Maximum-Demand Graphs for Eternal Security

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Abstract

Informally, a set of guards positioned on the vertices of a graph G is called eternally secure if the guards are able to respond to vertex attacks by moving a single guard along a single edge after each attack regardless of how many attacks are made. The smallest number of guards required to achieve eternal security is the eternal security number of G, denoted $es(G)$, and it is known to be no more than $\theta_v(G)$, the vertex clique cover number of G. We investigate conditions under which $es(G) = \theta_v(G)$.

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1 Introduction

Burger, Cockayne, Gründlingh, Mynhardt, van Vuuren, and Winterbach [1, 2], introduced a dynamic form of domination which has been designated *eternal security* by Goddard, Hedetniemi, and Hedetniemi [4]. Let S be a subset of the vertices of graph $G = (V, E)$. An *attack* on G is a choice of a vertex in $V(G)$. A *response by S* (or simply a *response* if S is understood) to an attack on v is a vertex

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$w \in S \cap N[v]$. We say response w to an attack on v *produces* the set $S' = (S - w) \cup \{v\}$. Informally, if such a response can be made no matter what vertex is attacked, and if the changing position of the guards can continue to respond forever, we say that the guards form an eternally secure set. The *eternal security number* $es(G)$ of G is the minimum cardinality of an eternally secure set and an *es-set* of G is an eternally secure set S such that $|S| = es(G)$.

This attack/response process can be viewed as a two person game in which one player, the attacker, selects a vertex to attack and the other player, the defender, decides on a response. If at any point the attacker makes an attack for which there is no response, the attacker wins. Otherwise, the defender wins. In this context a set is eternally secure if, under optimum play by both players, the defender wins.

Let $\beta_v(G)$ be the vertex independence number of graph G and $\theta_v(G)$ be its vertex clique cover number. It is easy to see that if S has a distinct vertex in every set of a clique cover of G, then S is eternally secure. Burger et al show the following basic inequality.

Theorem 1 [2] *For graph G, $\beta_v(G) \leq es(G) \leq \theta_v(G)$.*

Many important classes of graphs have $es(G) = \theta_v(G)$ and it was originally conjectured that this is true for all graphs. However, Goddard et al [4] have produced an infinite collection of counterexamples. Nevertheless, it is interesting to find families where this equality holds. We refer to members of such families as *maximum-demand graphs.*

In order to formalize the concept of eternal security, Goddard, et al [4] gave the following definition.

Definition 2 *Let $S_0 \subseteq V$. Then S_0 is an* eternally secure set *if for every positive integer k and any sequence $v_1, v_2, \ldots, v_k$ of vertices of G, there is a sequence of sets $S_1, S_2, \ldots, S_k$ and a sequence of vertices (guards) $u_1, u_2, \ldots, u_k$ such that $v_i \notin S_{i-1}$, $u_i \in S_{i-1}$, $u_i v_i \in E$, and $S_i = (S_{i-1} - \{u_i\}) \cup \{v_i\}$ is a dominating set.*

References [2] and [4] employ the notation $\gamma_\infty(G)$ and $\sigma_1(G)$, respectively, for $es(G)$.

We find definition 2 unsatisfactory for two reasons. First, the definition states that for an arbitrary vertex sequence $v_1, v_2, \ldots, v_k$

of vertices of G we can find a sequence of sets $S_1, S_2, \ldots, S_k$ and a sequence of vertices (guards) $u_1, u_2, \ldots, u_k$ such that $v_i \notin S_{i-1}$ .... However, if S_0 is not empty then any sequence with $v_1 \in S_0$ will not satisfy this condition. Of course this problem can be addressed by replacing the condition "$v_i \notin S_{i-1}$, $u_i \in S_{i-1}$, $u_i v_i \in E$" with the condition "$u_i \in S_{i-1} \cap N[v_i]$".

Second, even with this change the definition does not reflect the dynamic nature of the attack/response process. In particular, the definition does not allow the attacker to base the next attack on the most recent response. For example, consider the graph G in Figure 1. Using the two player game interpretation of eternal security the set $S = \{a, b, c, d\}$ is not eternally secure. The first player attacks vertex z. If the second player responds with c producing the set $\{a, b, z, d\}$, then the first player attacks vertex x and the second player must respond with a procuding the set$\{x, b, z, d\}$. At this point the first player attacks w and the second player has no response. Similarly, if the second player responds to the attack on z with d, then the first player attacks vertex y which forces the response b at which point the second player has no response to an attack on v.

