Classes of Hamilton Cycles in the 5-Cube

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Abstract

A Hamilton cycle in an n-cube is said to be k-warped if its k-paths have their edges running along different parallel 1-factors. No Hamilton cycle in the n-cube can be n-warped. The equivalence classes of Hamilton cycles in the 5-cube are represented by the circuits associated to their corresponding minimum change-number sequences, or minimum H-circuits. This makes feasible an exhaustive search of such Hamilton cycles allowing their classification according to class cardinalities, distribution of change numbers, duplicity, reversibility and k-warped representability, for different values of $k < n$. This classification boils down to a detailed enumeration of a total of 237675 equivalence classes of Hamilton cycles in the 5-cube, exactly four of which do not traverse any sub-cube. One of these four classes is the unique class of 4-warped Hamilton cycles in the 5-cube. In contrast, there is no 5-warped Hamilton cycle in the 6-cube. On the other hand, there is exactly one class of Hamilton cycles in the graph of middle levels of the 5-cube. A representative of this class possesses an elegant geometrical and symmetrical disposition inside the 5-cube.

1 Introduction

Given a positive integer n, the *n-cube* Q_n is defined as the graph whose vertex set is $\{0,1\}^n$ and whose edge set is formed by the pairs of vertices $(x_0,\ldots,x_{n-1})$, $(y_0,\ldots,y_{n-1})$ that differ in just one coordinate $i \in \{0,1,\ldots,n-1\}$, ($x_i \neq y_i$, but $x_j = y_j$ for $j \neq i$). A *Hamilton cycle*, (or H-*cycle*), of a finite connected graph G is a cycle whose length coincides with the vertex-set cardinality of G. If $G = Q_n$, this cardinality is 2^n. If we want to distinguish between the two orientations of an H-cycle, then the two resulting objects are called *Hamilton circuits*, (or H-*circuits*); see

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[1]. An H-cycle and its reverse cycle are the same object, but an H-circuit differs from its reverse H-circuit. This means that an evaluation of the number of H-circuits in an n-cube must double that of H-cycles.

It is easy to see that there is just one equivalence class of H-cycles in Q_2 and also in Q_3. In [1] and in [8, page 226], it is cited that there are 9 equivalence classes of H-cycles in Q_4. In the present paper, we deal with the following question.

Question 1 *How many equivalence classes of Hamilton cycles exist in Q_5 and how can they be classified according to class cardinality, distribution of change numbers, duplicity, reversibility and representability into having their k-paths running along different parallel 1-factors (represented by different change numbers), where $k \leq n$?*

We find, in Theorem 2 and its Corollary 3 of Section 4 and in tables presented in Section 7, answers to Question 1, with a total of 237,675 equivalence classes of Hamilton cycles in the 5-cube. Recently, we learned that this total is mentioned, still without a proof, by D. Knuth in [11, page 49], who encouraged us to present the present work.

Our results lead as well, via counting the number of elements in each equivalence class, to the total number of H-circuits of Q_5. This number happens to be exactly twice the number 906545760 of H-cycles of Q_5, claimed originally in [4], according to [2, page 162]; also cited in [6, 9, 14].

We also relate with the work of W. H. Mills in [12], showing in Corollary 4, Section 5, after having defined some elementary tools, that the Hamilton cycles in exactly four of the mentioned equivalence classes do not traverse any r-sub-cube of Q_5, for $1 < r < 5$, allowing as well to pose a general Question 5 relating our mentioned tools to Hamilton cycles in Q_n that do not traverse any r-sub-cube, for $1 < r < n$.

The approach taken to obtain the results, via minimum change-number sequences, seems absent in the literature, so we present it in Section 6. Section 7 presents the distribution of the equivalence classes of H-cycles in Q_5 with respect to invariants defined in Section 2.

Variations of our approach allowed, in Sections 8 and 9, to show that there is no H-cycle in Q_6 of a type existing in Q_5, and that there is just one equivalence class of H-cycles in the middle-levels of Q_5, ([13]), that we accompany with some comments on its associated symmetry.

2 Minimum Change-Number Sequences

Let $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ stand for the set of coordinate directions of Q_n. The edge set of Q_n splits into n 1-factors $f_0, f_1, \ldots, f_{n-1}$, each containing

2^{n-1} parallel edges. In fact, if $i \in \mathbb{Z}_n$, then the edges of f_i are parallel along the coordinate direction i.

