (see Appendix E).

5.4 Completing the Proof of Theorem 5.1

The proof of Theorem 5.1 will require a few technical Lemmas. First, Lemma 5.5 follows directly from the definition of L_s .

Lemma 5.5. *Let $(i, j; k), (i', j'; k') \in L_s$ where $n = 2^s$ and*

$$\begin{aligned} 0 &\leq i < n/2 \\ 0 &\leq j < n/2 \leq j' < n \end{aligned}$$

Then there exists an integer i' with $n/2 \leq i' < n$ such that

$$\{(i, j; k), (i, j'; k'), (i', j; k'), (i', j'; k')\}$$

is an intercalate in L_s .

Using ideas in the proof of Lemma 4.3, we also have:

Corollary 5.6. *Let $(i, j; k), (i, j'; k') \in L_s$ where $n = 2^s$, $i, j, j' \equiv 0 \pmod{4}$ and*

$$0 \leq i < n/2$$

$$0 \leq j < n/2 \leq j' < n$$

Then there exists an integer $i' \equiv \pmod{4}$ with $n/2 \leq i' < n$ such that

$$\mathcal{Q}_{i,j}^4(L_s) \cup \mathcal{Q}_{i,j'}^4(L_s) \cup \mathcal{Q}_{i',j}^4(L_s) \cup \mathcal{Q}_{i',j'}^4(L_s) \approx L_3$$

Lemma 5.7. *Let P be a partial latin square contained in the latin square L of order n . Let $R_1, R_2, C \subseteq N_n$ and define $S = (R_1 \cup R_2) \times C$ so that*

$$|C| = |R_1 \cup R_2|$$

$$Q = \{(i, j; k) \mid (i, j) \in S, (i, j; k) \in P\}$$

$$L' = \{(i, j; k) \mid (i, j) \in S, (i, j; k) \in L\}$$

where L' is a latin subsquare of L . If Q strongly completes (extends) to

$$\hat{Q} = Q \cup \{(i, j; k) \mid (i, j; k) \in L', i \in R_1, c \in C\}$$

and $\mathcal{A}(P, R_1 \times C, L) = \mathcal{A}(Q, R_1 \times C, L')$ then P has a unique extension to

$$\hat{P} = P \cup \{(i, j; k) \mid (i, j; k) \in L, i \in R_1, c \in C\}. \quad (9)$$

Remark 5.8. The subsquare Q has strong completion through rows R_1 only. This is useful if rows R_2 of the arrays of alternatives are not equal. On the other hand, if we set $R_2 = \emptyset$ then the lemma says that P extends to $P \cup L'$.

Proof of Lemma 5.7. Since Q has strong completion through rows R_1 in L' there must be a sequence

$$(Q^{(1)}, r_1, c_1), (Q^{(2)}, r_2, c_2), \dots, (Q^{(m)}, r_m, c_m)$$

such that $Q^{(1)} = Q$, $Q^{(m+1)} = \hat{Q}$, and $Q^{(i)} \subsetneq Q^{(i+1)}$, $r_i \in R_1$, $c_i \in C$ for each i . Also, the pairs (r_i, c_i) are distinct and $|\mathcal{A}(Q^{(i)}, S, L')_{r_i, c_i}| = 1$ for each i .

Define the sequence $(P^{(i)}, r_i, c_i) = (P \cup Q^{(i)}, r_i, c_i)$ for $1 \leq i \leq m$. It is obvious that $P^{(i)} \subsetneq P^{(i+1)}$. To show that this is a strong completion we

need $|\mathcal{A}(P^{(i)}, S, L)_{r_i, c_i}| = 1$ for each i . First, $|\mathcal{A}(P^{(1)}, S, L)_{r_1, c_1}| = 1$ by the definition of Q . Now suppose that $|\mathcal{A}(P^{(i)}, S, L)_{r_i, c_i}| = 1$ and

$$\mathcal{A}(P^{(i)}, S, L)_{r, c} = \mathcal{A}(Q^{(i)}, S, L')_{r, c} \text{ for each } r \in R_1, c \in C$$

where $i > 1$ is fixed. Fill the cell (r_i, c_i) in $Q^{(i)}$ with the (unique) symbol $c_i \in \mathcal{A}(Q^{(i)}, S, L')_{r_i, c_i}$ to get $Q^{(i+1)}$. Do the same in $P^{(i)}$. Now let

$$\mathcal{A}(P^{(i+1)}, S, L)_{r, c} = \begin{cases} \mathcal{A}(P^{(i)}, S, L)_{r, c}, & \text{if } r \neq r_i \text{ and } c \neq c_i \\ \mathcal{A}(P^{(i)}, S, L)_{r, c} \setminus \{e_i\}, & \text{otherwise} \end{cases}$$

Since $\mathcal{A}(Q^{(i+1)}, S, L)_{r, c}$ has the same definition (i.e. the symbol e_i deleted from the corresponding row and column) it follows that $\mathcal{A}(P^{(i+1)}, S, L)_{r, c} = \mathcal{A}(Q^{(i+1)}, S, L)_{r, c}$ for each r, c . Hence P has strong completion to \hat{P} . \square

Lemma 5.9. $G(k, k', s)$ has strong completion top down.