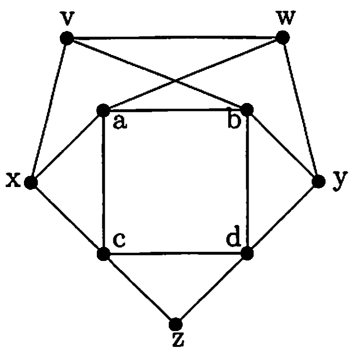


Figure 1: Illustration of Definition Conflict

On the other hand, the set $S = \{a, b, c, d\}$ is eternally secure according to Definition 2, even with the above suggested change. Suppose S is not eternally secure. Under this assumption there exists a sequence of vertices of G such that no response is possible to the attack on v_k. We choose $v_1, v_2, \ldots, v_k$ to be such a sequence with k as small as possible. By this choice of k, $v_1 \notin \{a, b, c, d\}$. If $v_1 = x$ let

$u_1 = c$. This produces a set S_1 with a distinct vertex in every set of a clique cover of G. Such a set is eternally secure which contradicts the assumption that S is not eternally secure. Similar contradictions are obtained if $v_1 = v$, $v_1 = w$, or $v_1 = y$.

Finally, suppose $v_1 = z$. The minimality of k ensures that $v_2 \notin \{a, b, z\}$. If $v_2 = c$, then letting $u_1 = c$ and $u_2 = z$ also contradicts the minimality of k; so, $v_2 \neq c$ and similarly $v_2 \neq d$. If $v_2 = x$, let $u_1 = d$ and $u_2 = c$; if $v_2 = y$, let $u_1 = c$ and $u_2 = d$; if $v_2 = v$, let $u_1 = c$ and $u_2 = b$; and if $v_2 = w$, let $u_1 = d$ and $u_2 = a$. In each of these cases we obtain a set S_2 with a distinct vertex in every set of a clique cover of G contradicting the choice of S.

Of course, for some graphs the attacker may be able to proceed without regard to the defender's responses, for example, when attacking in succession the vertices of an independent set. In such cases we may refer to a *sequence of attacks*. If $v_1, v_2, \ldots, v_k$ is a sequence of attacks and $S_1, S_2, \ldots, S_k$ is the sequence of vertices (guards) produced by responses $u_1, u_2, \ldots, u_k$, we will say that the sequence of attacks *produces* S_k.

Motivated by an approach mentioned in [4], we avoid the above difficulty by employing an algorithmic definition. The input is a graph $G = (V, E)$. The algorithm defines a function μ_G on a subset of $P(V(G))$ where $P(V(G))$ is the power set of $V(G)$.

ALGORITHM ES

$\mathcal{E} := P(V(G))$
$\mathcal{N} := \emptyset$
$\mathcal{T} := \{S \in \mathcal{E}$: S not a dominating set$\}$
$\mu_G(S) := 0$ for all $S \in \mathcal{T}$
$k := 0$
while $\mathcal{T} \neq \emptyset$
    $k := k + 1$
    $\mathcal{N} := \mathcal{N} \cup \mathcal{T}$
    $\mathcal{E} := \mathcal{E} - \mathcal{T}$
    $\mathcal{T} := \{S \in \mathcal{E}$: there is an attack on some vertex such that all responses
            produce a set in $\mathcal{N}\}$
    $\mu_G(S) := k$ for all $S \in \mathcal{T}$

Note that the algorithm must terminate since $P(V(G))$ is finite. Furthermore, $\mathcal{E}$ is nonempty upon termination since V itself will never be placed in $\mathcal{N}$. We now can give the revised definition of eternally secure sets.

Definition 3 *Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is* eternally secure *if it remains in $\mathcal{E}$ upon termination of ALGORITHM ES when applied to G. The value $es(G)$ is the smallest cardinality of a set in $\mathcal{E}$.*

We make the following observation regarding the function μ_G.

Observation 4 *If S is not an eternally secure set, then either $\mu_G(S) = 0$ and S does not dominate G, or $\mu_G(S) > 0$ and there exists a vertex v such that $\mu_G(S') < \mu_G(S)$ for every S' produced by a response to an attack on v.*

For a starting defense S of guards which is not eternally secure, $\mu_G(S)$ is a measure of the number of steps in the game, assuming optimum play by both participants. In the above example, $\mu_G(\{x, z, b, d\}) = 0$ and $\mu_G(\{a, c, z, y\}) = 0$ since neither of these two sets dominate G. Consequently, $\mu_G(\{a, z, b, d\}) = 1$ since a is the only response by $\{a, z, b, d\}$ to an attack on x and $\mu_G(\{a, c, z, b\}) = 1$ since b is the only response by $\{a, c, z, b\}$ to an attack on y. Thus, $\mu_G(\{a, b, c, d\}) = 2$ since the responses by this set to an attack on z produce either $\{a, z, b, d\}$ or $\{a, c, z, b\}$.

The value of $\mu_G(S)$ found by ALGORITHM ES also can be determined by the following recursive definition.

Definition 5 *Let $G = (V, E)$ be a graph. For a non eternally secure set $S \subseteq V$*

$$\mu_G(S) = \begin{cases} 0 & \text{if } S \text{ is nondominating} \\ k & \text{if there is an attack on some vertex such that} \\ & \quad \text{any response produces a set } S' \text{ with } \mu_G(S') \leq k-1, \text{and} \\ & \quad \text{some response produces a set } S' \text{ with } \mu_G(S') = k-1 \end{cases}$$

In subsequent discussions we may speak informally of responses to attacks in terms of the movement of a guard. In most such cases the original and final sets of guards form eternally secure sets.