Let us denote each parallel 1-factor f_i just by the corresponding coordinate direction $i \in \mathbb{Z}_n$. An H-circuit C of Q_n is completely determined by the sequence $s(C)$ of these coordinate directions, for the successive edges of C, starting from the null vertex 0. In fact, let $p^0, p^1, \ldots, p^{2^n-1}$ be the consecutive vertices of an H-circuit C in Q_n, where $p^0 = 0$ and p^{2^n-1} is adjacent to ^{0}. Any two consecutive vertices p^j and p^{j+1}, where subindices $j \pm 1$ are taken mod 2^n, differ in exactly one component, say $p^j_k \neq p^{j+1}_k$, with $p^j_\ell = p^{j+1}_\ell$, for $\ell \in \mathbb{Z}_n \setminus \{k\}$. Then, following [12] we say that $k = k_j$ is the j-th *change number* of C. But k_j coincides with some coordinate direction $i \in \mathbb{Z}_n$ with $k_{j-1} \neq i \neq k_{j+1}$, (justifying the name of change number, because an H-circuit cannot have contiguous edges in a common parallel 1-factor f_i). Accordingly, we say that the sequence $s(C) = k_0 k_1 \ldots k_{2^n-1}$ is the *change-number sequence*, (or *CNS*), of C. Also, observe that $s(C)$ can be identified with the sequence of 1-factors its component coordinate directions represent, so we could say that $s(C)$ is a *change 1-factor sequence* of C.

There is just one H-cycle in Q_2, represented by the CNS $s(C_2) = 0101$ of an H-circuit C_2. Also, given an H-circuit C_{n-1} in Q_{n-1}, for $2 < n \in \mathbb{Z}$, an H-circuit C_n can be constructed from C_{n-1} by interspersing the new coordinate direction $n-1$ of Q_n (which is not a direction of Q_{n-1}) between each two change numbers present in $s(C_{n-1})$. This way, from $s(C_2) = 0101$, we get $s(C_3) = s^2(C_2) = 02120212$, $s(C_4) = s^2(C_3) = s^3(C_2) = 03231323032313, \ldots, s(C_n) = s^2(C_{n-1}) = \ldots = s^{n-1}(C_2) = \ldots$.

By considering lexicographically the coordinate directions $0, 1, \ldots, n-1$ in their ascending order, it is seen that there is exactly one *minimum CNS*, (or *MCNS*), of an H-circuit in each equivalence class of H-circuits in Q_n.

There is exactly one equivalence class of H-circuits in Q_3, represented by $s(C_3)$, with MCNS $s'(C_3) = 01020102$.

3 Case $n = 4$ and Some Invariants

Let $A_n = |Aut(Q_n)|/2^{n-2}$. We will need $A_4 = 96$ and $A_5 = 480$. Given a MCNS s, let $\mathcal{E}(s)$ be the cardinality of the equivalence class of H-circuits that s represents, and let

(a) $p =$ distribution of change numbers of s in non-increasing order;

(b) $\xi = \log_2(\mathcal{E}(s)/A_n)$;

(c) $q = y(es)$, if $s = t^2$ and length$(t) = 2^{n-1}$; $q = n(ot)$, otherwise;

(d) $r = 1$, if s^{-1} is equivalent to s; $r = 0$, otherwise.

Properties **(c)** and **(d)** may be referred as the *duplicity* and the *reversibility* of s. We use the 9 classes of H-cycles in Q_4 cited in [1] and [8, page 226] to exemplify items **(a)**-**(d)**. When arranging lexicographically the MCNS of H-circuits associated to the 9 classes, from left to right and from top to bottom, starting from $s^2(s'(C_3))$ and accompanied by the expressions (p, ξ, q, r), we get

010201030102010З	$(4211, 1, y, 1)$	0102010302120213	$(3221, 4, n, 0)$
0102010310121013	$(3311, 2, n, 1)$	0102013102010232	$(3221, 2, n, 1)$
0102013201020132	$(3221, 2, y, 1)$	0102101301021013	$(3311, 3, n, 1)$
0102101302012023	$(3221, 3, n, 1)$	0102030130321013	$(3221, 1, y, 1)$
0102032123031213	$(2222, 3, n, 0)$		

Counting multiples of $A_4 = 96$, we get a total of 2688 H-circuits in Q_4, or $2688/2 = 1344$ H-cycles, as cited in [2, 5]. Also note that the two of those 9 classes having $r = 0$ correspond to 4 classes of H-circuits in Q_4, while the remaining 7 classes of H-cycles, with $r = 1$, coincide with whole classes of H-circuits. Thus, there are 11 classes of H-circuits in Q_4 corresponding to the 9 classes of H-cycles.

To apply the invariant (p, ξ, q, r) to the case $n = 5$ later on, we need to say now the following. The list of 9 CNS's and associated (p, ξ, q, r)'s, for $n = 4$, was obtained

1. via an exhaustive search that produced 37 CNS's,
2. then reducing the 37 resulting sequences to the minimum sequences in the corresponding equivalence classes of H-circuits,
3. and finally deleting, from the resulting 37-term list, the repeated sequences.

This leaves us with the 9 classes in question.

4 Counting Results via Warped Paths

We show in what follows that the list approach essayed for $n = 4$ works for $n = 5$ as well, provided that the H-circuits are divided into three subsets, according to the following considerations.

