On d-Antimagic Labelings for a Special Class of Plane Graphs

Martin Bača
Department of Appl. Mathematics, Technical University
Letná 9, 042 00 Košice, Slovak Republic
Martin.Baca@tuke.sk

Edy Tri Baskoro
Department of Mathematics, Institut Teknologi Bandung
Jalan Ganesa 10, Bandung, Indonesia
ebaskoro@dns.math.itb.ac.id

Yus M. Cholily
Department of Mathematics
Muhammadiyah University of Malang
Jl. Tlogomas 246, Malang, Indonesia

Abstract

A bijection $\lambda : V \cup E \cup F \rightarrow \{1, 2, 3, \ldots, |V|+|E|+|F|\}$ is called a d-antimagic labeling of type $(1,1,1)$ of plane graph $G(V, E, F)$ if the set of s-sided face weights is $W_s = \{a_s, a_s+d, a_s+2d, \ldots, a_s+(f_s-1)d\}$ for some integers s, a_s and d, where f_s is the number of s-sided faces and the face weight is the sum of the labels carried by that face and the edges and vertices surrounding it. In this paper we examine the existence of d-antimagic labelings of type $(1,1,1)$ for a special class of plane graphs C_a^b.

1 Introduction and Definitions

All graphs in this paper will be finite and plane. The plane graph $G = (V, E, F)$ has vertex set $V = V(G)$, edge set $E = E(G)$ and face set $F = F(G)$. We write v for $|V(G)|$, e for $|E(G)|$ and f for $|F(G)|$. A general reference for graph theoretic notions are [13] and [14].

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A bijection $\lambda : V(G) \cup E(G) \cup F(G) \to \{1, 2, 3, \ldots, v+e+f\}$ is called a labeling of type $(1,1,1)$ and a labeling of type $(1,1,0)$ is a bijection from the set $\{1, 2, 3, \ldots, v+e\}$ onto the vertices and edges of plane graph $G(V, E, F)$.

Specially, if we label only vertices or only edges or only faces, we call such a labeling a *vertex labeling* or an *edge labeling* or a *face labeling*, respectively.

The *weight* of a face under a labeling is the sum of the labels (if present) carried by that face and the edges and vertices surrounding it.

A labeling of a plane graph G is called *d-antimagic* if for every number s, the set of s-sided face weights is $W_s = \{a_s, a_s+d, a_s+2d, \ldots, a_s+(f_s-1)d\}$ for some integers a_s and d ($a_s > 0$, $d \geq 0$), where f_s is the number of s-sided faces. We allow different sets W_s for different s.

d-antimagic labeling is natural extension of the notion of *magic* labeling introduced by Ko-Wei Lih [11]. Ko-Wei Lih [11] studied *magic* (0-antimagic) labelings of type $(1,1,0)$ for wheels, friendship graphs and prisms. 0-antimagic labelings of type $(1,1,1)$ for m-antiprisms, grid graphs and hexagonal planar maps are given in [1, 2, 3]. Kathiresan et al. and Qu [8, 9, 12] described consecutive (1-antimagic) labelings for the special classes of plane graphs.

Other types of antimagic labelings were studied by Hartsfield and Ringel [7] and by Bodendiek and Walther [5, 6]. A survey of results and open problems on antimagic labelings is [4].

2 Construction of plane graph C_a^b

Let $I = \{1, 2, 3, \ldots, a\}$ and $J = \{1, 2, 3, \ldots, b\}$ be index sets. Let y_1, $y_2, \ldots, y_a$ be the fixed vertices. We connect the vertices y_i and y_{i+1} by means of b internally disjoint paths $P_i^j = \{y_i, x_{i,j,1}, x_{i,j,2}, \ldots, x_{i,j,i}, y_{i+1}\}$ of length $i+1$ each, where $i \in I$ and $j \in J$. We make the convention that $y_{a+1} = y_1$ to simplify later notations. The resulting graph embedded in the plane we denote by C_a^b with the vertex set $V(C_a^b) = \{y_i : i \in I\} \bigcup \bigcup_{i \in I} \bigcup_{j \in J} \{x_{i,j,k} : 1 \leq k \leq i\}$ and the edge set $E(C_a^b) = \bigcup_{i \in I} \{y_i x_{i,j,1} : j \in J\} \bigcup \bigcup_{i \in I} \bigcup_{j \in J} \{x_{i,j,k} x_{i,j,k+1} : 1 \leq k \leq i-1\} \bigcup \bigcup_{i \in I} \{x_{i,j,i} y_{i+1} : j \in J\}$. Let us denote the face set of C_a^b by $F(C_a^b) = \bigcup_{i \in I} \{f_{i,j} : j \in J - \{b\}\} \bigcup \{f_{int}, f_{ext}\}$ where $f_{i,j}$ is $(2i+2)$-sided face determined by paths P_i^j and P_i^{j+1}, $i \in I, j \in J - \{b\}$, and f_{int} is the *internal* $\frac{a(a+3)}{2}$-sided face determined by cycle on vertices $\{y_i : i \in I\} \bigcup \bigcup_{i \in I} \{x_{i,b,k} : 1 \leq k \leq i\}$ and f_{ext} is the *external* $\frac{a(a+3)}{2}$-sided face determined by cycle on vertices