References [2] and [4] present many examples of maximum-demand graphs. These include paths, cycles, bipartite graphs, complete multipartite graphs, perfect graphs, and graphs for which the vertex clique cover number is at most three. Other examples include the Cartesian products $K_m \times K_n$, $P_m \times P_n$, $C_4 \times C_5$, and $C_5 \times C_5$. Burger et al [2] prove $7mn/23 \leq es(C_m \times C_n) \leq \theta_v(C_m \times C_n)$ in general. In Section 2 we show $C_m \times C_n$ is a maximum-demand graph for all values of m and n. Section 3 discusses the situation for the coalescence of two graphs, Section 4 presents results pertaining to a minimum non maximum-demand graph, and Section 5 shows that graphs with no subgraph homeomorphic to K_4 are maximum-demand graphs. We conclude with some open questions.

2 Eternal Security Number of $C_m \times C_n$ and $P_m \times C_n$

If m and n are both even, $C_m \times C_n$ is bipartite and thus is a maximum-demand graph. It is straightforward to demonstrate $C_m \times C_n$ is a maximum-demand graph for $n \leq 3$ and $m \geq n$. Thus we may assume $n \geq 4$ and $m \geq 5$ is odd. The following theorem solves all the remaining cases.

Theorem 6 *Let n and m be integers where $n \geq 4$ and $m \geq 5$ is odd. Then $C_m \times C_n$ is a maximum-demand graph.*

Proof: It is easy to see that $\theta_v(C_m \times C_n) = \lceil \frac{mn}{2} \rceil$; hence, by Theorem 1 $es(C_m \times C_n) \leq \lceil \frac{mn}{2} \rceil$. We claim that $es(C_m \times C_n)$ has the same value. Without loss of generality assume that either n is even, or n is odd and $n \geq m$. Embed the vertices of $C_m \times C_n$ on the m by n lattice $\{(i,j) : 0 \leq i \leq m-1, 0 \leq j \leq n-1\}$ such that the vertices in any row or column form a cycle of $C_m \times C_n$. Designate the vertex at (i,j) by $v_{i,j}$. For $0 \leq t \leq m-1$, let $d_t = \{v_{t,0}, v_{t+1,1}, v_{t,2}, v_{t+1,3}, v_{t,4}, \ldots, v_{t+1,n-1}\}$ if n is even and let $d_t = \{v_{t,0}, v_{t+1,1}, \ldots, v_{t+m-1,m-1}, v_{t+m-2,m}, v_{t+m-1,m+1}, v_{t+m-2,m+2}, \ldots, v_{t+m-1,n-1}\}$ if n is odd. In both cases the addition in the first subscript is modulo m. Figure 2 gives two examples, with the vertices of d_0 marked with circles and those of d_2 with squares. Note that $d_t \cup d_{t+2}$, with the subscript modulo m, is an independent set and the closed neighborhood of d_1 is $d_0 \cup d_1 \cup d_2$. Let S be an eternally secure set for $C_m \times C_n$. Let S_1 be the set produced by attacking the vertices of $d_0 \cup d_2$ in any order. The independence of $d_0 \cup d_2$ implies that S_1 contains all of these $2n$ vertices. Let S_2 be the set which is produced by then attacking the vertices of d_1 in any order. All responses to the attacks on d_1 must come from $d_0 \cup d_1 \cup d_2$. Without loss of generality, $\lceil \frac{n}{2} \rceil$ of the vertices in d_0 do not respond to these attacks. Thus, $|S_2 \cap d_0 \cup d_1| \geq n + \lceil \frac{n}{2} \rceil$. Let S_3 be the set produced by then attacking the vertices in $d_3, d_5, ...d_{m-2}$ in any order. These vertices form an independent set and are disjoint from the closed neighborhood of $d_0 \cup d_1$; hence, all of these $\frac{n(m-3)}{2}$ vertices must be in S_3. Furthermore, no vertex in $d_0 \cup d_1$ can respond to any of these attacks. It follows that $|S_3| \geq n + \lceil \frac{n}{2} \rceil + \frac{n(m-3)}{2} = \lceil \frac{mn}{2} \rceil$. By the definition of response, $|S_3| = |S|$; hence, since S was an arbitrary eternally secure set, $es(C_m \times C_n) \geq \lceil \frac{mn}{2} \rceil$ and the result follows □

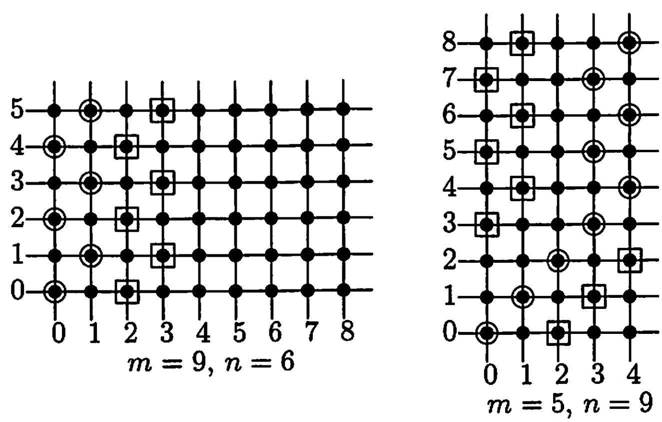


Figure 2: Examples showing sets d_0 and d_2

The following general observation is useful in determining $es(P_m \times C_n)$.