Proof. The case $s = 2$ and base case $s = 3$ are given earlier. So suppose that the theorem is true for all L_t where $3 \leq t < s$. Let $\delta = 2^{s-2}$. There are four cases depending on where the row swap occurs.

Case 1: $k, k' \in N_\delta^3$. Write $l = k - 2\delta, l' = k' - 2\delta, h = k - 3\delta, h' = k' - 3\delta$. Then

$$G(k, k', s) = \begin{array}{c} \begin{array}{|c|c|} \hline E(k, k')_{s-1}^0 & P_{s-1}^1 \\ \hline G(l, l', s-1)^1 & G(l, l', s-1)^0 \\ \hline \end{array} \\ \\ \begin{array}{|c|c|c|c|} \hline E(l, l')_{s-2}^0 & L_{s-2}^1 & L_{s-2}^2 & P_{s-2}^3 \\ \hline E(l, l')_{s-2}^1 & E(l, l')_{s-2}^0 & P_{s-2}^3 & P_{s-2}^2 \\ \hline E(l, l')_{s-2}^2 & P_{s-2}^3 & E(l, l')_{s-2}^0 & P_{s-2}^1 \\ \hline G(h, h', s-2)^3 & G(h, h', s-2)^2 & G(h, h', s-2)^1 & G(h, h', s-2)^0 \\ \hline \end{array} \\ \\ (10) \end{array}$$

First we show that the top δ rows of $G(k, k', s)$ have unique completion top down. Since the cells $N_\delta^0 \times (N_\delta^1 \cup N_\delta^2)$ are completely filled in, we need to show that the cells $N_\delta^0 \times (N_\delta^0 \cup N_\delta^3)$ have unique completion. We will use Lemma 5.7. Let

$$R_1 = N_\delta^0, \quad R_2 = N_\delta^3, \quad C = N_\delta^0 \cup N_\delta^3, \quad L = \alpha_{k, k'} L_s, \quad P = G(k, k', s)$$

Then Q and L' are defined to be

$$Q = \begin{array}{|c|c|} \hline E(l, l')_{s-2}^0 & P_{s-2}^3 \\ \hline G(h, h', s-2)^3 & G(h, h', s-2)^0 \\ \hline \end{array} = G(l, l', s-1)$$

$$L' = \begin{array}{|c|c|} \hline L_{s-2}^0 & L_{s-2}^3 \\ \hline L_{s-2}^3 & L_{s-2}^0 \\ \hline \end{array} = \alpha_{l, l'} L_{s-1}$$

By the inductive hypothesis Q has unique completion top down to L' . From (10) we see that $\mathcal{A}(P, R_1 \times C, L) \cap (N_\delta^1 \cup N_\delta^2) = \emptyset$ and $\mathcal{A}(P, R_1 \times C, L) = \mathcal{A}(Q, R_1 \times C, L')$. Now Lemma 5.7 gives the strong top down extension $G^+(k, k', s)$ of $G(k, k', s)$:

L_{s-2}^0	L_{s-2}^1	L_{s-2}^2	L_{s-2}^3
$E(l, l')_{s-2}^1$	$E(l, l')_{s-2}^0$	P_{s-2}^3	P_{s-2}^2
$E(l, l')_{s-2}^2$	P_{s-2}^3	$E(l, l')_{s-2}^0$	P_{s-2}^1
$G(h, h', s-2)^3$	$G(h, h', s-2)^2$	$G(h, h', s-2)^1$	$G(h, h', s-2)^0$

We will now show that

$$\mathcal{A}(G^+(k, k', s), N_n \times N_n, \alpha_{k, k'} L_s)_{r, c} \subseteq N_\delta^1 \cup N_\delta^3 \text{ for } r \in N_\delta^1, c \in N_\delta^0 \quad (11)$$

$$\mathcal{A}(G^+(k, k', s), N_n \times N_n, \alpha_{k, k'} L_s)_{r, c} \subseteq N_\delta^1 \cup N_\delta^3 \text{ for } r \in N_\delta^1, c \in N_\delta^2 \quad (12)$$

