ON INTEGER-MAGIC SPECTRA OF CATERPILLARS

EBRAHIM SALEHI and PATRICK BENNETT
Department of Mathematical Sciences
University of Nevada Las Vegas
Las Vegas, NV 89154-4020.
ebrahim.salehi@unlv.edu
patrick.bennett@unlv.edu

ABSTRACT. For any $h \in \mathbb{N}$, a graph $G = (V, E)$ is said to be h-magic if there exists a labeling $l : E(G) \longrightarrow \mathbb{Z}_h - \{0\}$ such that the induced vertex set labeling $l^+ : V(G) \longrightarrow \mathbb{Z}_h$ defined by

$$l^+(v) = \sum_{uv \in E(G)} l(uv)$$

is a constant map. For a given graph G, the set of all $h \in \mathbb{Z}_+$ for which G is h-magic is called the integer-magic spectrum of G and is denoted by $IM(G)$. The concept of integer-magic spectrum of a graph was first introduced in [4]. But unfortunately, this paper has a number of incorrect statements and theorems. In this paper, first we will correct some of those statements, then we will determine the integer-magic spectra of caterpillars.

Key Words: magic, non-magic, integer-magic spectrum.
2000 Mathematics Subject Classification: 05C78

1. INTRODUCTION

In this paper all graphs are connected, finite, simple, and undirected. For an abelian group A, written additively, any mapping $l : E(G) \longrightarrow A - \{0\}$ is called a *labeling*. Given a labeling on the edge set of G one can introduce a vertex set labeling $l^+ : V(G) \longrightarrow A$ by

$$l^+(v) = \sum_{uv \in E(G)} l(uv).$$

A graph G is said to be A-*magic* if there is a labeling $l : E(G) \longrightarrow A - \{0\}$ such that for each vertex v, the sum of the labels of the edges incident with v are all equal to the same constant; that is, $l^+(v) = c$ for some fixed $c \in A$. In general, a graph G may admit more than one labeling to become A-magic; for example, if $|A| > 2$ and $l : E(G) \longrightarrow A - \{0\}$ is a magic labeling of G with sum c, then $\lambda : E(G) \longrightarrow A - \{0\}$, the *inverse labeling* of l, defined by $\lambda(uv) = -l(uv)$ will provide another magic labeling of G with sum $-c$. A graph $G = (V, E)$ is called *fully magic* if it is A-magic for every abelian group A. For example, every regular graph is fully magic. A graph $G = (V, E)$ is called *non-magic* if for every abelian group A, the graph is not A-magic. The most obvious class of non-magic graphs is P_n ($n \geq 3$), the path of order n. As a result, any graph with a pendant path of length $n \geq 3$ would be non-magic. Here is another example of a non-magic graph: Consider the graph H Figure 1. Given any abelian group A, a typical magic labeling of H is illustrated in that figure. The combination of conditions $l^+(u) = l^+(v) = l^+(w) = x$ imply that $y = z = 0$, which is not an acceptable magic labeling. Thus H is not A-magic. We will generalize this fact in 3.2.

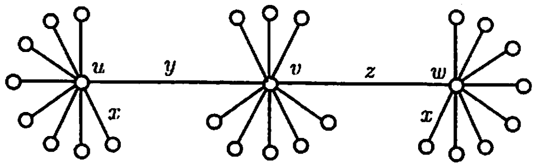

FIGURE 1. An example of a non-magic graph.

Certain classes of non-magic graphs are presented in [1]. The original concept of A-magic graph is due to J. Sedlacek [11, 12], who defined it to be a graph with a real-valued edge labeling such that

(1) distinct edges have distinct nonnegative labels; and
(2) the sum of the labels of the edges incident to a particular vertex is the same for all vertices.

Jenzy and Trenkler [3] proved that a graph G is magic if and only if every edge of G is contained in a (1-2)-factor. $\mathbb{Z}$-magic graphs were considered by Stanley [13, 14], who pointed out that the theory of magic labeling can be put into the more general context of linear homogeneous diophantine equations. Recently, there has been considerable research articles in graph labeling, interested readers are directed to [2, 15]. For convenience, the notation 1-magic will be used to indicate $\mathbb{Z}$-magic and $\mathbb{Z}_h$-magic graphs will be referred to as h-magic graphs. Clearly, if a graph is h-magic, it is not necessarily k-magic ($h \neq k$).