$\{y_i : i \in I\} \bigcup \bigcup_{i\in I}\{x_{i,1,k} : 1 \le k \le i\}$.

So, $v = \frac{ab(a+1)}{2} + a$, $e = \frac{ab(a+3)}{2}$ and $f = a(b-1)+2$.

If we omit the paths $P_a^j = \{y_a, x_{a,j,1}, x_{a,j,2}, \dots, x_{a,j,a}, y_1\}$, $j \in J$, we obtain a plane graph defined in [10] by Kathiresan and Ganesan as P_a^b. Kathiresan and Ganesan [10] studied d-antimagic labelings of type $(1,1,1)$ for the plane graph P_a^b and described d-antimagic labelings for $d \in \{0,1,2,3,4,6\}$.

In the present article we deal with d-antimagic labelings of type $(1,1,1)$ for C_a^b and we show that plane graph C_a^b has d-antimagic labeling for $d \in \{0,1,2,3\}$.

3 Vertex labelings

If $a \ge 3$, $b \ge 2$ and $i \in I$, $j \in J$, $1 \le k \le i$, we construct a vertex labeling $\lambda_t : V(C_a^b) \to \{1,2,3,\dots,v\}$, $t \in \{1,2,3\}$, as follows.

$$\lambda_1(y_i) = \lambda_2(y_i) = \lambda_3(y_i) = i,$$

$$\lambda_1(x_{i,j,k}) = \begin{cases} \frac{bi(i-1)}{2} + a + b - \frac{j-1}{2} & \text{if } i \text{ and } j \text{ are odd, } k=1 \\ \frac{bi(i-1)}{2} + a + \lceil \frac{b}{2} \rceil - \frac{j-2}{2} & \text{if } i \text{ is odd, } j \text{ is even, } k=1 \\ \frac{bi(i-1)}{2} + a + b(k-1) + j & \text{if } i \text{ is even, } k \text{ is odd, or} \\ & \text{if } i \text{ is odd, } i \ge 3,\ k \text{ is even} \\ \frac{bi(i-1)}{2} + a + kb + 1 - j & \text{if } i \text{ and } k \text{ are even, or} \\ & \text{if } i \text{ and } k \text{ are odd, } i,k \ge 3 \end{cases}$$

$$\lambda_2(x_{i,j,k}) = \begin{cases} \frac{bi(i-1)}{2} + a + \lceil \frac{b}{2} \rceil - \frac{j-1}{2} & \text{if } i \text{ and } j \text{ are odd, } k=1 \\ \frac{bi(i-1)}{2} + a + b - \frac{j-2}{2} & \text{if } i \text{ is odd, } j \text{ is even, } k=1 \\ \frac{bi(i-1)}{2} + a + b(k-1) + j & \text{if } i \text{ is even, } k \text{ is odd, or} \\ & \text{if } i \text{ is odd, } i \ge 3,\ k \text{ is even} \\ \frac{bi(i-1)}{2} + a + kb + 1 - j & \text{if } i \text{ and } k \text{ are even, or} \\ & \text{if } i \text{ and } k \text{ are odd, } i,k \ge 3 \end{cases}$$

$$\lambda_3(x_{i,j,k}) = \begin{cases} \frac{bi(i-1)}{2} + a + b(k-1) + j & \text{if } k \text{ is odd} \\ \frac{bi(i-1)}{2} + a + kb + 1 - j & \text{if } k \text{ is even.} \end{cases}$$

4 Edge labelings

In this section, we provide constructions of the edge labelings $\delta_t : E(C_a^b) \to \{1, 2, 3, \ldots, e\}$, $t \in \{1, 2, 3, 4, 5\}$, in the following way.