The H-circuit of a MCNS in Q_n is called a *minimum H-circuit.* Given a positive integer $k \leq n$, an H-cycle or H-circuit C of Q_n is said to be k-warped if the k-paths of C have their edges running along different parallel 1-factors (of $\{f_1, \ldots, f_n\}$).

Let $C'_{n,k} = \{k\text{-warped minimum H-circuits of } Q_n\}$. Let $C_{n,k} \subseteq C'_{n,k}$ be the subset of $C'_{n,k}$ in which each H-cycle of Q_n represented by an H-circuit of $C'_{n,k}$ is present just once, exactly by the least of the two H-circuits it stands for.

While $|C'_{2,2}| = |C_{2,2}| = 1$, it is not difficult to see that $|C'_{n,n}| = |C_{n,n}| = 0$ for $n > 2$, because no Hamilton cycle of Q_n is n-warped. Also, from what was presented above, it is seen that $|C'_{3,2}| = |C_{3,2}| = 1$, $|C'_{4,2}| = 11$ and $|C_{4,2}| = 9$, but $|C'_{4,3}| = |C_{4,3}| = 0$.

Theorem 2 *The cardinalities of the nonzero $C_{5,i}$'s and $C'_{5,i}$'s are:*

(A) $|C'_{5,4}| = |C_{5,4}| = 1$;

(B) $|C'_{5,3}| = |C_{5,3}| = 3$;

(C) $|C'_{5,2}| = 473037$ *and* $|C_{5,2}| = 237671$.

In fact, there are exactly $z_1 = 2313$ members of $C_{5,2}$ with $r = 1$, so that

$$|C'_{5,2}| = z_1 + 2(|C_{5,2}| - z_1).$$

Corollary 3 *There are exactly 237675 equivalence classes of H-cycles in Q_5. They are distributed as follows:*

$c_0 =$	2	*classes of* $h_0 =$	480	*H-circuits; total*	960;
$c_1 =$	16	*classes of* $h_1 =$	960	*H-circuits; total*	15360;
$c_2 =$	90	*classes of* $h_2 =$	1920	*H-circuits; total*	172800;
$c_3 =$	3024	*classes of* $h_3 =$	3840	*H-circuits; total*	11612160;
$c_4 =$	234543	*classes of* $h_4 =$	7680	*H-circuits; total*	1801290240.

This amounts to 1813091520 *H-circuits in* Q_5.

To corroborate our calculations, notice that the number of H-circuits in Corollary 3 equals exactly twice the 906545760 H-cycles mentioned in the Introduction.

In the five cardinality categories in which the classes of H-cycles were divided in Corollary 3, there are respectively $d_0 = 2$, $d_1 = 15$, $d_2 = 80$, $d_3 = 2216$ and $d_4 = 0$ classes with $r = 1$, totalling the $z_1 = 2313$ classes in Theorem 2. Let c'_j = number of classes of H-cycles corresponding to the c_j classes of h_j H-circuits in Corollary 3, for $j = 0, 1, 2, 3, 4$. The equation at the end of Theorem 2 can be subdivided into equations $c'_j = d_j + 2(c_j - d_j)$, namely

$$\begin{array}{rrlrlrll} c'_0 = & 2 & + \; 2(& 2 & - & 2) & = & 2, \\ c'_1 = & 15 & + \; 2(& 16 & - & 15) & = & 17, \\ c'_2 = & 80 & + \; 2(& 90 & - & 80) & = & 100, \\ c'_3 = & 2216 & + \; 2(& 3024 & - & 2216) & = & 3832, \\ c'_4 = & 0 & + \; 2(& 234543 & - & 0) & = & 469086. \end{array}$$

This allows to verify that $|C'_{5,2}| = 473037 = c'_0 + c'_1 + c'_2 + c'_3 + c'_4$.

5 Relation with the Work of W. H. Mills

We indicate the proofs of Theorem 2 and Corollary 3. We start by relating with a concept of W. H. Mills in [12].

The four classes representing $|C_{5,3}| = 3$ and $|C_{5,4}| = 1$ have respective associated MCNS's and expressions (p, ξ, q, r) as follows:

$$\begin{array}{ll} 01201320420310324034012431042134 & (43333, 3, n, 1) \\ 01201320432431240120132043243124 & (43333, 1, y, 1) \\ 01201321342340240120132134234024 & (43333, 0, y, 1) \\ 01230124032103240123012403210324 & (43332, 1, y, 1) \end{array}$$

where ξ uses $A_5 = 480$. Thus, these four classes represent altogether 6240 H-cycles, and 6240 H-circuits of Q_5, because $r = 1$ for each of them.