Definition 1.1. *For a given graph G the set of all positive integers h for which G is h-magic is called the* integer-magic spectrum *of G and is denoted by $IM(G)$.*

Since any regular graph is fully magic, then it is h-magic for all positive integers $h \geq 2$; therefore, $IM(G) = \mathbb{N}$. On the other hand, the graph H, Figure 1, is non-magic, hence $IM(H) = \emptyset$. The integer-magic spectra of certain classes of graphs resulted by the amalgamation of cycles and stars have already been identified [5], and in [6] the integer-magic spectra of the trees of diameter at most four have been completely characterized. Also, the integer-magic spectra of some other graphs have been studied in [7, 8, 9, 10].
The concept of integer-magic spectrum of a graph was first introduced in [4]. But unfortunately, this paper has a number of incorrect statements and theorems. In the following sections, first we present corrections to this paper, then we will determine the integer-magic spectra of caterpillars.

2. CORRECTIONS

The paper [4] has one initial incorrect statement, which is used repeatedly in proofs. The statement claims that if a graph G is $\mathbb{N}$-magic, then G is k-magic for all $k > 2$. This is not necessarily true, as demonstrated in Figure 2.

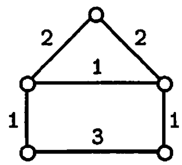

FIGURE 2. An $\mathbb{N}$-magic graph that is not 3-magic.

Correction 2.1. *If a graph G has an $\mathbb{N}$-magic labeling $l : E(G) \to \mathbb{N}$, then G is k-magic as long as k does not divide $l(e)$ for every $e \in E(G)$*

The following observation will be used as well:

Observation 2.2. *If a graph G has a $\mathbb{Z}$-magic labeling $l : E(G) \to \mathbb{Z}$, then G is k-magic as long as k does not divide $l(e)$ for every $e \in E(G)$*

Proof. In order to construct a k-magic labeling, we start with the $\mathbb{Z}$-magic labeling of G, and replace every edge label $l(e)$ with $l(e) \pmod{k}$. Since k does not divide any $l(e)$, none of these new labels are 0. □

Corollary 2.3. *If G is $\mathbb{Z}$-magic, then G is k-magic for sufficiently large k.*

Proof. If G has a $\mathbb{Z}$-magic labeling ℓ, then G is k-magic as long as $k > \ell(e)$ for every edge e. So G is k-magic for every k larger than $max\{\ell(e)\}$ □

Theorems 3 and 7 in [4] discuss the double star $DS(m, m)$. These theorems are incorrect. The general case of double-stars $DS(m, n)$ will be discussed later in 3.3. However, here is the correct version from [6]:

Correction 2.4. *If $m > 2$, then $IM(\,DS(m,m)\,) = \mathbb{N} - \{\; h > 1 \;:\; h|(m-2)\;\}$.*

For example, $DS(11, 11) = \mathbb{N} - \{3, 9\}$.
Theorems 4 and 13 in [4] discuss the wheel W_n. Although the statements of these theorems are correct, but the proof of theorem 4 is invalid, and the proof for Theorem 13 is omitted. In particular, Figure 3 in [4] does not provide a 3-magic labeling of the graph W_7. Here are the theorems, with proofs [9]:

Correction 2.5. *If $n \geq 3$, then $IM(W_n) = \mathbb{N} - \{1 + (-1)^n\}$.*

Proof. We will consider two cases:
Case I. $n = 2k+1$ is odd. We observe that the degree set of W_{2k+1} is $\{3, 2k+1\}$, hence it is h-magic for all even numbers h; we simply label all the edges by $h/2$. Also, if $h > k$, then we label all the cycle edges by k and spokes by 1. This is a magic labeling of W_{2k+1} with sum $n = 2k+1$.

Now, we may assume that h is odd and is at most k. If $\gcd(k, h) = \delta$, $1 \le \delta < h$, then we label the cycle edges by δ and spokes by x, where x is the nonzero solution of the equation $kx \equiv \delta \pmod{h}$. This provides a magic labeling of W_{2k+1} with sum $x + 2\delta$.
Finally, if $h|k$, we label $h+1$ consecutive spokes by 1 and the rest of them by $h-1$. For cycle edges, we label those that are adjacent to the spokes labeled $h-1$ by 1 the remaining by $h-1$ and 1, alternatively. This would be a magic labeling of W_{2k+1} with sum 1. Therefore, W_{2k+1} is h-magic for all $h \ge 1$; that is $IM(W_{2k+1}) = \mathbb{N}$.
Case II. $n = 2k$ is even. We observe that the degree set of W_{2k} is $\{3, 2k\}$, hence it cannot be 2-magic. Next we label all the spokes by x and the cycle edges by a, b, alternatively. The requirement of having the same number for the sum of the edges incident with vertices will provide the equation

