Superior Distance in Graphs

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Abstract

The main objective of this paper is to introduce a generalization of distance called superior distance in Graphs. For two vertices u and v of a connected graph, we define $D_{u,v} = N[u] \cup N[v]$. We define a $D_{u,v}$ - walk as a u-v walk that contains every vertex of $D_{u,v}$ .The Superior Distance $d_D(u,v)$ from u to v is the length of a shortest $D_{u,v}$ – walk. In this paper, first we give the bounds for the superior diameter of a graph and a property that relates the superior eccentricities of adjacent vertices. Finally we investigate those graphs that are isomorphic to the superior center of some connected graph and those graphs that are isomorphic to the superior periphery of some connected graph.

Key Words : superior distance, superior radius, superior diameter, superior center, superior periphery.

1 Introduction and Definitions

By a graph we mean a non-trivial finite undirected connected graph without loops and multiple edges. As usual V(G) denotes the set of vertices of a graph G, and E(G) denotes the set of edges of G. The distance between vertices *u* and *v* is the length of a shortest path in G between *u* and *v*. The *eccentricity* $e(u)$ of a vertex *u* is given by $e(u) = max\{ \ d(u,v): v \in V(G) \ \}$.The *radius* $r(G)$ and the *diameter* $d(G)$ are defined as follows : $r(G) = min \{ \ e(u): u \in V(G) \}$ and $d(G) =$

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$max\ \{e(u): u \in V(G)\}$. A vertex v of V(G) is called an *eccentric vertex* of a vertex u if $d(u,v) = e(u)$. *A* vertex v is called a *center vertex* if $e(v)$ = r(G)and a *peripheral vertex if* $e(v) = d(G)$. If C and P are the set of all center vertices and peripheral vertices respectively, then ‹C› is called the *center* $C(G)$ of G and ‹P› is called the *periphery* $P(G)$ of G.In [5], Santhakumaran discussed three generalizations of the radius and diameter namely (V,ξ) *- radius* r_1 ,(V,ξ) *- diameter* d_1, (ξV) *- radius* r_2, $(\xi\text{-}V)$ *- diameter* d_2, (ξ,ξ) *- radius* r_3, $(\xi\xi)$ *- diameter* d_3 of a connected graph G. He proved that for any connected graph $r_i \le d_i \le 2r_i + 1$ for i = 1,2,3. In [4] Parthasarathi and Nandakumar studied the properties of eccentric vertices of a graph. For general notation and terminology, we follow Harary[1,2].

If X and Y are two cities, then for a taxi driver the distance between the two cities is the actual distance between the two cities. However for a mobusal bus driver, the distance between the same cities is just higher than the usual distance since he has to cover some important places in and around the two cities to pick up and drop the passengers. So a mobusal bus driver has to find a shortest rout that begins from X and ends at Y and passes through each of the neighbouring places of X and Y.

In this paper we discuss a variation of distance that models the bus route just described. For a simple connected graph G and for two vertices u and v of G, let $D_{u,v} = N[u] \cup N[v]$. We define a $D_{u,v}$*- walk* as a *u-v walk* in G that contains every vertex of $D_{u,v}$.

The *superior distance* $d_D(u,v)$ from u to v is the length of a shortest $D_{u,v}$ *- walk.* For each vertex v of a simple connected graph G, we define the *superior eccentricity* of v as $e_D(v) = max\ \{ d_D(u,v): u \in V(G)\}$. A vertex v of a graph G is said to be a *superior eccentric vertex* of a vertex u if $d_D(u,v) = e_D(u)$. A vertex u *is superior eccentric vertex of G if it is a superior eccentric vertex of some vertex v.* It is interesting to note that the superior distance from u to itself is greater than zero. The superior distance in a simple connected graph G is a generalization of distance because $d_D(u,v) = d(u,v)$ *if* $N(u)$ *and* $N(v)$ are singleton sets. However, the superior distance is generally not a metric since $d_D(u,v) \neq 0$.

To illustrate the ideas presented above consider the graph $K_{1,3}$ with the end vertices labeled u,v and w and the fourth vertex labeled x.