If $a \geq 3$, $b \geq 2$ and $i \in I$, $j \in J$, $1 \leq k < i$, then

$$\delta_1(y_i x_{i,j,1}) = \begin{cases} \frac{b(i-1)(i+2)}{2} + j & \text{if } i \text{ is odd} \\ \frac{b(i-1)(i+2)}{2} + \frac{j+1}{2} & \text{if } i \text{ is even, } j \text{ is odd} \\ \frac{b(i-1)(i+2)}{2} + \lceil \frac{b}{2} \rceil + \frac{j}{2} & \text{if } i \text{ and } j \text{ are even} \end{cases}$$

$$\delta_1(x_{i,j,i} y_{i+1}) = \begin{cases} \frac{bi(i+3)}{2} + 1 - j & \text{if } i \text{ is odd} \\ \frac{bi(i+3)}{2} - b + j & \text{if } i \text{ is even} \end{cases}$$

$$\delta_1(x_{i,j,k} x_{i,j,k+1}) = \begin{cases} \frac{bi(i+1)}{2} + kb + 1 - j & \text{if } k \text{ is odd} \\ \frac{bi(i+1)}{2} + b(k-1) + j & \text{if } k \text{ is even} \end{cases}$$

$$\delta_2(y_i x_{i,j,1}) = \begin{cases} j & \text{if } i = 1 \\ \frac{ab(a+3)}{2} - b + \frac{j+1}{2} & \text{if } i = 2, j \text{ is odd} \\ \frac{ab(a+3)}{2} - \frac{b-1}{2} + \frac{j}{2} & \text{if } i = 2, j \text{ is even} \\ \frac{b(i^2+i-4)}{2} + j & \text{if } i \text{ is odd, } i \geq 3 \\ \frac{b(i^2+i-4)}{2} + \frac{j+1}{2} & \text{if } i \text{ is even, } i \geq 4, j \text{ is odd} \\ \frac{b(i^2+i-4)}{2} + \lceil \frac{b}{2} \rceil + \frac{j}{2} & \text{if } i \text{ and } j \text{ are even, } i \geq 4 \end{cases}$$

$$\delta_2(x_{i,j,i} y_{i+1}) = \begin{cases} \frac{b(i^2+3i-2)}{2} + 1 - j & \text{if } i \text{ is odd, } i \geq 3 \\ 2b + 1 - j & \text{if } i = 1 \\ \frac{b(i-1)(i+4)}{2} + j & \text{if } i \text{ is even} \end{cases}$$

$$\delta_2(x_{i,j,k} x_{i,j,k+1}) = \begin{cases} \frac{bi(i+1)}{2} + b(k-1) + 1 - j & \text{if } k \text{ is odd} \\ \frac{bi(i+1)}{2} + b(k-2) + j & \text{if } k \text{ is even} \end{cases}$$

$$\delta_3(y_i x_{i,j,1}) = \delta_1(y_i x_{i,j,1})$$

$$\delta_3(x_{i,j,i}y_{i+1}) = \frac{bi(i+3)}{2} + 1 - j$$

$$\delta_3(x_{i,j,k}x_{i,j,k+1}) = \begin{cases} \frac{bi(i+1)}{2} + b(k-1) + j & \text{if } i \text{ is even, } k \text{ is odd, or} \\ & \text{if } i \text{ is odd, } k \text{ is even} \\ \frac{bi(i+1)}{2} + bk + 1 - j & \text{if } i \text{ and } k \text{ are even, or} \\ & \text{if } i \text{ and } k \text{ are odd} \end{cases}$$