Let i, j, u, v be integers such that $i, j, u, v \equiv 0 \pmod{4}$ and

$$i, i+3 \in N_\delta^1 \quad j, j+3 \in N_\delta^0 \quad v, v+3 \in N_\delta^3$$

Then by Lemma 5.2, $\mathcal{Q}_{i, j}^4(G^+(k, k', s)) \supseteq E(k, k')_2^w$ for some (k, k') and integer w . Also, $\mathcal{Q}_{u, v}^4(G^+(k, k', s)) \supseteq P_2^x$ for some integer x . So to restrict the array of alternatives we apply Lemma 5.3 to the subsquare (which exists due to Corollary 5.6 for some $x \in N_\delta^2$)

$$\mathcal{Q}_{i, j}^4(G^+(k, k', s)) \cup \mathcal{Q}_{i, v}^4(G^+(k, k', s)) \cup \mathcal{Q}_{x, j}^4(G^+(k, k', s)) \cup \mathcal{Q}_{x, v}^4(G^+(k, k', s))$$

and Lemma 5.4 to the subsquare (which exists due to Corollary 5.6 for some $y \in N_\delta^2$)

$$\mathcal{Q}_{i, j+\delta}^4(G^+(k, k', s)) \cup \mathcal{Q}_{i, v-\delta}^4(G^+(k, k', s)) \cup \mathcal{Q}_{y, j+\delta}^4(G^+(k, k', s)) \cup \mathcal{Q}_{y, v-\delta}^4(G^+(k, k', s))$$

which gives

$$\begin{aligned} \mathcal{A}(G^+(k, k', s), N_n \times N_n, \alpha_{k, k'} L_s)_{r, c} \cap N_\delta^2 &= \emptyset & \text{for } r \in N_\delta^1, c \in N_\delta^0 \\ \mathcal{A}(G^+(k, k', s), N_n \times N_n, \alpha_{k, k'} L_s)_{r, c} \cap N_\delta^0 &= \emptyset & \text{for } r \in N_\delta^1, c \in N_\delta^2 \end{aligned}$$

which imply (11) and (12). Now consider these two subsquares of $G^+(k, k', s)$:

$E(l, l')^1_{s-2}$		P^3_{s-2}	
$G(h, h', s-2)^3$		$G(h, h', s-2)^1$	

(13)

	$E(l, l')^0_{s-2}$		P^2_{s-2}
$G(h, h', s-2)^2$		$G(h, h', s-2)^0$	

(14)

We can now interleave the application of Lemma 5.7 with the inductive hypothesis to show that the top halves of these subsquares completes strongly top down. Suppose row r , for $r \in N_\delta^1$ has been completed in (13). Then the array of alternatives for subsquares identified in (14) is restricted such that the inductive hypothesis applies. In other words,

$$A(G^+(k, k', s), N_n \times N_n, \alpha_{k, k'} L_s)_{r, c} \subseteq N_\delta^0 \cup N_\delta^2 \text{ for } c \in N_\delta^1$$

$$A(G^+(k, k', s), N_n \times N_n, \alpha_{k, k'} L_s)_{r, c} \subseteq N_\delta^0 \cup N_\delta^2 \text{ for } c \in N_\delta^3$$

Hence $G^+(k, k', s)$ strongly extends top down to $G^{++}(k, k', s)$:

L^0_{s-2}	L^1_{s-2}	L^2_{s-2}	L^3_{s-2}
L^1_{s-2}	L^0_{s-2}	L^3_{s-2}	L^2_{s-2}
$E(l, l')^2_{s-2}$	P^3_{s-2}	$E(l, l')^0_{s-2}$	P^1_{s-2}
$G(h, h', s-2)^3$	$G(h, h', s-2)^2$	$G(h, h', s-2)^1$	$G(h, h', s-2)^0$

Finally, interleave the application of Lemma 5.7 with $R_2 = \emptyset$ to the subsquares

$$R_1 \times C = \{2\delta, 2\delta + 1, \dots, 4\delta - 1\} \times \{0, 1, \dots, 2\delta - 1\}$$

and

$$R_1 \times C = \{2\delta, 2\delta + 1, \dots, 4\delta - 1\} \times \{2\delta, 2\delta + 1, \dots, 4\delta - 1\}$$

which finishes the completion of $G(k, k', s)$.