An H-cycle C of a connected graph G is said *to traverse a subgraph* H of G if the vertices of H span a sub-path of C. An H-cycle of Q_n is a *Mills cycle*, (or *M-cycle*), if it does not traverse any r-sub-cube of Q_n, for $1 < r < n$. Mills commented in [12] that all H-cycles of Q_4 traverse squares and that there is just one equivalence class of H-cycles in Q_4 not traversing any 3-sub-cube, represented by the last CNS in the 9-list above. He also showed that the second cycle representing $C_{5,3} = 3$ above, with $(p, \xi, q, r) = (43333, 0, y, 1)$, is an M-cycle. In fact, the four cycles realizing both $|C_{5,3}| = 3$ and $|C_{5,4}| = 1$, represented by the 4 CNS's displayed above, are M-cycles. Moreover, [12] made explicit at least one M-cycle in Q_n, for each positive integer n, and we observe that this cycle is in $C_{n,3}$. We distinguish these facts as follows.

Corollary 4 *The four cycles in Q_5 realizing $|C_{5,3}| = 3$ and $|C_{5,4}| = 1$, represented in the 4-list above, are M-cycles. Moreover, the M-cycle constructed by Mills in Q_n is in $C_{n,3}$.*

Comparing these facts with our tools, a question can be made:

Question 5 *For every integer $n > 5$, is every member of $C_{n,r}$ with $r > 2$ an M-cycle?*

6 Details of Approach for Case $n = 5$

Now we will describe the procedures that allowed the computations involved. Every representative of **(A)** $C_{5,2}$, **(B)** $C_{5,3}$ and **(C)** $C_{5,4}$ is given by some MCNS starting respectively with **(A)** 0102..., **(B)** 0120... and **(C)** 0123.... In each of these three instances, we use an exhaustive recursion search via backtracking, in order to determine all H-circuits of Q_5.

Let each vertex $(x_0, x_1, x_2, x_3, x_4)$ of Q_5 be denoted by the integer

$$x = x_0 + 2x_1 + 4x_2 + 8x_3 + 16x_4.$$

Given a vertex x of Q_5 and a coordinate direction $d \in \{0,1,2,3,4\}$, let $f(x,d)$ be the end-vertex of an edge along d whose other end-vertex is x. In the selected programming language (TurboPascal), we consider the function f as an array. Additionally, we introduce arrays c, v, m defined from the set $\{0,\ldots,32 = 2^5\}$ onto the integers and initialized by $c(i) = 5, m(i) = 0, v(i) = -1$, for $i = 0,\ldots,32$, in order to construct recursively CNS's, step by step, through augmenting partial CNS's, as follows.

1. Let $c(i)$ be the i-th change number of a partial CNS s,
2. let $v(i)$ be the second end-vertex of the edge realizing $c(i)$ in a partial H-circuit candidate S realizing s and departing from the vertex 0, and
3. let $m(v(i)) = 1$.

Initialization also includes, in each one of our three instances:

(A)

	$c(0) = 0$,	$c(1) = 1$,	$c(2) = 0$,	$c(3) = 2$,	so
$v(0) = 0$,	$v(1) = 1$,	$v(2) = 3$,	$v(3) = 2$,	$v(4) = 6$,	and
$m(0) = 1$,	$m(1) = 1$,	$m(3) = 1$,	$m(2) = 1$,	$m(6) = 1$;	

(B)

	$c(0) = 0$,	$c(1) = 1$	$c(2) = 2$,	$c(3) = 0$,	so
$v(0) = 0$,	$v(1) = 1$,	$v(2) = 3$	$v(3) = 7$,	$v(4) = 6$,	and
$m(0) = 1$,	$m(1) = 1$,	$m(3) = 1$	$m(7) = 1$,	$m(6) = 1$;	

(C)

	$c(0) = 0$,	$c(1) = 1$	$c(2) = 2$,	$c(3) = 3$,	so
$v(0) = 0$,	$v(1) = 1$,	$v(2) = 3$	$v(3) = 7$,	$v(4) = 15$,	and
$m(0) = 1$,	$m(1) = 1$,	$m(3) = 1$	$m(7) = 1$,	$m(15) = 1$.	

The recursive step, denoted STEP(j, c, v, m), consists in essaying $c(j-1) = i$, for each $i = 0,1,2,3,4$ different from:

(A) $c(j-2)$;

(B) $c(j-2)$ and $c(j-3)$;

(C) $c(j-2)$, $c(j-3)$ and $c(j-4)$.

Let $k = f(v(j-1), i)$ be a new candidate terminal vertex for S. After checking that k is not yet in S, we set $v(j) = k$, $m(k) = 1$ and if $j < 2^5 - 1 = 31$ then $j := j+1$ and apply (recursively) STEP(j, c, v, m).

In case of returning (to the previous stage of STEP(j, c, v, m)), the running must clearly be stepped back to $j := j-1$. However, if $j = 31$, then, in case that $v(j) = 1, 2, 4, 8, 16$, that is if $v(j)$ is in the set of neighbors of the first vertex, 0, of S, then the final change number of s is clearly set respectively equal to $0, 1, 2, 3, 4$.