$$\delta_4(x_{i,j,i}y_{i+1}) = \delta_2(x_{i,j,i}y_{i+1})$$

$$\delta_4(x_{i,j,k}x_{i,j,k+1}) = \delta_2(x_{i,j,k}x_{i,j,k+1})$$

$$\delta_4(y_i x_{i,j,1}) = \begin{cases} j & \text{if } i = 1 \\ \frac{ab(a+3)}{2} - \frac{j-1}{2} & \text{if } i = 2,\ j \text{ is odd} \\ \frac{ab(a+3)}{2} - \lceil \frac{b}{2} \rceil - \frac{j-2}{2} & \text{if } i = 2,\ j \text{ is even} \\ \frac{b(i^2+i-4)}{2} + j & \text{if } i \text{ is odd, } i \geq 3 \\ \frac{b(i^2+i-4)}{2} + \frac{j+1}{2} & \text{if } i \text{ is even, } i \geq 4,\ j \text{ is odd} \\ \frac{b(i^2+i-4)}{2} + \lceil \frac{b}{2} \rceil + \frac{j}{2} & \text{if } i \text{ and } j \text{ are even, } i \geq 4 \end{cases}$$

$$\delta_5(y_i x_{i,j,1}) = \begin{cases} \frac{b(i-1)(i+2)}{2} + j & \text{if } i \text{ is odd} \\ \frac{b(i-1)(i+2)}{2} + b + 1 - j & \text{if } i \text{ is even} \end{cases}$$

$$\delta_5(x_{i,j,i}y_{i+1}) = \delta_3(x_{i,j,i}y_{i+1})$$

$$\delta_5(x_{i,j,k}x_{i,j,k+1}) = \delta_3(x_{i,j,k}x_{i,j,k+1}).$$

5 The results

With the vertex labelings and the edge labelings of the previous sections in hand, we investigate d-antimagic labelings of the plane graph C_a^b.

First, let us denote the weight of the $(2i+2)$-sided face $f_{i,j}$ and the external (internal) $\frac{a(a+3)}{2}$-sided face under a vertex labeling λ and an edge labeling δ as follows:

$$w(f_{i,j}) = \lambda(y_i) + \sum_{k=1}^{i} \lambda(x_{i,j,k}) + \lambda(y_{i+1}) + \sum_{k=1}^{i} \lambda(x_{i,j+1,k}) +$$

$$\delta(y_i x_{i,j,1}) + \sum_{k=1}^{i-1} \delta(x_{i,j,k} x_{i,j,k+1}) + \delta(x_{i,j,i} y_{i+1}) + \delta(y_i x_{i,j+1,1}) +$$

$$\sum_{k=1}^{i-1} \delta(x_{i,j+1,k} x_{i,j+1,k+1}) + \delta(x_{i,j+1,i} y_{i+1})$$

for $i \in I$ and $j \in J - \{b\}$,

$$w(f_{ext}) = \sum_{i=1}^{a} \lambda(y_i) + \sum_{i=1}^{a} \sum_{k=1}^{i} \lambda(x_{i,1,k}) + \sum_{i=1}^{a} \delta(y_i x_{i,1,1}) +$$

$$\sum_{i=1}^{a} \sum_{k=1}^{i-1} \delta(x_{i,1,k} x_{i,1,k+1}) + \sum_{i=1}^{a} \delta(x_{i,1,i} y_{i+1}),$$

$$w(f_{int}) = \sum_{i=1}^{a} \lambda(y_i) + \sum_{i=1}^{a} \sum_{k=1}^{i} \lambda(x_{i,b,k}) + \sum_{i=1}^{a} \delta(y_i x_{i,b,1}) +$$

$$\sum_{i=1}^{a} \sum_{k=1}^{i-1} \delta(x_{i,b,k} x_{i,b,k+1}) + \sum_{i=1}^{a} \delta(x_{i,b,i} y_{i+1}).$$

Let $W_i = \{w(f_{i,j}) : j \in J - \{b\}\}$, $i \in I$, be a set of the $(2i+2)$-sided face weights of C_a^b.

Theorem 1 *For $a \geq 3$ and $b \geq 2$, the plane graph C_a^b has a 0-antimagic labeling of type $(1,1,1)$.*

Proof Let us distinguish three cases.

Case 1. a is odd and b is even

Label the vertices and the edges of C_a^b by λ_1 and $v+\delta_1$. One can check that the obtained labeling successively assumes values $1, 2 \ldots, v+e$, every set W_i, $i \in I$, consists of an arithmetic sequence of difference 1, and $w(f_{ext}) - w(f_{int}) = b - 1$.

Define a face labeling $\sigma_1 : F(C_a^b) \to \{v+e+1, v+e+2, \ldots, v+e+f\}$ as follows:

$$\sigma_1(f_{i,j}) = \begin{cases} v+e+f-b+j & \text{if } i=1 \\ v+e+f-(i-1)(b-1)-1-j & \text{if } i \text{ is even} \\ v+e+f-i(b-1)-2+j & \text{if } i \text{ is odd, } i \geq 3 \end{cases}$$

for $i \in I$ and $j \in J - \{b\}$,

$\sigma_1(f_{ext}) = v+e+f-b$,

$\sigma_1(f_{int}) = v+e+f$.