Case 2: $k, k' \in N_\delta^2$. Write $l = k - \delta, l' = k' - \delta, h = k - 2\delta, h' = k' - 2\delta$. Then

$$G(k, k', s) =$$

L^0_{s-2}	$E(l, l')^1_{s-2}$	L^2_{s-2}	P^3_{s-2}
L^1_{s-2}	L^0_{s-2}	P^3_{s-2}	P^2_{s-2}
$A(h, h')^2_{s-2}$	$G(h, h', s-2)^3$	$A(h, h')^0_{s-2}$	$G(h, h', s-2)^1$
P^3_{s-2}	P^2_{s-2}	P^1_{s-2}	P^0_{s-2}

and the reasoning is simpler than Case 1.

Case 3: $k, k' \in N_\delta^1$. Write $l = k - \delta$, $l' = k' - \delta$.

$$G(k, k', s) = \begin{array}{|c|c|c|c|} \hline L_{s-2}^0 & L_{s-2}^1 & E(k, k')_{s-2}^2 & P_{s-2}^3 \\ \hline A(l, l')_{s-2}^1 & A(l, l')_{s-2}^0 & G(l, l', s-2)^3 & G(l, l', s-2)^2 \\ \hline L_{s-2}^2 & P_{s-2}^3 & L_{s-2}^0 & P_{s-2}^1 \\ \hline P_{s-2}^3 & P_{s-2}^2 & P_{s-2}^1 & P_{s-2}^0 \\ \hline \end{array}$$

Case 4: $k, k' \in N_\delta^0$.

$$G(k, k', s) = \begin{array}{|c|c|c|c|} \hline A(k, k')_{s-2}^0 & A(k, k')_{s-2}^1 & A(k, k')_{s-2}^2 & G(k, k', s-2)^3 \\ \hline L_{s-2}^1 & L_{s-2}^0 & P_{s-2}^3 & P_{s-2}^2 \\ \hline L_{s-2}^2 & P_{s-2}^3 & L_{s-2}^0 & P_{s-2}^1 \\ \hline P_{s-2}^3 & P_{s-2}^2 & P_{s-2}^1 & P_{s-2}^0 \\ \hline \end{array}$$

This completes the proof. \square

Lemma 5.10. $G(k, k', s)$ is a 2-critical greedy critical set.

Proof. The case $s = 2$ and base case $s = 3$ were given earlier. Suppose that the result is true for all $G(k, k', t)$ where $3 \leq t < s$. Write $\delta = 2^{s-2}$. There are four cases for k, k' . First, suppose that $k, k' \in N_\delta^0$. Write down $G = G(k, k', s)$:

$$G = \begin{array}{|c|c|c|c|} \hline A(k, k')_{s-2}^0 & A(k, k')_{s-2}^1 & A(k, k')_{s-2}^2 & G(k, k', s-2)^3 \\ \hline L_{s-2}^1 & L_{s-2}^0 & P_{s-2}^3 & P_{s-2}^2 \\ \hline L_{s-2}^2 & P_{s-2}^3 & L_{s-2}^0 & P_{s-2}^1 \\ \hline P_{s-2}^3 & P_{s-2}^2 & P_{s-2}^1 & P_{s-2}^0 \\ \hline \end{array}$$

Now identify the following subquases:

1. $Q_{2\delta, 0}^{2\delta}(G) \approx P_{s-1}$
2. $Q_{2\delta, 2\delta}^{2\delta}(G) \approx P_{s-1}$
3. $Q_{0, 2\delta}^{2\delta}(G) \approx G(k, k', s-1)$
4. $Q_{0, 0}^\delta(G) \cup Q_{0, 3\delta}^\delta(G) \cup Q_{3\delta, 0}^\delta(G) \cup Q_{3\delta, 3\delta}^\delta(G) \approx G(k, k', s-1)$
5. $Q_{0, \delta}^\delta(G) \cup Q_{0, 3\delta}^\delta(G) \cup Q_{2\delta, \delta}^\delta(G) \cup Q_{2\delta, 3\delta}^\delta(G) \approx G(k, k', s-1)$
6. $Q_{\delta, 0}^\delta(G) \cup Q_{\delta, 2\delta}^\delta(G) \cup Q_{3\delta, 0}^\delta(G) \cup Q_{3\delta, 2\delta}^\delta(G) \approx P_{s-1}$
7. $Q_{\delta, \delta}^\delta(G) \cup Q_{\delta, 3\delta}^\delta(G) \cup Q_{3\delta, \delta}^\delta(G) \cup Q_{3\delta, 3\delta}^\delta(G) \approx P_{s-1}$