Each time this final goal of an instance of STEP(j, c, v, m) is attained, the resulting s is written into a line of the text file created to the purpose of keeping the exhaustive results and the running is sent back to the previous stage of STEP(j, c, v, m).

This way, we get three different text files, LIST(**I**), one per each of the cases **I** = **A**, **B**, **C** considered. The number of lines of each of the resulting files is: (**A**) 7635740; (**B**) 24; (**C**) 8.

Next, we consider in turn each of the lines of LIST(**I**), which represents a CNS s. There are $2|Aut(Q_n)| = 7680$ transformations of s into equivalent sequences (that is, representing the same H-cycle), not necessarily all distinct. They are obtained by

1. switching $s = s^0 = (s_0, s_1, \dots, s_{31})$ successively to
$$s^1 = (s_1, s_2, \dots, s_{31}, s_0), \dots, s^{31} = (s_{31}, s_0, \dots, s_{30});$$
2. applying the permutations π of the set $\{0, 1, 2, 3, 4\}$ to each s^i in item 1 above, thus obtaining a CNS $\pi(s^i)$, for $i = 0, \dots, 31$;
3. considering both $\pi(s^i)$ and the reverse sequence $\overline{\pi(s^i)}$.

Considering the natural order of $\{0, 1, 2, 3, 4\}$, the 7680 CNS's described for s have a lexicographic order, from which we select the minimum one, denoted by $min(s)$. Accordingly, each LIST(**I**) is transformed into a corresponding file MIN(**I**) of MCNS's $min(s)$, for **I**=**A**, **B**, **C**.

In order to get the exact number of (minimum representatives of) equivalence classes of CNS's, or of their corresponding H-cycles, repeated elements must be eliminated from MIN(**I**) . This task is done readily for cases (**B**) and (**C**) because of the small cardinalities of MIN(**B**) and MIN(**C**), yielding corresponding adjusted, or fitted, files FIT(**B**) and FIT(**C**), which contain respectively the three MCNS's making up $C_{5,3}$ and the one making up $C_{5,4}$, summing up the four sequences in Section 5.

However, MIN(**A**) must be split successively into a collection of smaller files, first by classifying the component MCNS's $s = s_0 s_1 \dots s_{31}$ according successively to their different values in the 5-th, 6-th, ..., 14-th coordinate if needed, until files of sizes less that 2^{16} are obtained. This produces 246 files that, because of the classification method, are ordered, say from file $F(1)$ up to file $F(246)$, so that each sequence of $F(i)$ is lexicographically previous to (or less than) each sequence of $F(j)$ if and only if $i < j$, for $1 \le i, j \le 246$.

Specifically, the mentioned subdivision of MIN(**A**) proceeds according to whether the 5-th coordinate s_4 of any of its MCNS's equals either 0 or

1 or 3, (because $s_0s_1s_2s_3 = 0102$ must be followed by a change number $c_4 \notin \{2,4\}$, since s starts with 0102, so $c_4 \neq 2$, and 4 cannot happen as an entry of s if 3 does not appear somewhere to its left). This splits MIN(**A**) into corresponding files $\text{MIN}_0(\mathbf{A})$, $\text{MIN}_1(\mathbf{A})$ and $\text{MIN}_3(\mathbf{A})$.

Each of the latter three files can then be split according to the values of the 6-th coordinate, yielding a finite collection of nonempty files $\text{MIN}_{a_4,a_5}(\mathbf{A})$, where $a_4 = 0,1,3$ and $a_5 \neq a_4$. Subsequently, the 7-th coordinate can be considered to refine the last stage of the subdivision into smaller files, and so on. However, when one of these split files reaches $\leq 2^{16}$, no further subdivision of it is taken, and the file is kept waiting in a list of files until a total subdivision of MIN(**A**) into files with $\leq 2^{16}$ lines is obtained. When this list is reached, it is constituted by the 246 mentioned files.

We rename these 246 files as $F(1), F(2), \dots, F(246)$ in the way specified above and apply to each one of them a procedure that deletes repeated elements, leaving one representative of a MCNS for each equivalence class of H-cycles of Q_5, as it was the case for MIN(**B**) and MIN(**C**), that yielded FIT(**B**) and FIT(**B**). In the present case, $F(1), F(2), \dots, F(246)$ yield this way corresponding adjusted, or fitted, files $G(1), G(2), \dots, G(246)$.

We apply a quicksort procedure Θ to each of $G(1), G(2), \dots, G(246)$. Such a Θ produces corresponding files $H(1), H(2), \dots, H(246)$, each appearing with the MCNS's represented by its lines in their correct lexicographical order. A unique file FIT(**A**) is obtained by concatenating $H(1), H(2), \dots, H(246)$ in that order, with a total of 237671 lines. A list of the first MCNS's in each of $H(1), H(2), \dots, H(246)$ can be found in [3].