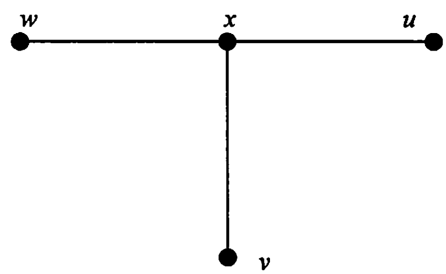


Figure 1

A shortest $D_{u,x}$ – walk is W : u,x,v,x,w,x. The length of W is 5 and so $d_D(u,x) = 5$. Similarly $d_D(v,x) = d_D(w,x) = 5$; $d_D(u,w) = d_D(w,v) = 2$; $d_D(x,x) = 6$.

2 Properties

The following observations will be used throughout this paper.

For $u, v, w \in V(G)$:

a) $d(u, v) \le d_D(u, v)$;

b) $d_D(u, v) > 0$; $d_D(u, v) = d_D(v, u)$;

c) $d_D(u, v) = d(u, v)$ *iff* $N(u)$ *and* $N(v)$ *are singleton sets;*

d) $d_D(u, v) \le d_D(u, w) + d_D(w, v) - d_D(w, w)$ *when* $N(u) \ne N(v)$

e) *Either if* $u = v$ *and* $\deg(u) > 1$ *or* $u \ne v$ *then* $d_D(u, v) \le d_D(u, u) + d_D(v, v) + d(u, v) - 4$ *and equality holds if G is a tree and* $u \ne v$;

f) $d_D(u,v) \le 2 \deg(u)$;

g) $e(u) \le e_D(u) - 1$;

h) *Superior eccentric vertex of a vertex u is itself iff* $\deg(u) = p - 1$

There are two particular values of the superior eccentricity. For a simple connected graph G, the *superior radius* $r_D(G)$ of G is defined by $r_D(G) = \min\{e_D(v) : v \in V(G)\}$ and the *superior diameter* $d_D(G)$ of G is defined by $d_D(G) = \max\{e_D(v) : v \in V(G)\}$. It is familiar that for a connected graph G and for an integer c such that $r(G) < c < d(G)$, there is a vertex v of G such that $e(v) = c$. But in the superior distance we have the following observation.

Observation 2.1 *For a connected graph G and for an integer c such that* $r_D(G) < c < d_D(G)$, *there may not exists a vertex v of G such that* $e_D(v) = c$.

The superior eccentricity of vertices of a tree is given below:

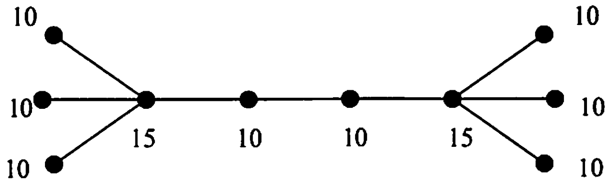


Figure 2

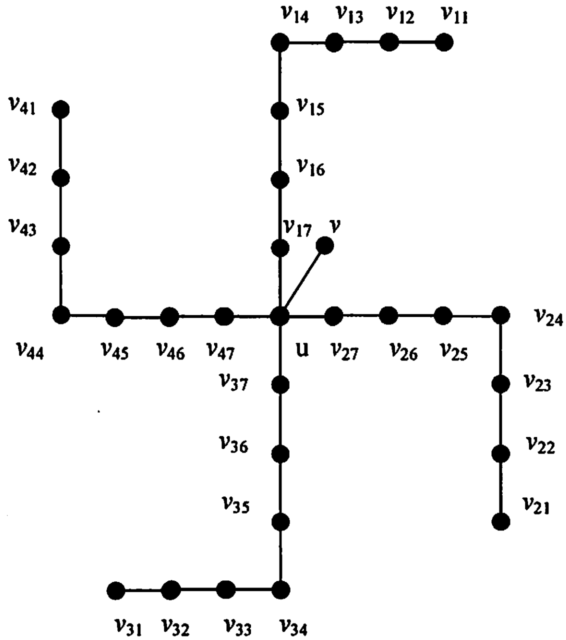
In this example, there does not exist a vertex v of G such that $e_D(v) = 12$

Theorem 2.2 *For a connected graph G, the superior radius satisfies the inequality* $r_D(G) \le d_D(G) \le 2r_D(G) - 2$.

Proof. The first inequality follows from the definitions. For the second inequality, let u and v be the vertices of G such that $d_D(u,v) = d_D(G)$ and let $w \in V(G)$ such that $e_D(w) = r_D(G)$. By observation *(d)*, $d_D(G) = d_D(u,v) \leq d_D(u,w) + d_D(w,v) - d_D(w,w)$. It is obvious that $d_D(w,w) \geq 2$. Thus we obtain $d_D(u,v) \leq e_D(w) + e_D(w) - 2$ and hence $r_D(G) \leq d_D(G) \leq 2\, r_D(G) - 2$.