Observation 7 *If H is a spanning subgraph of graph G, then $es(H) \geq es(G)$.*

Using this observation and the fact than $\theta_v(P_m \times C_n) = \theta_v(P_m \times P_n) = \theta_v(C_m \times C_n)$, we have the following result.

Theorem 8 *The graphs $P_m \times C_n$ and $P_m \times P_n$ are maximum-demand graphs.*

Of course, the fact that $P_m \times P_n$ is a maximum-demand graph was known previously. [4]

3 On the Coalescence of Two Graphs

Let H and K be graphs with designated vertices v_H and v_K, respectively. The *coalescence* G of H and K, written $G = H \circ K$, is the graph obtained by identifying v_H and v_K. We designate the common vertex in G by v, and, for convenience, also employ v for v_H and v_K when discussing either H or K alone. Furthermore we call the copies of H and K in G by the same symbols H and K, respectively.

Lemma 9 *Let v be a vertex in a graph G. Then $es(G-v) \geq es(G)-1$.*

Proof: Any eternally secure set of $G-v$ along with v is an eternally secure set of G. □

Lemma 10 *Let $G = H \circ K$. Then $\theta_v(H) + \theta_v(K) - 1 \leq \theta_v(G) \leq \theta_v(H) + \theta_v(K)$ and $es(H) + es(K) - 1 \leq es(G) \leq es(H) + es(K)$.*

Proof: The upper bounds are immediate since the unions of es-sets or vertex clique covers for H and K forms an eternal security set or vertex clique cover, respectively, of G. The lower bound for $\theta_v(G)$ follows since it can be smaller than $\theta_v(H) + \theta_v(K)$ only if v is in a singleton clique of a minimum vertex clique cover of either H or K, and then the total number of cliques required is reduced only by one from $\theta_v(H) + \theta_v(K)$.

Suppose $es(G) \leq es(H) + es(K) - 2$ and let S be an es-set of G. Note that no vertex of $H - v$ can respond to an attack on a vertex of $K - v$, so, $|S \cap V(K)| \geq es(K - v)$ and, by Lemma 9, $|S \cap V(K)| \geq es(K)-1$. Similarly, $|S \cap V(H)| \geq es(H)-1$. The upper bound on $es(G)$ implies at least one of the inequalities is an equality. Without loss of generality, we assume $|S \cap V(H)| = es(H) - 1$ which implies $S \cap V(H)$ is not an eternally secure set for H. Among all such sets we choose S so that $\mu_H(S \cap V(H))$ is a minimum. Since $S \cap V(H)$ is not an eternally secure set for H, there exists a vertex $x \in V(H)-S$ such that either (i) $S \cap V(H)$ does not dominate x or (ii) for every

S' produced by a response to an attack on x, $\mu_H(S' \cap V(H)) < \mu_H(S \cap V(H))$. Thus, no vertex of $S \cap V(H)$ can respond to an attack on x without contradicting the definition of S. If $x \neq v$, then all responses from S to attacks on x must come from $S \cap V(H)$ and we contradict S being an es-set for G. Therefore, $v = x \notin S$ and $S \cap V(H) = S \cap V(H - v)$. If now we restrict attacks to K, then the vertices of $S \cap V(H)$ cannot respond. Therefore, $S - (S \cap V(H))$ is an eternally secure set for K. Again we have a contradiction since $|S - (S \cap V(H))| \leq (es(H) + es(K) - 2) - (es(H) - 1) = es(K) - 1$. □

We show next that if H and K are maximum-demand graphs, then there is a condition under which $H \circ K$ is a maximum-demand graph.

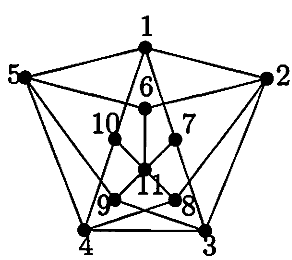
Proposition 11 *Let $G = H \circ K$ where H, K, $H - v$, and $K - v$ are all maximum-demand graphs. Then G is a maximum-demand graph.*

Proof: By Lemma 10 we may restrict our attention to the case $es(G) = es(H) + es(K) - 1$ and $\theta_v(G) = \theta_v(H) + \theta_v(K)$. Observe that no minimum vertex clique cover of either H or K contains v in a singleton set. Let S be an es-set of G.