If we combine labelings λ_1, $v+\delta_1$ and σ_1 we obtain labeling of type $(1,1,1)$ such that all $(2i+2)$-sided faces, for each $i \in I$, have the same weight and f_{ext} has weight one less than f_{int}.

If we swap the edge label $v + \delta_1(x_{a,1,a}y_1) = v+e$ with the face label $\sigma_1(f_{a,1}) = v+e+1$ then the face weight of $f_{a,1}$ will remain the same, but the face weight of f_{ext} will be increased by one. Thus the resulting labeling of type $(1,1,1)$ is 0-antimagic.

Case 2. a is even

If a and b are even then label the vertices and the edges of C_a^b by λ_1 and $v+\delta_1$. If a is even and b is odd then label the vertices and the edges of C_a^b by λ_2 and $v+\delta_1$. In both these cases we obtain a labeling of type $(1,1,0)$ where the external face f_{ext} has the same weight as the internal face f_{int} and the weights of $(2i+2)$-sided faces, for each $i \in I$, constitute an arithmetic progression of difference 1.

Define a new face mapping $\sigma_2 : F(C_a^b) \to \{v+e+1, v+e+2, \ldots, v+e+f\}$ by

$$\sigma_2(f_{i,j}) = \begin{cases} v+e+f-(i-1)(b-1)-1-j & \text{if } i \text{ is even} \\ v+e+f-i(b-1)-2+j & \text{if } i \text{ is odd} \end{cases}$$

for $i \in I$ and $j \in J - \{b\}$,

$\sigma_2(f_{ext}) = v+e+f$,

$\sigma_2(f_{int}) = v+e+f-1$.

It can be seen that the labelings λ_1, $v+\delta_1$ and σ_2, and also the labelings λ_2, $v+\delta_1$ and σ_2, combine to labeling of type $(1,1,1)$ where $(2i+2)$-sided faces, for each $i \in I$, have common weight and the face f_{int} has weight one less than the face f_{ext}.

If we swap the edge label $v + \delta_1(x_{a,b,a}y_1) = v + e$ with the face label $\sigma_2(f_{a,b-1}) = v + e + 1$ then the face weight of $f_{a,b-1}$ will remain the same, but the face weight of f_{int} will be increased by one i.e. after swaping the faces f_{int} and f_{ext} obtain the same weights. It follows that the resulting labeling is a 0-antimagic of type $(1, 1, 1)$.

Case 3. a and b are odd

Define a face labeling $\sigma_3 : F(C_a^b) \rightarrow \{v + e + 1, v + e + 2, \dots, v + e + f\}$ such that for $i \in I$, $j \in J - \{b\}$

$$\sigma_3(f_{i,j}) = \begin{cases} v+e+f-b-1+j & \text{if } i = 1 \\ v+e+b-j & \text{if } i = 2 \\ v+e+f-(i-1)(b-1)-2+j & \text{if } i \text{ is odd, } i > 1 \\ v+e+f-(i-2)(b-1)-1-j & \text{if } i \text{ is even, } i > 2 \end{cases}$$

$\sigma_3(f_{ext}) = v + e + f,$

$\sigma_3(f_{int}) = v + e + f - 1.$

Now, label the vertices of C_a^b by the labeling λ_2, the edges by the labeling $v + \delta_2$ and the faces by the labeling σ_3. It is easy to verify that under the resulting labeling of type $(1, 1, 1)$ the $(2i + 2)$-sided faces, for each $i \in I$, have common weight and the internal face f_{int} has weight $\lceil \frac{b}{2} \rceil$ less than the external face f_{ext}. Therefore we swap the edge label $v + \delta_2(y_2x_{2,b,1}) = v + e - \frac{b-1}{2}$ with the face label $\sigma_3(f_{2,b-1}) = v + e + 1$ that do not change the face weight of $f_{2,b-1}$, but the weight of f_{int} will be increased by $\lceil \frac{b}{2} \rceil$. It means that the $\frac{a(a+3)}{2}$-sided faces have the same weights and the resulting labeling is a 0-antimagic of type $(1, 1, 1)$. □

Theorem 2 *For $a \geq 3$ and $b \geq 2$, the plane graph C_a^b has a 2-antimagic labeling of type* $(1, 1, 1)$.