We now consider the sharpness of the bounds of the theorem 2.2.

Define an n-longated star $K_n^*(1,m)$, $n \geq 2m - 1$, formed by subdividing $n - 1$ times each edge of $K_{1,m}$ and introduce a new vertex v and join it with the vertex u (say) of degree m



$K_7^*(1,4)$
Figure 3

For $m \geq 2$, $n \geq 2m - 1$, we define $G_m \cong K_n^*(1,m)$. Label the vertices of G_m by $v_{11}, v_{12}, \ldots, v_{1n}, v_{21}, v_{22}, \ldots, v_{2n}, \ldots, v_{m1}, v_{m2}, \ldots, v_{mn}$ other than u and v as shown in fig.3. $v_{11}, v_{21}, \ldots, v_{m1}, v$ are the end vertices of G_m. Set $E_m = \{ v_{i2} : i = 1,2,\ldots, m\}$. E_m is the set consisting of all superior eccentric vertices of G_m. From the above construction we get the following relations:

$e_D(v_{i1}) = 2n+1$ for $i = 1,2,\ldots, m$, $e_D(v) = r_D(G_m) = n+2, e_D(u) = 2m+n+1$.

$e_D(v_{ij}) = 2n + 4 - j$ for $i = 2,3,4,\ldots,m$ and $j = 2,3,\ldots, m$.

Now $d_D(G_m) = 2n+2 = 2(n+2) - 2 = 2\, r_D(G_m) - 2$

Therefore, for m ≥2, n ≥ 2m -1, the class $\mathscr{F}_1 = \{G_m : m \geq 2,\ n \geq 2m - 1\}$ is an infinite class of graphs that verifies the sharpness of the upper bound in theorem 2.2.

Consider the set $\mathscr{F}_2$ of all cycles C_m on m vertices, $m \geq 3$. $r_D(C_3) = d_D(C_3) = 3$, $r_D(C_4) = d_D(C_4) = 4$, $r_D(C_5) = d_D(C_5) = 5$. The superior radius and superior diameter of cycles C_m, $m > 5$ are given by

$$r_D(C_m) = d_D(Cm) = \begin{cases} m/2 + 4 & \text{if } m \text{ is even} \\ (m-1)/2 + 4 & \text{if } m \text{ is odd} \end{cases}$$

Hence we have another infinite class of graphs $\mathscr{F}_2 = \{C_m : m \geq 3\}$ that verifies the sharpness of the lower bound in theorem 2.2.

A useful property that relates the superior eccentricities of adjacent vertices in a graph is given in the next corollary.

Theorem 2.3 *If G is a connected graph of ordern* $p \geq 3$ *and* $u,v \in V(G)$ *with* $e_D(v) \leq e_D(u)$, *then* $e_D(u) - e_D(v) \leq d_D(u,u) + d(u,v) - 4$.

Proof. There exists $u_1 \in V(G)$ such that $d_D(u,u_1) = e_D(u)$.
Since $d_D(u,u_1) \leq d_D(u,v) + d_D(v,u_1) - d_D(v,v)$, $e_D(u) \leq d_D(u,v) + d_D(v,u_1) - d_D(v,v)$.
This implies that $e_D(u) - e_D(v) \leq d_D(u,v) - d_D(v,v) \leq d_D(u,u) + d_D(v,v) + d(u,v) - 4 - d_D(v,v)$ by observation *(e)*. Thus we obtain $e_D(u) - e_D(v) \leq d_D(u,u) + d(u,v) - 4$

Corollary 2.4 *If G is a connected graph of order* $p \geq 3$ *and* $uv \in E(G)$ *with* $e_D(v) \leq e_D(u)$ *then* $e_D(u) - e_D(v) \leq 2\deg(u) - 3$.

Proof. By the above theorem 2.3,

We obtain $e_D(u) - e_D(v) \leq d_D(u,u) + d(u,v) - 4$
$= d_D(u,u) - 3$
$\leq 2\deg(u) - 3$ *by observation (f)*

Next we give a definition on trees and give some propositions related with this definition.

Let T be any double star. Then there are two vertices u and v such that each pendant vertex is adjacent with either u or v. For a given double star T, define the T_n, $n \geq 0$, by subdividing uv 'n' times. It is obvious that T_0 is the given double star.