$$(2k-1)x \equiv a + b \pmod{h}. \tag{2.1}$$

If $\gcd(2k-1, h) = \delta \ge 3$, then we choose $a = 1$, $b = -1$, and $x = h/\delta$. this would be a magic labeling of W_{2k} with sum x.
If $\gcd(2k-1, h) = 1$, then we choose $a = b = 1$ and notice that the equation $(2k-1)x \equiv 2 \pmod{h}$ has a nonzero solution for x. We label all the spokes with this x, the result is a magic labeling of W_{2k} in $\mathbb{Z}_h$ with sum $x + 2$. Therefore, $IM(W_{2k}) = \mathbb{N} - \{2\}$. □

Theorem 10 in [4], which discusses even coronas, is incorrect. Here is the correct version:

Correction 2.6. *The corona $C_n@K_1$, where n is even, has integer-magic spectrum $\mathbb{N}$*

Proof. Consider the $\mathbb{Z}$-magic labeling in which all pendant edges are labeled 1, and the edges of C_n are alternately labeled $1, -1$. This shows directly that the graph is $\mathbb{Z}$-magic, and since all the edge labels have absolute value 1, by Observation 2.2, the graph is k-magic for all $k > 1$. □

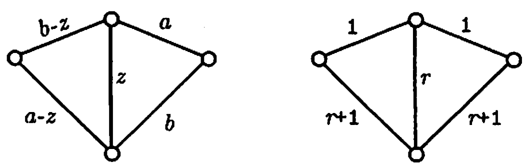

FIGURE 3. A typical magic labeling of F_4.

Theorem 12 in [4], which discusses fans $F_n = P_n + K_1$ is incorrect. For example, $IM(F_3) = 2\mathbb{N} - \{2\}$; A typical magic labeling of $F_3 \cong K_4 - e$ is illustrated in Figure 3, for which we require that $a + b - 2z = a + b$ or $2z \equiv 0 \pmod{h}$; that is, h has to be even. On the other hand, if $h = 2r$, then F_4 is 4-magic (Figure 3). Therefore, $IM(F_4) = 2\mathbb{N} - \{2\}$.

Section 4 in [4] has title: "**Graphs G with $IM(G) = \{1\} \bigcup \{4+2k : k = 1, 2 \ldots\}$.**" The corollary 2.3 indicates that there is no graph G with such an integer-magic spectrum. Obviously, this means that Theorem 15 in [4] is not correct. This theorem discusses one particular graph that is illustrated in Figure 4.

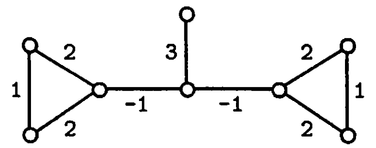

FIGURE 4. $IM(G) = \mathbb{N} - \{2, 3\}$.

Correction 2.7. *The graph G of Figure 4 has integer-magic spectrum $\mathbb{N} - \{2, 3\}$*

Proof. Note that the $\mathbb{Z}$-magic labeling, which by Observation 2.2, implies that G is k-magic for all $k > 3$. Clearly, this graph is not 2 or 3-magic. □

Theorem 18 in [4] discusses the integer-magic spectrum of stars $ST(n) = K(1, n)$, which is incorrect. The correct versions is as follows [6]:

Correction 2.8. *Let $n \geq 3$, and $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorization of $n-1$. Then*

$$IM(K_{1,n}) = \bigcup_{i=1}^{k} p_i \mathbb{N}.$$

3. CATERPILLARS

Caterpillar is a tree having the property that the removal of its end-vertices results in a path (the spine). We use $CR(a_1, a_2, \cdots, a_n)$ to denote the caterpillar with a P_n-spine, where the ith vertex of P_n has degree a_i. Since $CR(1, a_1, \cdots, a_n, 1) = CR(a_1, \cdots, a_n)$ and $a_i \neq 1$ $(2 \leq i \leq n-1)$, we will assume that $a_i \geq 2$. Furthermore, if $a_1 = 2$ or $a_n = 2$, then we will have a graph with a P_3 pendant, which is nonmagic.