Case 1. Suppose $v \notin S$. The condition $es(G) = es(H) + es(K) - 1$ implies either $|S \cap V(H)| \leq es(H) - 1$ or $|S \cap V(K)| \leq es(K) - 1$. Assume without loss of generality that $|S \cap V(H)| \leq es(H) - 1$. By construction, no vertex of $K - v$ can respond to an attack on $H - v$; so, $v \notin S$ implies that $S \cap V(H) = S \cap V(H - v)$ is an eternally secure set for $H - v$. Thus, $es(H - v) \leq es(H) - 1$. By assumption $H - v$ is a maximum-demand graph and we have $\theta_v(H - v) = es(H - v) \leq es(H) - 1 = \theta_v(H) - 1$. Therefore, we can add $\{v\}$ to a minimum clique cover of $H - v$ to obtain a minimum clique cover of H, contrary to the above observation.

Case 2. Next suppose $v \in S$. Here the condition $es(G) = es(H) + es(K) - 1$ implies either $|S \cap V(H)| \leq es(H)$ or $|S \cap V(K)| \leq es(K)$. Assume without loss of generality that $|S \cap V(H)| \leq es(H)$. Suppose v is required to respond at some point to an attack on $H - v$ producing an es-set S' of G. In that event, $v \notin S'$ and we proceed as in Case 1 by substituting S' for S. Otherwise, $S \cap V(H - v)$ is an eternally secure set for $H - v$ and $es(H - v) \leq |S \cap V(H - v)| \leq es(H) - 1$. This leads to the same contradiction as in Case 1. □

Proposition 11 is not valid if the condition that $H - v$ and $K - v$ are maximum-demand graphs is removed. One of the examples due to Goddard, Hedetniemi, and Hedetniemi [4] shows that $3 = es(G) < \theta_v(G) = 4$ if G is the complement of the Grötzsch graph shown in Figure 3. We employ this fact in showing that the proposition does not extend.



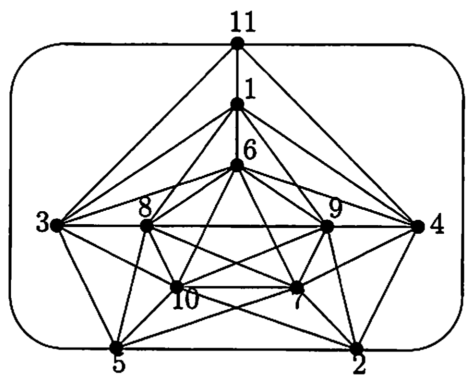


Figure 3: The Grötzsch graph and its complement

Create a graph H by adding a pendant vertex v adjacent to vertex 4 in the complement of the Grötzsch graph. Because v, 1, and 5 are independent, there exists an es-set S of H that contains them. Suppose that $S = \{v, 1, 5\}$. Attack vertex 3. Vertex v cannot respond since it is not adjacent to vertex 3. If vertex 1 responds, vertex 9 is not dominated. If vertex 5 responds, vertex 2 is not protected. Thus $es(H) \geq 4$. It is easy to see that $\theta_v(H) = 4$, so $es(H) = 4$. Let $K = H$ and form G as the coalescence of H and K with the vertex v's identified. Then three vertices form an eternal security set of $H - v$ and four of K. It follows that $es(G) = 7$ but $\theta_v(G) = 8$ so Proposition 11 does not extend. Notice that if F is the graph formed by coalescing n copies of H at the vertex v, $es(F) = 3n + 1$ and $\theta_v(F) = 4n$, so the difference in the two values can be arbitrarily large. This arbitrary difference also can be shown by the examples in [4].

4 Adding Paths

This section is concerned with when it is possible to construct maximum-demand graphs by adding paths to a known maximum-demand graph. The following four lemmas provide useful tools.

Lemma 12 *Suppose H is an induced subgraph of graph G. If S is an eternally secure set of G such that $|S \cap V(H)|$ is maximized, then $S \cap V(H)$ is an eternally secure set of H.*

Proof: Suppose not. Choose S to be an eternally secure set of G such that $|S \cap V(H)|$ is maximized, $S \cap V(H)$ is not an eternally secure set of H, and $\mu_H(S \cap V(H))$ is minimized. Since $S \cap V(H)$ is not an eternally secure set of H, there exists a vertex $x \in H$ such that either (i) $S \cap V(H)$ does not dominate x or (ii) if $w \in S \cap V(H)$ is a response to an attack on x, then $\mu_H(((S \cap V(H)) \cup x) - \{w\}) < \mu_H(S \cap V(H))$. The choice of S precludes (ii); hence, (i) holds and no vertex of $S \cap V(H)$ can respond to an attack on x. Therefore, the response from S to an attack on x comes from $G - H$ which contradicts the maximality of $|S \cap V(H)|$. □

Lemma 13 *Suppose graph G is obtained from graph H by adding a path P of length at least two between distinct vertices v and w. Suppose v and w are contained in every es-set S of G such that $|S \cap V(H)|$ is maximized. Either $|S \cap V(H)| \geq es(H) + 1$ or $es(H - \{v, w\}) = es(H) - 2$. Furthermore, if v and w are adjacent, $|S \cap V(H)| \geq es(H) + 1$.*