However, we must remark that six of the 246 files $G(1)$, $G(2)$, ..., $G(246)$, were in practice too large for the quicksort procedure Θ to be applicable, with more than 2000 lines each, so each of them needed to be split into a collection of smaller files, repeating once more the procedure of the previous paragraph, so as for Θ to be practicable in each split file, after which the corresponding $H(j)$ could be obtained by an adequate concatenation. These six files and their data can also be found in [3].

By concatenating FIT(**A**), FIT(**B**) and FIT(**C**), a file FIT containing exactly 237675 MCNS's is obtained, corresponding each to a MCNS. These yields the claimed total of 237675 classes of H-cycles of Q_5.

7 Invariant Distribution of the MCNS's of Q_5

It can be seen that the first component p of the invariant (p, ξ, q, r) of a MCNS, that behaves as a partition of 16, cannot have three or more 1's. Thus partitions 85111 and 76111 are impossible in this context. The other partitions of 16 do happen for the 237675 MCNS's found. We indicate each of them with a single symbol, as follows, (where $85111 = a$ and $76111 = f$ are not to be present).

$$\begin{array}{lllll} 84211 = b, & 83311 = c, & 83221 = d, & 82222 = e, & 75211 = g, \\ 74311 = h, & 74221 = i, & 73321 = k, & 73222 = j, & 66211 = l, \\ 65311 = m, & 65221 = n, & 64411 = o, & 64321 = p, & 64222 = q, \\ 63331 = r, & 63322 = s, & 55411 = t, & 55321 = u, & 55222 = v, \\ 54421 = w, & 54331 = x, & 54322 = y, & 53332 = z, & 44431 = \alpha, \\ 44422 = \beta, & 44332 = \gamma, & 43333 = \delta. & & \end{array}$$

A table follows in which each line is headed by a triple pqr and each column is headed by an integer ξ, where p, q, r and ξ are as defined in (a-d) above. The table shows the number of sequences s in FIT having the indicated values of pqr and ξ. From the table, the number of lines of FIT having $r = 1$ is 2313, from which 2309 were present in FIT(A) and the other four arise from the three lines of FIT(B) and the line of FIT(C), their contributions indicated with a star. We rearrange the end of the table in smaller sub-tables to accommodate too wide 5-digit numbers, when $\xi = 4$.

	1	2	3	4		1	2	3	4		1	2	3	4
$by0$					$cy0$					$dy0$				
$bn0$					$cn0$					$dn0$			1	1
$by1$	1				$cy1$	1				$dy1$		1		
$bn1$					$cn1$		1			$dn1$		1	1	
$ey0$					$gy0$			1		$hy0$			2	
$en0$					$gn0$				1	$hn0$				25
$ey1$					$gy1$					$hy1$				
$en1$			1		$gn1$			4		$hn1$			7	
$iy0$			1		$jy0$					$ky0$			4	
$in0$			5	92	$jn0$			3	209	$kn0$			8	214
$iy1$					$jy1$					$ky1$				
$in1$			8		$jn1$			15		$kn1$			22	
$ly0$					$my0$			3		$ny0$			4	
$ln0$				1	$mn0$				119	$nn0$			11	419
$ly1$	3				$my1$		1			$ny1$		1		
$ln1$		4	4		$mn1$			1		$nn1$		1	15	
$oy0$		1	1		$py0$			2		$qy0$				
$on0$			6	113	$pn0$				3866	$qn0$				178
$oy1$	2	1			$py1$					$qy1$				
$on1$		3	27		$pn1$			2		$qn1$			22	
$ry0$			2		$sy0$		1	4		$ty0$		1	1	
$rn0$			35	1363	$sn0$			22	5529	$tn0$			8	223
$ry1$		2			$sy1$	2	5			$ty1$	1	3		
$rn1$		2	53		$sn1$			102		$tn1$		5	44	

	1	2	3	4
$uy0$			1	
$un0$			39	367
$uy1$		2		
$un1$		2	79	

	1	2	3	4
$vy0$				
$vn0$			3	1671
$vy1$				
$vn1$		4	95	

	1	2	3	4
$wy0$			14	
$wn0$			47	7111
$wy1$		2		
$wn1$		2	95	

	2	3	4
$xy0$		24	
$xn0$		67	14955
$xy1$	2		
$xn1$	2	113	

	3	4
$yy0$	7	
$yn0$	51	40882
$yy1$		
$yn1$	89	0

	2	3	4
$zy0$		7	
$zn0$		8	2764
$zy1$	2		
$zn1$		162	

	2	3	4
$\alpha y0$		7	
$\alpha n0$		97	9359
$\alpha y1$	2		
$\alpha n1$	2	158	

	3	4
$\beta y0$		
$\beta n0$		12498
$\beta y1$		
$\beta n1$	266	

	1	2	3	4
$\gamma y0$		6	18	
$\gamma n0$			85	77127
$\gamma y1$	4•	6		
$\gamma n1$		1	45	