With these subsquares, the inductive hypothesis, and Corollary 4.2, we see that each entry $x \in G(k, k', s)$ is 2-essential and that there exists a trade $T_x \subset \alpha_{k,k'}L_s$ such that $G(k, k', s) \cap T_x = \{x\}$ and x is the least element of T_x . Further, $G(k, k', s)$ has strong completion by Lemma 5.9 so $G(k, k', s)$ is a critical set. Finally, Lemma 3.4 shows that $G = \text{gcs}(\alpha_{k,k'}L_s)$. We omit the remaining three cases where k, k' are in $N_\delta^1, N_\delta^2, N_\delta^3$ since the reasoning is very similar. \square

Theorem 5.1. *Let $\alpha_{k,k'}$ be a row isotopism on a latin square of order $n = 2^s$, defined by*

$$\alpha_{k,k'}(i) = \begin{cases} k', & i = k \\ k, & i = k' \\ i, & \text{otherwise.} \end{cases}$$

where

$$|k - k'| < 3 \text{ and } j \leq k < k' < j + 4 \text{ for some } j \equiv 0 \pmod{4} \quad (15)$$

Then $\text{gcs}(\alpha_{k,k'}L_s)$ is 2-critical, strong, and completes top down to $\alpha_{k,k'}L_s$.

Proof. The Theorem follows from Lemmas 5.9 and 5.10. \square

6 Conclusion

We believe that a stronger version of the theorem is true, where (2) is weakened.

Conjecture 6.1. *Let $\alpha_{k,k'}$ be a row isotopism on a latin square of order $n = 2^s$, defined by*

$$\alpha(i) = \begin{cases} k', & i = k \\ k, & i = k' \\ i, & \text{otherwise.} \end{cases}$$

where $|k - k'| < 3$ Then $\text{gcs}(\alpha_{k,k'}L_s)$ is 2-critical, strong, and completes top down.

We have verified the conjecture by computer search for $2 \leq s \leq 5$ and all possible $\alpha_{k,k'}$.

References

- [1] N J Cavenagh, C Hamalainen, D Donovan, A Khodkar, On Greedy Critical Sets in Latin Squares, Proceedings of the Fifteenth Australasian Workshop on Combinatorial Algorithms (AWOCA2004), 1157–163.
- [2] D Donovan, F Chin-Mei, A Khodkar, *An investigation of 2-critical sets in latin squares*, Ars Combinatoria 72 (2004), 223–234.
- [3] D R Stinson and G H J Van Rees, *Some large critical sets*, Congressus Numerantium 34 (1982), 441–456.
- [4] A D Keedwell, *Critical sets for Latin squares, graphs and block designs: a survey*, Congressus Numerantium 113 (1996), 231–245.
- [5] A Khodkar, *On smallest critical sets for the elementary abelian 2-group*, Utilitas Mathematica 54 (1998), 45–50.
- [6] A D Keedwell, *A note on critical sets for the elementary abelian 2-group*, Utilitas Math 49 (1996), 69–84.

Appendix A

Suppose we wish to calculate $\text{gcs}(\alpha_{60,62}L_6)$.

- First, $s = 6$ so $n = 2^5 = 64$, and $\delta = 2^{s-2} = 2^4 = 16$. Use (6) to write down $G(60, 62, 6)$:

$$G(60, 62, 6) = \begin{array}{|c|c|} \hline \frac{E(60, 62)_5^0}{G(60 - 32, 62 - 32, 5)^1} & \frac{P_5^1}{G(60 - 32, 62 - 32, 5)^0} \\ \hline \frac{E(60, 62)_5^0}{G(28, 30, 5)^1} & \frac{P_5^1}{G(28, 30, 5)^0} \\ \hline \end{array}$$

- Now use (6) twice more:

$$G(28, 30, 5) = \begin{array}{|c|c|} \hline \frac{E(28, 30)_4^0}{G(12, 14, 4)^1} & \frac{P_4^1}{G(12, 14, 4)^0} \\ \hline \end{array}$$

$$G(12, 14, 4) = \begin{array}{|c|c|} \hline \frac{E(12, 14)_3^0}{G(4, 6, 3)^1} & \frac{P_3^1}{G(4, 6, 3)^0} \\ \hline \end{array}$$

The subsquares P_3 and $G(4, 6, 3)$ are base cases and can be looked up in the later appendices and the main part of the paper.