Proof: Suppose v and w are contained in every es-set S of G such that $|S \cap V(H)|$ is maximized and let S be such a set. By Lemma 12, $|S \cap V(H)| \geq es(H)$. Suppose $|S \cap V(H)| = es(H)$. If we restrict our attacks to $H - \{v, w\}$, then v and w cannot respond without violating the assumption that they are in every es-set with maximum intersection with H. This implies that $S \cap V(H - \{v, w\})$ is an eternally secure set for $H - \{v, w\}$. Thus, $es(H - \{v, w\}) \leq |S \cap V(H - \{v, w\})| = |S \cap V(H)| - 2 = es(H) - 2$. Since adding

$\{v, w\}$ to any eternally secure set of $H - \{v, w\}$ produces an eternally secure set for H, we have $es(H - \{v, w\}) + 2 \geq es(H)$; hence, $es(H - \{v, w\}) = es(H) - 2$. If v and w are adjacent, then $S \cap V(H - v)$ is an eternally secure set of H which implies $|S \cap V(H)| \geq es(H) + 1$. □

The proof of Lemma 13 actually suffices for any cutset $\{v, w\}$ but we do not require that result here.

Lemma 14 *Let $n \geq 3$ and suppose G is obtained from graph H by adding a path $P = \langle v_1, v_2, ..., v_n \rangle$ between distinct vertices v_1 and v_n of H. Let S be an es-set for G. If $|S \cap V(H)| \geq es(H) + 1$ then $es(G) \geq es(H) + \lceil \frac{n}{2} \rceil - 1$.*

Proof: Attacks on the $\lceil \frac{n}{2} \rceil - 2$ independent vertices $v_3, v_5, ..., v_{2\lceil \frac{n}{2} \rceil - 3}$ cannot be responded to by vertices in H. Thus $|S - V(H)| \geq \lceil \frac{n}{2} \rceil - 2$ and $es(G) = |S| = |S \cap V(H)| + |S - V(H)| \geq es(H) + 1 + \lceil \frac{n}{2} \rceil - 2 = es(H) + \lceil \frac{n}{2} \rceil - 1$. □

Lemma 15 *Suppose $n \geq 4$ is an even integer and G is obtained from graph H by adding a path $P = \langle v_1, v_2, ..., v_n \rangle$ between distinct vertices v_1 and v_n of H. If there exists an es-set S for G such that $|S \cap V(H)|$ is maximized and $|S \cap \{v_1, v_n\}| \leq 1$, then $es(G) \geq es(H) + \frac{n}{2} - 1$.*

Proof: Without loss of generality suppose v_1 is not in S. By Lemma 12, $S \cap V(H)$ is an eternally secure set for H, so $|S \cap V(H)| \geq es(H)$. Attacks on the independent vertices $v_2, v_4, \ldots, v_{n-2}$ cannot be responded to by vertices of H which implies $|S - V(H)| \geq \frac{n}{2} - 1$. Thus, $es(G) = |S| = |S \cap V(H)| + |S - V(H)| \geq es(H) + \frac{n}{2} - 1$. □

Theorem 16 *Let $n \geq 4$ and suppose G is obtained from graph H by adding a path $P = \langle v_1, v_2, ..., v_n \rangle$ between distinct vertices v_1 and v_n of H. If H, $H - v_1$, $H - v_n$, and $H - \{v_1, v_n\}$ are maximum-demand graphs, then G is a maximum-demand graph.*

Proof: Adding $\{v_2, v_3\}, \{v_4, v_5\}, \ldots, \{v_{2\lceil \frac{n}{2} \rceil - 2}, v_{2\lceil \frac{n}{2} \rceil - 1}\}$ to a clique cover of H provides a clique cover of G, so $\theta_v(G) \leq \theta_v(H) + \lceil \frac{n}{2} \rceil - 1$.

If $es(G) \geq es(H) + \lceil \frac{n}{2} \rceil - 1$, the theorem follows from the assumption $es(H) = \theta_v(H)$ and the inequality $es(G) \leq \theta_v(G)$. Consequently, we assume that $es(G) \leq es(H) + \lceil \frac{n}{2} \rceil - 2$. Let S be an es-set for G such that $|S \cap V(H)|$ is maximized. By Lemma 12, $S \cap V(H)$ is an eternally secure set of H. By the contrapositive to Lemma 14, $|S \cap V(H)| \leq es(H)$; hence, $|S \cap V(H)| = es(H)$ and $|S - V(H)| \leq \lceil \frac{n}{2} \rceil - 2$. Also attacks on the $\lceil \frac{n}{2} \rceil - 2$ independent vertices $v_3, v_5, \ldots, v_{1+2(\lceil \frac{n}{2} \rceil - 2)}$ cannot be responded to by vertices of H; hence, $|S - V(H)| = \lceil \frac{n}{2} \rceil - 2$ and $es(G) = es(H) + \lceil \frac{n}{2} \rceil - 2$.