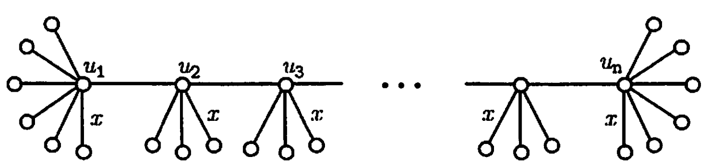

FIGURE 5. A Caterpillar of diameter $n+1$ (P_n-spine).

Theorem 3.1. *Given a caterpillar $G = CR(a_1, a_2, \ldots, a_n)$, let $\sigma = \dfrac{(-1)^n - 1}{2} - \sum_{i=1}^{n} (-1)^i a_i$ and $c_i = \dfrac{3 - (-1)^i}{2} - a_i + a_{i-1} - \cdots + (-1)^i a_1$ $(1 \leq i \leq n-1)$. Also,* *let C be the set of all positive divisors of c_i $\forall i = 1, \cdots, n-1$. Then*

$$IM(G) = \begin{cases} \emptyset & \text{if } \sigma | c_i \text{ for some } i = 1, \cdots, n-1; \\ \mathbb{N} - C & \text{if } \sigma = 0; \\ \bigcup_{d \in D} d\mathbb{N} & \text{otherwise,} \end{cases}$$

where D is the set of all positive divisors d of σ with the property that $d \nmid c_i$ $\forall i = 1, \cdots, n-1$.

Proof. Let $l : E(G) \to \mathbb{Z}_h$ be a magic labeling of $G = CR(a_1, \cdots a_n)$ and let y_i be the label of $u_i u_{i+1}$ $(1 \le i \le n-1)$, as illustrated in Figure 5. Note that in any magic labeling of G all the terminal edges have the same label x, which is then equal to the vertex sum. The graph G is h-magic if and only if we can find nonzero elements $x, y_i \in \mathbb{Z}_h$ such that $l^+(u_i) = x$. This will provide a homogeneous system of n equations with n unknowns $y_1 + (a_1 - 2)x \equiv 0 \pmod h$ and $y_i + y_{i-1} + (a_i - 3)x \equiv 0 \pmod h$, that will result in

$$\sigma x \equiv 0 \pmod h, \tag{3.1}$$

$$y_i \equiv c_i x \pmod h \tag{3.2}$$

where $\sigma = \dfrac{(-1)^n - 1}{2} - \sum_{i=1}^{n}(-1)^i a_i$ and $c_i = \dfrac{3-(-1)^i}{2} - a_i + a_{i-1} - \cdots + (-1)^i a_1$.
We observe that if $\sigma | c_i$ for some i, then $y_i = 0$ and the graph would be nonmagic. In particular, if $c_i = 0$ for some i, then the graph is nonmagic. Assume that $c_i \neq 0$.
If $\sigma = 0$, then equation (3.1) is automatically satisfied. Choose $x = 1$ and note that equation (3.2) has $y_i = 0$ as its solution if and only if h be a divisor of c_i. Therefore, to avoid this solution we must exclude all the divisors of c_i $(1 \le i \le n-1)$. In this case, the interger-magic spectrum of the caterpillar would be $\mathbb{N} - C$.
Finally, suppose $\sigma \neq 0$ and $\sigma \nmid c_i$ for all $i = 1, \cdots, n-1$. We claim that $IM(G) = \bigcup_{d \in D} d\mathbb{N}$, where D is the set of all positive divisors d of σ with the property that $d \nmid c_i, \forall i = 1, \cdots, n-1$.
Suppose $h \in IM(G)$. Then equation (3.1) has a nonzero solution for x if and only if $\gcd(\sigma, h) = d > 1$, and h/d divides x. Also, $d \nmid c_i, \forall i$. Because, if $d | c_i$ for some $i = 1, \cdots, n-1$, then $d(h/d) | c_i x$ or $h | y_i$ and $y_i \equiv 0 \pmod h$. Therefore, $h = dk$, where $d \in D$.
On the other hand, let $h = dk$ with $d \in D$ and $k \in \mathbb{N}$. Note that $d \in D$ implies that $d > 1$. We choose $x = h/d \not\equiv 0 \pmod h$. Since $d \nmid c_i$, then $d(h/d) \nmid c_i x$ or $h \nmid y_i$ and $y_i \not\equiv 0 \pmod h$. Therefore, $h \in IM(G)$. □

Corollary 3.2. *Using the notations of theorem 3.1, $G = CR(a_1, a_2, \ldots, a_n)$ is nonmagic if and only if $\sigma | c_i$ for some $i = 1, 2, \cdots, n-1$.*