	0	1	2	3	4
$\delta y0$		1	1	4	
$\delta n0$				106	25675
$\delta y1$	2•	1•	3		
$\delta n1$			8	354•	

We list the MCNS's with $\xi \leq 1$ from FIT(**A**), that are not previously presented (in Section 5), where the initial subsequences 0102 are eliminated to save space, together with the corresponding triples pqr.
With $\xi = 0$, (1 sequence out of 2): 34321012434201023432101243420 $\delta y1$
With $\xi = 1$, (14 sequences out of 16):

01030102010401020103010201040	$by1$	01030201020401020103020102040	$cy1$
01031012101401020103101210140	$ly1$	01031014101301020103101410130	$ly1$
01032120212401020103212021240	$ty1$	01310201024201020131020102420	$oy1$
01314241423201020131424142320	$\gamma y1$	01320102014201020132010201420	$oy1$
01324142413201020132414241320	$\gamma y1$	10130102101401021013010210140	$ly1$
10340302301401021034030230140	$sy1$	30140302103401023014030210340	$sy1$
31304143212401023130414321240	$\delta y0$	34023134023401023402313402340	$\gamma y1$

The remaining sequences of FIT(**A**), which can be found in [3], are:

- 90 sequences with $\xi = 2$,
- 3023 sequences with $\xi = 3$ out of 3024, and
- 234543 sequences with $\xi = 4$.

8 A Nonexistence Result on Q_6

An adaptation of the exhaustive recursion search via backtracking in Section 5 to the case of the 6-cube Q_6 was essayed for the exhaustive search of elements of $C_{6,5}$. However, the computational language indicated a stack overflow error. The essayed program was then modified to obtain just an exhaustive list $P_{6,43}$ of sub-paths of H-cycles in Q_6 of length 43, where the stack overflow error was not present. The resulting list $P_{6,43}$ contained 1182 lines. By readapting and applying a modified concatenated exhaustive search for the elements of $C_{6,5}$ departing from $P_{6,43}$, no such an element was produced, from where we concluded that $C_{6,5} = 0$.

Theorem 6 $C_{6,5} = 0$.

In view of this theorem, it is seen that Question 5 does not lead to the production of H-cycles in $C_{n,n-1}$, for every $n \geq 5$, such as it was shown for Mills cycles in Q_n, for every $n \geq 5$, in [12]. However, the Mills cycles produced in [12] were all in $C_{n,3}$, for every $n \geq 5$.

9 Uniqueness in the Middle-Levels of Q_5

A second adaptation of the search in Section 5, to the graphs M_{2k+1} formed by the *middle levels* of Q_{2k+1}, that is induced by the vertices of Q_n having weights $\in \{k, k+1\}$, ([13]), allowed to obtain an interesting contrasting result. To compare with the relatively high counting of equivalence classes of Hamilton cycles in Q_5, we have just the following.

Theorem 7 *There is only one equivalence class of Hamilton cycles in the middle-levels graph M_5 of Q_5.*

Proof. M_5 has regular degree 3. Using the mentioned adaptation, we found that there is only one equivalence class of H-cycles in M_5. The MCNS of an H-circuit C for this class can be represented as follows,

024	103	214	023	104	312	403	124	301	234
012	304	213	402	134	201	423	014	302	413

where each triple t_p of numbers represents the coordinate directions of the edges incident to the corresponding vertex $p = p^i$ in C, ($i = 0, 1, \ldots, 19$ taken mod 20), with the direction along which C continues set as the first, or leftmost, position in t_p and the remaining two directions presented lexicographically. It is seen easily that the starting vertex is $p^0 = 01010$, with support $\{1, 3\} \subset \mathbb{Z}_4$. This yields Theorem 7.

We will give now a symmetric presentation of a representative of the equivalence class in Theorem 7. Another way of presenting the MCNS in the the proof of Theorem 7 is as follows, where the last, or rightmost, position in each t_p equals the edge direction previous to p in C,

$$\begin{array}{cccccccccc} 024 & 130 & 241 & 032 & 140 & 321 & 403 & 124 & 301 & 243 \\ 012 & 340 & 213 & 402 & 134 & 201 & 432 & 014 & 320 & 413 \end{array}$$

This is called a *1-factorable presentation* of C. Note now that the middle positions of these t_p's forms a 1-factor g of M_5.