Proposition 2.5 *In* T_n, $d_D(u,v) = 2q(T_n) - d(u,v)$.

Proof. Let W be the $D_{u,v}$ – *walk* and let P be the unique u-v path in T_n. Since W is a $D_{u,v}$ - walk, it contains every edge of P. Every edge incident with a pendant vertex will be counted twice in W. Therefore

$$2q(T_n) = d_D(u,u) + d_D(v,v) + 2[d(u,v) - 2]$$
$$= d_D(u,v) + d(u,v) \text{ by observation (e)}$$

Hence $d_D(u,v) = 2q(T_n) - d(u,v)$

Corollary 2.6 *If* $T_0 \cong K_{1,m}$ *,* $m \geq 1$*, then* $e_D(u) = 2q(T_0)$ where u is the center vertex and $e_D(w) = 2q(T_0) - 1$ for every pendant vertex w of T_0.

proposition 2.7 *For any vertex w of* T_n*,* $e_D(w) \leq 2(2\Delta + r - 3)$ *where r is the radius of* T_n.

Proof. Let w be a pendant vertex of T_n. Then superior eccentric vertex w_1 of w is either u or v. Observations (e) and (f) give us

$$d_D(w,w_1) = d_D(w,w) + d_D(w_1,w_1) + d(w,w_1) - 4$$
$$< 2\, deg(w) + 2\, deg(w_1) + 2r - 4$$
$$< 2\Delta + 2\Delta + 2r - 6, \text{ since w is a pendant vertex.}$$
$$= 2\,(2\Delta + r - 3)$$

Let w be a vertex of T_n which is not a pendant and $w \neq u \neq v$. Then superior eccentric vertex w_1 of w is either u or v. By observations (e) and (f) we obtain the following relation.

$$d_D(w,w_1) < 2\,deg(w) + 2deg(w_1) + 2r - 6$$
$$< 2\,\Delta + 2\Delta + 2r - 6$$
$$= 2(2\Delta + r - 3)$$

If $w = u$, then the superior eccentric vertex w_1 of w is either v *or* u *itself*. Then observations (e) and (f) give the following relation: if $w_1 = v$, then

$$d_D(u,v) = 2\,deg(u) + 2deg(v) + d(u,v) - 4$$
$$\leq 2\Delta + 2\,\Delta + (2r - 2) - 4$$

If $w_1 = u$ then obviously $d_D(u,u) < 2(2\Delta + r - 3)$

If $w = v$, then the superior eccentric vertex w_1 of w is either u or v itself. The discussion given in the above case will be used to obtain the conclusion, $d_D(v,u) \leq 2(2\Delta + r - 3)$. Thus in all the cases we have proved that $d_D(w,w_1) \leq 2(2\Delta + r - 3)$ for any vertex w of T_n where w_1 is a superior eccentric vertex of w.

3 The Superior Center and Superior Periphery of a Graph

Having defined $r_D(G)$ and $d_D(G)$ as extension of $r(G)$ and $d(G)$ for a graph G, it is natural to define extensions of the center $C(G) = \langle \{v \in V(G) : e(v) = r(G)\} \rangle$ and the periphery $P(G) = \langle \{v \in V(G) : e(v) = d(G)\} \rangle$. For a graph G, the *superior center* of G is defined by $C_D(G) = \langle \{v : e_D(v) = r_D(G)\} \rangle$ and the *superior periphery* is defined by $P_D(G) = \langle \{ v \in V(G) : e_D(v) = d_D(G)\} \rangle$.

We first investigate those graphs that are isomorphic to the superior center of some connected graph. Let G be a graph of order p. We define a graph G^* from G by $G^* = G \vee K_2$ and let the vertices of K_2 be u and v.

Theorem 3.1 *For a graph G there exists a connected graph H such that* $C_D(H) \cong G$ *if any one of the following conditions hold.*

1. For each $w \in V(G)$*,* G^* *contains a* $w - u$ *path which passes through v and*

passes through all the vertices of G; and for each $w \in V(G)$, G^* contains a $w-v$ path which passes through u and passes through all the vertices of G.
2. On each $u-v$ walk, each vertex of G lies exactly once and either u or v appear two times.