If n is even, the contrapositive to Lemma 15 implies $|S \cap \{v_1, v_n\}| \geq 2$ and hence the hypothesis of Lemma 13 is satisfied. Thus, by Lemma 13, $es(H - \{v_1, v_n\}) = es(H) - 2$. Adding $\{v_1, v_2\}, \{v_3, v_4\}, \ldots, \{v_{2\lceil \frac{n}{2} \rceil - 1}, v_{2\lceil \frac{n}{2} \rceil}\}$ to a clique cover of $H - \{v_1, v_n\}$ provides a clique cover of G; so, $\theta_v(G) \leq \theta_v(H - \{v_1, v_n\}) + \lceil \frac{n}{2} \rceil = es(H - \{v_1, v_n\}) + \lceil \frac{n}{2} \rceil = es(H) - 2 + \lceil \frac{n}{2} \rceil = es(G)$ and the theorem follows in this case.

Suppose n is odd. If every such S contains both v_1 and v_n, then the hypothesis of Lemma 13 holds and we may proceed as in the even case replacing $\{v_{2\lceil \frac{n}{2} \rceil - 1}, v_{2\lceil \frac{n}{2} \rceil}\}$ with $\{v_{2\lceil \frac{n}{2} \rceil - 1}\}$. Otherwise, we may assume without loss of generality that v_1 is not in S. Produce S' from S by attacking the $\lceil \frac{n}{2} \rceil - 2$ independent vertices $v_2, v_4, \ldots, v_{n-3}$. Since $v_1 \notin S$, these attacks cannot be responded to by vertices of H. Thus $|S' \cap V(H)| = |S \cap V(H)| = es(H)$ and $|S' - V(H)| = |S - V(H)| = \lceil \frac{n}{2} \rceil - 2$; hence $S' - V(H) = \{v_2, v_4, \ldots, v_{n-3}\}$. Thus, v_n must be in S' in order to dominate v_{n-1}, and furthermore it cannot respond to attacks on $H - v_n$ without leaving v_{n-1} unguarded. It follows that $S' \cap V(H - v_n)$ is an eternally secure set for $H - v_n$. Hence, $es(H - v_n) \leq es(H) - 1$. By assumption $\theta_v(H - v_n) = es(H - v_n)$. Also, adding the $\lceil \frac{n}{2} \rceil - 1$ cliques $\{v_2, v_3\}, \{v_4, v_5\}, \ldots, \{v_{n-1}, v_n\}$ to a clique cover of $H - v_n$ produces a clique cover of G. Therefore, $\theta_v(G) \leq \theta_v(H - v_n) + \lceil \frac{n}{2} \rceil - 1 \leq es(H) - 1 + \lceil \frac{n}{2} \rceil - 1 = es(G)$ and the theorem follows. □

Theorem 17 *Let G be obtained from graph H by adding a path $P = \langle v_1, v_2, v_3 \rangle$ between adjacent vertices v_1 and v_3. If H and $H - \{v_1, v_3\}$ are maximum-demand graphs, then G is a maximum-demand graph.*

Proof: Clearly $\theta_v(G) \leq \theta_v(H) + 1$ and $es(H) \leq es(G) \leq es(H) + 1$. If $es(G) = es(H) + 1$, the result follows from the assumption that

$es(H) = \theta_v(H)$. Thus suppose $es(G) = es(H)$. Since v_1 and v_3 are adjacent, the contra-positive to Lemma 13 implies there exists an es-set S of G such that $|S \cap V(H)|$ is maximized and either v_1 or v_3 is not in S. Without loss of generality we assume $v_1 \notin S$. Since $|S \cap V(H)| = es(H) = es(G) = |S|$, the vertex $v_2 \notin S$. This implies that $v_3 \in S$ or v_2 would be unguarded. Note that v_3 cannot respond to any attack on $H-P$ without leaving v_2 unguarded. Thus, $S \cap V(H-P)$ is an eternally secure set for $H-P$ which implies $es(H-P) \leq es(H)-1$. By assumption, $\theta_v(H-P) = es(H-P)$. Thus $\theta_v(H-P) \leq es(H) - 1 = \theta_v(H) - 1$. Adding the clique $\{v_1, v_3\}$ to any vertex clique cover of $H-P$ creates a vertex clique cover of H, so $\theta_v(H) \leq \theta_v(H-P)+1$. We conclude that $\theta_v(H) = \theta_v(H-P)+1$ and there exists a minimum clique cover C of H such that $\{v_1, v_3\} \in C$. Then, $(C - \{v_1, v_3\}) \cup \{v_1, v_2, v_3\}$ is a clique cover of G of size $\theta_v(H)$. Thus, $\theta_v(G) \leq \theta_v(H) = es(H) = es(G)$ and the result follows. □

It clearly is not true in general that $es(G) = \theta_v(G)$ if G is obtained from H by adding a path of length one between two vertices of H, even if every subgraph K of H satisfies $es(K) = \theta_v(K)$. Simply consider the complement of the Grötzsch graph minus an edge. The only remaining case to consider is adding a path of length two between two independent vertices. This case remains open.