Let $a \in \mathbb{Z}_4$. Denote each edge in $M_5 \cap f_a$ with end-vertex supports $\{b, c\}$ and $\{a, b, c\}$ by (a, bc). Then g is composed by the following five edge pairs in the respective 1-factors f_i, for $i = 0, 2, 1, 4, 3$:

$$(0,21),(0,24),(2,14),(2,13),(1,43),(1,40),(4,30),(4,32),(3,02),(3,01).$$

Then the permutation (12)(34) of $\mathbb{Z}_4$, which transforms the 1-factorable presentation of C given above into

$$\begin{array}{cccccccccc} 014 & 240 & 132 & 041 & 230 & 412 & 304 & 213 & 402 & 134 \\ 021 & 430 & 124 & 301 & 243 & 102 & 341 & 023 & 410 & 324 \end{array}$$

with starting-vertex support $\{2, 4\}$, allows to express g simply as

$$\{w_i = (i,(i+1)(i+2)), z_i = (i,(i+1)(i+3)); i \in \mathbb{Z}_4\},$$

with numbers taken mod 5. This edge notation can be used to represent the successive edges of C, and we do so now, interspersing between each two such edges a capital letter representing, and indexed with, the edge of g incident to their common vertex:

$$\begin{array}{llll} (1,03), A_{(2,03)}, & (4,03), F_{(3,40)}, & (0,34), I_{(2,34)}, & (1,34), C_{(3,14)}, \\ (4,31), B_{(0,31)}, & (2,31), G_{(1,23)}, & (3,12), J_{(0,12)}, & (4,12), D_{(1,42)}, \\ (2,14), C_{(3,14)}, & (0,14), H_{(4,01)}, & (1,04), F_{(3,40)}, & (2,40), E_{(4,20)}, \\ (0,42), D_{(1,42)}, & (3,42), I_{(2,34)}, & (4,23), G_{(1,23)}, & (0,23), A_{(2,03)}, \\ (3,20), E_{(4,20)}, & (1,20), J_{(0,12)}, & (2,01), H_{(4,01)}, & (3,01), B_{(0,31)}. \end{array}$$

The edges of g can be interpreted as expressing every third edge in an H-circuit of the *graph of proper levels* $Q_5 \setminus \{0,1\} = Q_5 \setminus \{00000, 11111\}$ of Q_5. This H-circuit is given by the CNS

$$040123212340434012101234323401$$

starting at the vertex with support $\{1, 2\}$, or in a more cyclical form, given by the successive concatenation of the paths

$$j(j-1)j(j+1)(j+2)(j-2),$$

where $j = 2i$ with $i = 0,1,2,3,4$. The remaining edges of $Q_5 \setminus \{0,1\}$ form, together with the edges of g, five 6-cycles C^i that have CNS's given respectively by

$$((i+2)(i-1)(i+1))^2 = (i+2)(i-1)(i+1)(i+2)(i-1)(i+1),$$

starting at the vertices v^i whose respective supports are the numbers $i \in \mathbb{Z}_4$. Note that C^i is isomorphic image of the middle-levels cycle of Q_3 under the isomorphism ϕ_i that sends $0 \in Q_3$ onto v^i and $1 \in Q_3$ onto the complement of v^{i-2}, for $i \in \mathbb{Z}_4$. The 3-sub-cubes $\phi_i(Q_3)$ and $\phi_{i+1}(Q_3)$ have solely the edge

$$u_i = (i+2, i(i+1))$$

in common, for $i \in \mathbb{Z}_4$. Thus, these 3-sub-cubes form a 5-cyclic chain of five 3-cubes, where each two contiguous 3-cubes share just an edge.

There is a 10-cycle C_{wu} formed by the concatenation of contiguous-edge pairs $w_{-i}u_{-i}$, for $i = 0,1,2,3,4$. On the other hand, there are just five edges of $Q_5 \setminus \{0,1\}$ not in the union of the 3-cubes $\phi_i(Q_3)$, and they form, together with the edges z_i, $(i \in \mathbb{Z}_4)$, a 10-cycle C_z disjoint from C_{wu}.

By coloring the H-circuit C with two colors, we obtain the two 1-factors

$$h_1 = \{(j,(j+2)(j-2)),(j+1,j(j+2)); j = 2i, i = 0,1,2,3,4\},$$

$$h_2 = \{(j-2,j(j+2)),(j+2,j(j+1)); j = 2i, i = 0,1,2,3,4\}.$$

We get a 1-factorization of M_5, formed by g, h_1 and h_2. The 1-factor h_1 forms, together with g, a Hamilton cycle D of M_5. The 1-factor h_2 forms, together with g, the union of the two 10-cycles C_{wu} and C_z. By expressing C similarly, it is seen that the permutations $\sigma_i = (i,i+1)(i+2,i-1)$ yield equivalences between C and D, for $i \in \mathbb{Z}_4$. Moreover, the antipodal map τ of Q_5 given by uniform binary complementation is closed in h_1 and exchanges g and h_2. Thus, τ offers another equivalence between $C = h_2 \cup h_1$ and $D = g \cup h_1$. It is also closed in C_{wu} and in C_z. It is seen that the total number of isomorphisms of Q_5 restricting to equivalences between C and D is 10, obtained by combining τ with the five σ_i's.

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