Proof. Suppose the condition 1 holds in G. Construct the graph H from the graph G^* by adding two vertices x and y and introducing the edges ux and xy. From this construction we observe that $\Delta(H) = deg(u) = deg(v) = p + 1$. The superior eccentric vertex of each vertex of H (except u and v) is either u or v. The superior eccentric vertex of u is v and that of v is u. By the assumption a $D_{x,v}$ – walk will be as follows : $x, u, w_1, w_2, \dots, w_p, v, y, v$; where $w_i \in V(G)$ for all i = 1, 2, ..., p. This is a shortest walk between x and v. The length of this walk is $p + 4$. Thus $e_D(x) = p + 4$. Similarly $e_D(y) = p + 4$. Now consider the $D_{u,v}$ – walk : $u, x, u, w_1, w_2 \dots, w_p, v, y, v$. The length of this shortest walk is $p + 5$. Thus $e_D(u) = e_D(v) = p + 5$.

Consider a vertex w_i of G for any i. u and v are adjacent to all the vertices w_i of G and hence the superior eccentric vertex of each w_i is either u or v. $w_i, u, w_1, w_2, \dots, w_{i-1}, w_{i+1}, \dots, w_p, v, y, v$ is a shortest $D_{wi,v}$-walk with length $p+3$ and hence $e_D(w_i) = p+3$ *for all* i = 1, 2, ..., p. The superior center vertices of H are the vertices of G. So we conclude that there exists a connected graph H such that $C_D(H) \cong G$.

Suppose condition 2 holds. By this assumption we can get $e_D(w_i) = p + 3$ for all i = 1, 2, ..., p and $e_D(x) = e_D(y) = p + 5$, $e_D(u) = e_D(v) = p + 6$ and hence $C_D(H) \cong G$.

H :

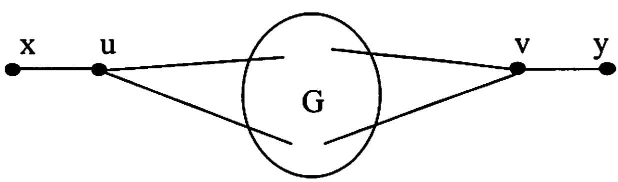


Figure 4

We now turn to superior periphery of a graph. The graphs G those are isomorphic to the periphery of a connected graph H are characterized next.

Theorem 3.2 *Let G be a graph of order p. Then there exists a connected graph H such that $P_D(H) \cong G$ if $\Delta(G) = p - 1$ and in G^*, for every pair of vertices u, v of G there is a $u-v$ path containing all the vertices of G.*

Proof. Construct the graph H from the graph G^* as follows:

(i) add two vertices x and y and introduce the edges ux and vy.

(ii) For each vertex w_i of G, i = 1, 2, ..., p, introduce two new vertices w_{i1}, w_{i2} and join them with w_i.

Superior eccentric vertex of each vertex of H is any one of the vertices of degree Δ of G in G^*. Consider any vertex w_i of degree Δ in G. x, u, w_1, w_2, ..., w_{i-1}, w_{i+1}, ..., v, w_i, w_{i1}, w_{i2}, w_i is a shortest $D_{x,wi}$ – walk. The length of this shortest walk is $p + 6$. Thus we obtain $e_D(x) = p + 6$. Similarly $e_D(w_{i1}) = e_D(w_{i2}) = p + 6$. Also it is obvious that $e_D(y) = p + 6$.

We use the same method to find the superior eccentricity of vertices of G. Consider any vertex w_i of G in G^*. w_i, w_{i1}, w_{i2}, w_i, u, w_1, w_2, ..., w_{i-1}, w_{i+1}, ... w_j, w_{j1}, w_{j2}, w_j *is a shortest* $D_{wi,wj}$ -walk with length $p+9$. Thus $e_D(w_i) = p + 9$ for all i. Also it is easy to find that $e_D(u) = e_D(v) = p + 7$. Thus there exists a connected graph H such that $P_D(H) \cong G$.

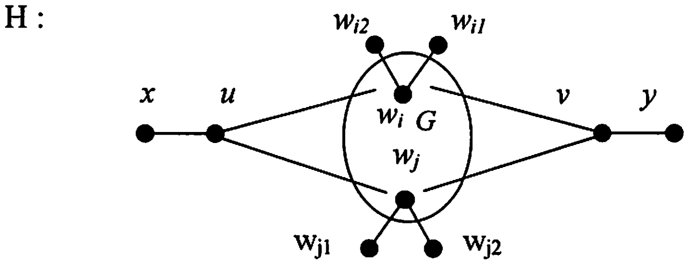


Figure 5

Open Problem :Characterize graphs G for which $d_D(G) = 2r_D(G) - 2$.

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