Double-stars are special cases of caterpillar whose spine is P_2. In fact, double-stars are trees of diameter 3 with two central vertices u and v plus leaves. Then as a corollary of theorem 3.1, for the integer-magic spectrum of double-stars $CR(m, n)$ we have:

Corollary 3.3. *Let C be the set of all positive divisors of $n-2$. Then*

$$IM(CR(m,n)) = \begin{cases} \emptyset & \text{if } (m-n)|(n-2); \\ I\!N - C & \text{if } m = n; \\ \bigcup_{d \in D} dI\!N & \text{otherwise,} \end{cases}$$

where D is the set of all positive divisors of $m-n$ that do not divide $n-2$.

Examples 3.4.

(a) $IM(CR(28,4)) = 4I\!N \cup 3I\!N - \{1,2\}$. Here, $m-n=24$, while $n-2=2$.
(b) $IM(CR(16,10)) = 3I\!N$. Here, $m-n=6$, while $n-2=8$.
(c) $IM(CR(10,21,17)) = \emptyset$. Here, $c_1=-8$, $c_2=-10$, $\sigma=5$, and $\sigma|c_2$.
(d) $IM(CR(12,9,6)) = 8I\!N$. Here, $c_1=-10$, $c_2=4$, and $\sigma=8$.
(e) $IM(CR(17,10,6)) = I\!N - \{1,2,3,4,5,8,15\}$. Here, $c_1=-15$, $c_2=8$, and $\sigma=0$.
(f) $IM(CR(5,6,8)) = \emptyset$. Here, $c_1=3$, $c_2=0$, and $\sigma=6$. The set of divisors of c_2 is $I\!N$.
(g) $IM(CR(7,5,14)) = 15I\!N$. Here, $c_1=-5$, $c_2=3$, and $\sigma=15$, and 15 is the only divisor of σ that does not divide c_1 and c_2.

References

[1] G. Bachman and E. Salehi, Non-Magic and K-Nonmagic Graphs, *Congressus Numerantium* **160** (2003), 97-108.
[2] J. Gallian, A Dynamic Survey in Graphs Labeling (ninth edition), *Electronic Journal of Combinatorics* (2005).
[3] S. Jezny and M. Trenkler, Characterization of Magic Graphs, *Czechoslovak Mathematical Journal* **33** (108), (1983), 435-438.
[4] S-M Lee, Alexander Nien-Tsu Lee, Hugo Sun, and Ixin Wen, On integer-magic spectra of graphs, JCMCC, **42** (2002), 177-185.
[5] S-M Lee, E. Salehi, Integer-Magic Spectra of Amalgamations of Stars and Cycles, *Ars Combinatoria* **67** (2003), 199-212.
[6] S-M Lee, E. Salehi, and H. Sun, Integer-Magic Spectra of Trees with Diameter at most Four, *Journal of Combinatorial Mathematics and Combinatorial Computing* **50** (2004), 3-15.
[7] S-M Lee and H. Wong, On Integer-Magic Spectra of Power of Paths, *Journal of Combinatorial Mathematics and Combinatorial Computing* **42** (2002), 187-194.
[8] R.M. Low and S-M Lee, On the Integer-Magic Spectra of Tessellation Graphs, *Australasian Journal of Combinatorics* **34** (2006), 195-210.
[9] E. Salehi, Integer-Magic Spectra of Cycle Related Graphs, *Iranian Journal of Mathematical Sciences and Informatics* **2** (2006), 53-63.
[10] E. Salehi and S-M Lee, Integer-Magic Spectra of Functional Extension of Graphs, to appear in the *Journal of Combinatorial Mathematics and Combinatorial Computing.*
[11] J. Sedlacek, On Magic Graphs, *Math. Slov.* **26** (1976), 329-335.
[12] J. Sedlacek, Some Properties of Magic Graphs, in Graphs, Hypergraph, *Bloc Syst. 1976, Proc. Symp. Comb. Anal. Zielona Gora* (1976), 247-253.
[13] R.P. Stanley, Linear Homogeneous Diophantine Equations and Magic Labelings of Graphs, *Duke Mathematics Journal* 40 (1973), 607-632.
[14] R.P. Stanley, Magic Labeling of Graphs, Symmetric Magic Squares, Systems of Parameters and Cohen-Macaulay Rings, *Duke Mathematics Journal* **40** (1976), 511-531.
[15] W.D. Wallis, Magic Graphs, *Birkhäuser Boston* 2001.