- Next, let $s = 6$, $\delta = 2^{6-2} = 16$, and apply (5) to get

$$E(60, 62)_5 = \begin{array}{|c|c|} \hline \frac{E(28, 30)_4^0}{E(28, 30)_4^1} & \frac{L_4^1}{E(28, 30)_4^0} \\ \hline \end{array}$$

Next, let $s = 5$, $\delta = 2^{5-2} = 2^3 = 8$. Then

$$E(28, 30)_4 = \begin{array}{|c|c|} \hline \frac{E(12, 14)_3^0}{E(12, 14)_3^1} & \frac{L_3^1}{E(12, 14)_3^0} \\ \hline \end{array}$$

Lastly, let $s = 4$, $\delta = 2^{4-2} = 2^2 = 4$. Then

$$E(12, 14)_3 = \begin{array}{|c|c|} \hline \frac{E(4, 6)_2^0}{E(4, 6)_2^1} & \frac{L_2^1}{E(4, 6)_2^0} \\ \hline \end{array}$$

The subsquares L_2 and $E(4, 6)_2$ are base cases and defined in the main section of the paper.

Here is $\text{gcs}(\alpha_{12,14}L_4)$:

0	1	-	3	4	5	6	7	8	9	10	11	12	13	14	-
1	0	3	2	5	4	7	6	9	8	11	10	13	12	-	-
2	3	0	1	6	7	4	5	10	11	8	9	14	-	12	-
3	2	1	0	7	6	5	4	11	10	9	8	-	-	-	-
4	5	-	7	0	1	-	3	12	13	14	-	8	9	10	-
5	4	7	6	1	0	3	2	13	12	-	-	9	8	-	-
6	7	4	5	2	3	0	1	14	-	12	-	10	-	8	-
7	6	5	4	3	2	1	0	-	-	-	-	-	-	-	-
8	9	-	11	12	13	14	-	0	1	-	3	4	5	6	-
9	8	11	10	13	12	-	-	1	0	3	2	5	4	-	-
10	11	8	9	14	-	12	-	2	3	0	1	6	-	4	-
11	10	9	8	-	-	-	-	3	2	1	0	-	-	-	-
14	-	12	-	10	-	8	-	6	-	4	-	2	-	0	-
-	12	15	-	-	8	11	-	-	4	7	-	-	0	3	-
12	13	-	-	8	9	-	-	4	5	-	-	0	1	-	-
-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-

Appendix B

$$\text{gcs}(\alpha_{4,6}L_3) = \begin{array}{|c|c|c|c|} \hline (0) & (1) & 2 & (3) \\ \hline (1) & (0) & (3) & (2) \\ \hline (2) & (3) & (0) & (1) \\ \hline (3) & (2) & (1) & (0) \\ \hline \end{array} \begin{array}{|c|c|c|c|} \hline (4) & (5) & (6) & 7 \\ \hline (5) & (4) & 7 & 6 \\ \hline (6) & 7 & (4) & 5 \\ \hline 7 & 6 & 5 & 4 \\ \hline \end{array} = \begin{array}{|c|c|} \hline E(4,6)_2 & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array}$$

$$\text{gcs}(\alpha_{5,6}L_3) = \begin{array}{|c|c|c|c|} \hline (0) & (1) & (2) & (3) \\ \hline (1) & (0) & (3) & (2) \\ \hline (2) & (3) & (0) & (1) \\ \hline (3) & (2) & (1) & (0) \\ \hline \end{array} \begin{array}{|c|c|c|c|} \hline (4) & (5) & (6) & 7 \\ \hline (5) & (4) & 7 & 6 \\ \hline (6) & 7 & (4) & 5 \\ \hline 7 & 6 & 5 & 4 \\ \hline \end{array} = \begin{array}{|c|c|} \hline E(5,6)_2 & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array}$$

$$\text{gcs}(\alpha_{5,7}L_3) = \begin{array}{|c|c|c|c|} \hline (0) & 1 & (2) & (3) \\ \hline 1 & 0 & (3) & (2) \\ \hline (2) & (3) & (0) & (1) \\ \hline (3) & (2) & (1) & (0) \\ \hline (4) & 5 & (6) & 7 \\ \hline 7 & (6) & (5) & 4 \\ \hline (6) & (7) & 4 & 5 \\ \hline 5 & 4 & 7 & 6 \\ \hline \end{array} \parallel \begin{array}{|c|c|c|c|} \hline (4) & (5) & (6) & 7 \\ \hline (5) & (4) & 7 & 6 \\ \hline (6) & 7 & (4) & 5 \\ \hline 7 & 6 & 5 & 4 \\ \hline (0) & 1 & (2) & 3 \\ \hline 3 & (2) & (1) & 0 \\ \hline (2) & (3) & 0 & 1 \\ \hline 1 & 0 & 3 & 2 \\ \hline \end{array} = \begin{array}{|c|c|} \hline E(5,7)_2 & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array}$$