Proof As in the proof of previous theorem, we will distinguish three cases.

Case 1. a is odd and b is even

If we label the vertices and the edges of C_a^b by λ_1 and $v + \delta_1$ we obtain a labeling of type $(1, 1, 0)$, where each set $W_i = \{w(f_{i,j}) : j \in J - \{b\}\}$, $i \in I$, consists of an arithmetic progression with difference 1, and $w(f_{ext}) - w(f_{int}) = b - 1$.

Define a new mapping $\sigma_4 : F(C_a^b) \rightarrow \{v + e + 1, v + e + 2, \dots, v + e + f\}$ by

$$\sigma_4(f_{i,j}) = \begin{cases} v+e+f-(i-1)(b-1)-1-j & \text{if } i \text{ is odd} \\ v+e+f-i(b-1)-2+j & \text{if } i \text{ is even} \end{cases}$$

for $i \in I$ and $j \in J - \{b\}$,

$\sigma_4(f_{ext}) = v + e + f - 1$,

$\sigma_4(f_{int}) = v + e + f$.

It can be seen that the labelings λ_1, $v + \delta_1$ and σ_4 combine to a labeling of type $(1, 1, 1)$ where the weights of $(2i + 2)$-sided faces, for each $i \in I$, constitute an arithmetic progression with difference 2 and the face f_{int} has weight $b - 2$ less than the face f_{ext}.

If we swap the edge value $v+\delta_1(x_{a,b,a}y_1) = v+e-b+1$ with the face value $\sigma_4(f_{a,b-1}) = v + e + 1$ then the face weight of $f_{a,b-1}$ will remain the same, but the face weight of f_{int} will be increased by b. Thus difference between the weights of f_{int} and f_{ext} is 2.

Case 2. a is even

Define a mapping $\sigma_5 : F(C_a^b) \to \{v + e + 1, v + e + 2, \ldots, v + e + f\}$ by

$\sigma_5(f_{i,j}) = \sigma_4(f_{i,j})$ for $i \in I$ and $j \in J - \{b\}$,

$\sigma_5(f_{ext}) = v + e + f$ and $\sigma_5(f_{int}) = v + e + f - 1$.

If a and b are even label the vertices, the edges and the faces of C_a^b by λ_1, $v + \delta_3$ and σ_5. If a is even and b is odd label the vertices, the edges and the faces by λ_2, $v + \delta_3$ and σ_5. In both these cases we obtain a labeling of type $(1, 1, 1)$ such that the weights of $(2i + 2)$-sided faces, for each $i \in I$, constitute an arithmetic progression of difference 2 and the weight of f_{ext} is one greater than the weight of f_{int}.

Swaping the edge label $v+\delta_3(x_{a,1,a}y_1) = v+e$ with the face label $\sigma_5(f_{a,1}) = v + e + 1$ does not change the face weight of $f(a, 1)$, but the weight of f_{ext} will be two greater than the weight of f_{int}.

Case 3. a and b are odd

Define set of labels on the faces $\sigma_6 : F(C_a^b) \to \{v+e+1, v+e+2, \ldots, v+e+f\}$ as follows:

$$\sigma_6(f_{i,j}) = \begin{cases} v+e+2b-j & \text{if } i = 1 \\ v+e+b-j & \text{if } i = 2 \\ v+e+f-(i-3)(b-1)+1-j & \text{if } i \text{ is odd, } i \geq 3 \\ v+e+f-(i-2)(b-1)+j & \text{if } i \text{ is even, } i \geq 4 \end{cases}$$

for $i \in I$ and $j \in J - \{b\}$,

$\sigma_6(f_{ext}) = v + e + b$,

$\sigma_6(f_{int}) = v + e + 2b$.

If we label the vertices, edges and faces in C_a^b by λ_2, $v + \delta_4$ and σ_6 and swap the edge value $v + \delta_4(y_2 x_{2,b,1}) = v + e - \frac{b-1}{2}$ with the face value $\sigma_6(f_{2,b-1}) = v + e + 1$, then the weights of $(2i + 2)$-sided faces, for each $i \in I$, and also the weights of f_{int} and f_{ext}, constitute the arithmetic progressions with difference 2. □

Theorem 3 *For $a \geq 3$, $b \geq 2$ and $d \in \{1, 3\}$, the plane graph C_a^b has a d-antimagic labeling of type $(1, 1, 1)$.*

Proof We divide the proof into two cases.