$$\text{gcs}(\alpha_{6,7}L_3) = \begin{array}{|c|c|c|c|} \hline 0 & (1) & (2) & (3) \\ \hline (1) & (0) & (3) & (2) \\ \hline 2 & (3) & 0 & (1) \\ \hline (3) & (2) & (1) & (0) \\ \hline 4 & (5) & (6) & 7 \\ \hline (5) & (4) & 7 & 6 \\ \hline (7) & 6 & (5) & 4 \\ \hline 6 & 7 & 4 & 5 \\ \hline \end{array} \parallel \begin{array}{|c|c|c|c|} \hline (4) & (5) & (6) & 7 \\ \hline (5) & (4) & 7 & 6 \\ \hline (6) & 7 & (4) & 5 \\ \hline 7 & 6 & 5 & 4 \\ \hline 0 & (1) & (2) & 3 \\ \hline (1) & (0) & 3 & 2 \\ \hline (3) & 2 & (1) & 0 \\ \hline 2 & 3 & 0 & 1 \\ \hline \end{array} = \begin{array}{|c|c|} \hline E(6,7)_2 & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array}$$

Appendix C

$$\text{gcs}(\alpha_{0,2}L_3) = \begin{array}{|c|c|c|c|} \hline (2) & (3) & (0) & (1) \\ \hline 1 & (0) & (3) & (2) \\ \hline (0) & (1) & (2) & (3) \\ \hline (3) & (2) & (1) & (0) \\ \hline (4) & (5) & (6) & 7 \\ \hline (5) & (4) & 7 & 6 \\ \hline (6) & 7 & (4) & 5 \\ \hline 7 & 6 & 5 & 4 \\ \hline \end{array} \parallel \begin{array}{|c|c|c|c|} \hline (6) & 7 & (4) & 5 \\ \hline 5 & (4) & (7) & 6 \\ \hline (4) & (5) & 6 & 7 \\ \hline 7 & 6 & 5 & 4 \\ \hline (0) & (1) & (2) & 3 \\ \hline (1) & (0) & 3 & 2 \\ \hline (2) & 3 & (0) & 1 \\ \hline 3 & 2 & 1 & 0 \\ \hline \end{array} = \begin{array}{|c|c|} \hline A(0,2)_2 & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array}$$

$$\text{gcs}(\alpha_{0,3}L_3) = \begin{array}{|c|c|c|c|} \hline (3) & 2 & 1 & (0) \\ \hline 1 & (0) & 3 & (2) \\ \hline 2 & 3 & (0) & (1) \\ \hline (0) & (1) & (2) & (3) \\ \hline (4) & (5) & (6) & 7 \\ \hline (5) & (4) & 7 & 6 \\ \hline (6) & 7 & (4) & 5 \\ \hline 7 & 6 & 5 & 4 \\ \hline \end{array} \parallel \begin{array}{|c|c|c|c|} \hline (7) & (6) & (5) & 4 \\ \hline (5) & 4 & (7) & 6 \\ \hline (6) & (7) & 4 & 5 \\ \hline 4 & 5 & 6 & 7 \\ \hline (0) & (1) & (2) & 3 \\ \hline (1) & (0) & 3 & 2 \\ \hline (2) & 3 & (0) & 1 \\ \hline 3 & 2 & 1 & 0 \\ \hline \end{array} = \begin{array}{|c|c|} \hline A(0,3)_2 & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array}$$

$$\text{gcs}(\alpha_{1,2}L_3) = \begin{array}{|c|c|c|c|} \hline (0) & (1) & (2) & (3) \\ \hline (2) & (3) & (0) & (1) \\ \hline (1) & (0) & (3) & (2) \\ \hline (3) & (2) & (1) & (0) \\ \hline \end{array} \begin{array}{|c|c|c|c|} \hline (4) & (5) & (6) & 7 \\ \hline (6) & 7 & (4) & 5 \\ \hline (5) & (4) & 7 & 6 \\ \hline 7 & 6 & 5 & 4 \\ \hline \end{array} = \begin{array}{|c|c|} \hline A(1,2)_2 & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array}$$

$$\text{gcs}(\alpha_{1,3}L_3) = \begin{array}{|c|c|c|c|} \hline (0) & (1) & (2) & (3) \\ \hline (3) & 2 & 1 & (0) \\ \hline 2 & 3 & (0) & (1) \\ \hline (1) & (0) & (3) & (2) \\ \hline \end{array} \begin{array}{|c|c|c|c|} \hline (4) & 5 & (6) & 7 \\ \hline 7 & (6) & (5) & 4 \\ \hline (6) & (7) & 4 & 5 \\ \hline 5 & 4 & 7 & 6 \\ \hline \end{array} = \begin{array}{|c|c|} \hline A(1,3)_2 & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array}$$

$$\text{gcs}(\alpha_{2,3}L_3) = \begin{array}{|c|c|c|c|} \hline (0) & (1) & (2) & (3) \\ \hline (1) & (0) & (3) & (2) \\ \hline 3 & (2) & 1 & (0) \\ \hline (2) & (3) & (0) & (1) \\ \hline \end{array} \begin{array}{|c|c|c|c|} \hline 4 & (5) & (6) & 7 \\ \hline (5) & (4) & 7 & 6 \\ \hline (7) & 6 & (5) & 4 \\ \hline 6 & 7 & 4 & 5 \\ \hline \end{array} = \begin{array}{|c|c|} \hline A(2,3)_2 & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array}$$

Appendix D

$$U(5,6) =$$

(0)	(1)	(2)	(3)	(4)	(5)	(6)	7
(1)	(0)	(3)	(2)	(5)	(4)	7	6
(2)	(3)	(0)	(1)	(6)	7	(4)	5
(3)	(2)	(1)	(0)	7	6	5	4
(4)	(5)	(6)	(7)	(0)	(1)	(2)	3
(5)	(4)	(7)	(6)	(1)	(0)	3	2
(6)	(7)	(4)	(5)	(2)	3	(0)	1
(7)	(6)	(5)	(4)	3	2	1	0

$$U(5,7) =$$

(0)	1	(2)	(3)	(4)	(5)	(6)	7
1	0	(3)	(2)	(5)	(4)	7	6
(2)	(3)	(0)	(1)	(6)	7	(4)	5
(3)	(2)	(1)	(0)	7	6	5	4
(4)	5	(6)	(7)	(0)	(1)	(2)	3
5	4	(7)	(6)	(1)	(0)	3	2
(6)	(7)	(4)	(5)	(2)	3	(0)	1
(7)	(6)	(5)	(4)	3	2	1	0

$$U(6,7) =$$

0	(1)	(2)	(3)	(4)	(5)	(6)	7
(1)	(0)	(3)	(2)	(5)	(4)	7	6
2	(3)	0	(1)	(6)	7	(4)	5
(3)	(2)	(1)	(0)	7	6	5	4
4	(5)	(6)	(7)	(0)	(1)	(2)	3
(5)	(4)	(7)	(6)	(1)	(0)	3	2
6	(7)	4	(5)	(2)	3	(0)	1
(7)	(6)	(5)	(4)	3	2	1	0

Appendix E

$$V(4,5) = V(4,6) =$$

(0)	(1)	2	(3)	(4)	(5)	(6)	7
(1)	(0)	(3)	(2)	(5)	(4)	7	6
(2)	(3)	(0)	(1)	(6)	7	(4)	5
(3)	(2)	(1)	(0)	7	6	5	4
(4)	(5)	(6)	7	(0)	(1)	2	(3)
(5)	(4)	7	6	(1)	(0)	(3)	(2)
(6)	7	(4)	5	(2)	(3)	(0)	(1)
7	6	5	4	(3)	(2)	(1)	(0)

$$V(5,6) =$$

(0)	(1)	(2)	(3)	(4)	(5)	(6)	7
(1)	(0)	(3)	(2)	(5)	(4)	7	6
(2)	(3)	(0)	(1)	(6)	7	(4)	5
(3)	(2)	(1)	(0)	7	6	5	4
(4)	(5)	(6)	7	(0)	(1)	(2)	(3)
(5)	(4)	7	6	(1)	(0)	(3)	(2)
(6)	7	(4)	5	(2)	(3)	(0)	(1)
7	6	5	4	(3)	(2)	(1)	(0)

$$V(5,7) =$$

(0)	1	(2)	(3)	(4)	(5)	(6)	7
1	0	(3)	(2)	(5)	(4)	7	6
(2)	(3)	(0)	(1)	(6)	7	(4)	5
(3)	(2)	(1)	(0)	7	6	5	4
(4)	(5)	(6)	7	(0)	1	(2)	(3)
(5)	(4)	7	6	1	0	(3)	(2)
(6)	7	(4)	5	(2)	(3)	(0)	(1)
7	6	5	4	(3)	(2)	(1)	(0)

$$V(6,7) =$$

0	(1)	(2)	(3)	(4)	(5)	(6)	7
(1)	(0)	(3)	(2)	(5)	(4)	7	6
2	(3)	0	(1)	(6)	7	(4)	5
(3)	(2)	(1)	(0)	7	6	5	4
(4)	(5)	(6)	7	0	(1)	(2)	(3)
(5)	(4)	7	6	(1)	(0)	(3)	(2)
(6)	7	(4)	5	2	(3)	0	(1)
7	6	5	4	(3)	(2)	(1)	(0)