Case 1. a is even

Label the vertices and the edges of C_a^b by λ_3 and $v+\delta_5$. There is no problem in seeing that we obtain a labeling of type $(1, 1, 0)$, where $w(f_{ext}) = w(f_{int})$ and every set $W_i = \{w(f_{i,j}) : j \in J - \{b\}\}$, $i \in I$, consists of an arithmetic progression with difference 2.

Now, we are able to arrange the face values $v+e+1, v+e+2, \ldots, v+e+f-2$ to the $(2i + 2)$-sided faces, $i \in I$, and the values $v + e + f - 1$, $v + e + f$ to the f_{ext}, f_{int} in such a way that the resulting labeling is 1-antimagic of type $(1, 1, 1)$.

If we label the $(2i + 2)$-sided faces by face labeling σ_2 and f_{ext} by the value $v + e + f - 1$, f_{int} by the value $v + e + f$, and swap the edge value $v + \delta_5(x_{a,1,a} y_1) = v + e$ $(v + \delta_5(x_{a,b,a-1} x_{a,b,a}) = v + e - b)$ with the face value $\sigma_2(f_{a,1}) = v + e + b - 1$ $(\sigma_2(f_{a,b-1}) = v + e + 1)$, then the resulting labeling of type $(1, 1, 1)$ is 3-antimagic.

Case 2. a is odd

Label the vertices and the edges of C_a^b by λ_3 and $v + \delta_5$. Again it is easy to verify that the labelings λ_3 and $v + \delta_5$ combine to labeling of type $(1, 1, 0)$, where set W_i , for every $i \in I$, consists of an arithmetic sequence of difference 2 and $w(f_{int}) - w(f_{ext}) = b - 1$.

(i) Label the faces $f_{i,j}$, $i \in I$, $j \in J - \{b\}$, by face labeling σ_4 and the external face by value $v+e+f$ and the internal face by value $v+e+f-1$. The weights of $(2i + 2)$-sided faces, $i \in I$, constitute an arithmetic progression with difference 1 and the face f_{ext} has weight $b - 2$ less than the face f_{int}.

If we swap the edge label $v + \delta_5(x_{a,1,a}y_1) = v + e$ with the face label $\sigma_4(f_{a,1}) = v + e + b - 1$ then the weight of $f_{a,1}$ remains the same, but the weight of f_{ext} is increased by $b-1$. Now, difference between the weights of f_{int} and f_{ext} is 1 and the resulting labeling of type $(1,1,1)$ is 1-antimagic.

(ii) Label the external face of C_a^b by value $v+e+f$, the internal face by value $v+e+f-b$ and the faces $f_{i,j}$, $i \in I$, $j \in J - \{b\}$, by face labeling σ_1. One can check that the face labeling and labelings λ_3, $v + \delta_5$ combine to labeling of type $(1,1,1)$ where $(2i+2)$-sided faces, $i \in I$, constitute an arithmetic sequence of difference 3 and the weight of f_{int} is one less than the weight of f_{ext}.

Therefore we swap the edge label $v+\delta_5(x_{a,1,a-1}x_{a,1,a}) = v+e-2b+1$ with the face label $\sigma_1(f_{a,1}) = v+e+1$ and the edge label $v+\delta_5(x_{a,b,a-1}x_{a,b,a}) = v+e-b+1$ with the face label $\sigma_1(f_{a,b-1}) = v+e+b-1$. This swaping does not change the face weights of the faces $f_{a,1}$ and $f_{a,b-1}$ but the weight of f_{ext} will be increased by two, so difference between the resulting weight of f_{ext} and the resulting weight of f_{int} will be 3. Thus we arrive at the desired result. □

6 Conclusion

In this paper, we have studied d-antimagic labelings of plane graph C_a^b. We have shown that for $a \geq 3$, $b \geq 2$ and $d \in \{0,1,2,3\}$, there exists d-antimagic labeling of type $(1,1,1)$.

We conclude with the following open problem.

Open Problem 1 *Find other possible values of the parameter d and corresponding d-antimagic labelings of type* $(1,1,1)$ *for* C_a^b.

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