5 Eternal Security Number of K_4 Minor Free Graphs

A graph is said to be K_4 minor free if it contains no subgraph homeomorphic to K_4. Let $\mathcal{F}$ be the set of graphs that do not contain a K_4 minor. We begin with a lemma for 2-connected graphs in $\mathcal{F}$.

Lemma 18 *Suppose G is 2-connected and does not contain K_4 as a minor. Then either G is a cycle or G contains a cycle with exactly two vertices of degree greater than two.*

Proof: If G is not a cycle, then obtain G' from G by suppressing all divalent vertices. Note that G' is still 2-connected and does not contain K_4 as a minor. Therefore G' is a series-parallel multigraph and can be constructed recursively from a K_2 by the operations of subdividing and doubling edges (see Diestel [3], p. 185). By construction G' contains no divalent vertices; hence, G' contains parallel edges. Any two parallel edges in G' correspond to a cycle in G which satisfies the conclusion of the Lemma. □

Armed with this lemma, we proceed to the result.

Theorem 19 *Every graph which does not contain a K_4 minor is a maximum-demand graph.*

Proof: Suppose not. Let G be a minimal member of $\mathcal{F}$ which is not a maximum-demand graph, that is, G is not a maximum-demand graph but every proper subgraph of G is a maximum-demand graph. Note that $es(G)$ and $\theta_v(G)$ are the sums of the corresponding values of the components of G. Thus, since G is minimal in $\mathcal{F}$, G is connected. If G has a cutvertex then there exist nonempty subgraphs H and K of G such that $G = H \circ K$. Since G is minimal in $\mathcal{F}$; H, K, and all subgraphs of H and K are maximum-demand graphs. Thus, by Proposition 11, G is a maximum-demand graph, a contradiction. Thus, we may assume that G is 2-connected.

If G is a cycle, we are done. Otherwise let G be a 2-connected minimum member of $\mathcal{F}$ with respect to being a maximum-demand graph. By Lemma 18, G contains a cycle C with exactly two vertices, x and y, of degree greater than two. Let P_1 and P_2 be the two internally disjoint paths in C with ends x and y. Let l_i be the length of P_i and assume, without loss of generality, that $l_1 \geq l_2$. Let H be the subgraph of G which is obtained by deleting the edges and internal vertices of P_1. Since G is minimum in $\mathcal{F}$, H and all of its subgraphs are maximum-demand graphs. If $l_1 \geq 3$, Theorem 16 implies G is a maximum-demand graph, a contradiction. If $l_1 = 2$ and $l_2 = 1$, Theorem 17 shows G is a maximum-demand graph, again a contradiction. If $l_1 = 1$, then P_1 and P_2 are parallel edges which implies $\theta_v(G) = \theta_v(H)$ and $es(G) = es(H)$. Since H is a maximum-demand graph, so is G, yet another contradiction.

The only remaining case to consider is $l_1 = l_2 = 2$. Let $V(P_1) = \{x, u, y\}$ and $V(P_2) = \{x, v, y\}$. Let S be an es-set for G such that $S \cap V(H - C)$ is maximized. Let S' be the response produced from S by attacking u and v. These attacks cannot be responded to by vertices in $H - C$. Thus $S' \cap V(H - C) = S \cap V(H - C)$. By

Lemma 12 $S' \cap V(H-C)$ is an eternally secure set for $H-C$. Thus, $es(G) = es(S') \geq es(H-C)+2$. Adding $\{x,u\},\{y,v\}$ to any clique cover of $H-C$ produces a clique cover of G, so $\theta_v(G) \leq \theta_v(H-C)+2$. Since G is minimum in $\mathcal{F}$, $es(H-C) = \theta_v(H-C)$. Therefore, $\theta_v(G) \leq es(H-C)+2 \leq es(G)$ which implies $\theta_v(G) = es(G)$, once again contradicting the assumption that G is not a maximum-demand graph. The theorem now follows. □

The following corollary is immediate.

Corollary 20 *Every series-parallel graph is a maximum-demand graph.*

6 Open Questions

The study of eternal security seems to be both difficult and rife with interesting questions. Here are three.

1. Goddard et al [4] wonder if toroidal graphs are maximum-demand graphs. We take this down a notch and pose the same question for planar graphs.

2. If S is an eternally secure set, does S contain a minimum eternally secure subset?

3. Goddard et al [4] show that if $\beta_v(G) = 2$, then $es(G) \leq 3$ and they ask if there is a similar bound on $es(G)$ when $\beta_v(G) = 3$. We feel this question is worth repeating and offer the following partial solution: If $\beta_v(G) = 3$ and there exists an independent set of vertices $S = \{a_1, a_2, a_3\}$ which is in no induced $K_{3,3}$, then $es(G) \leq 15$. To see this let A_i be the set of vertices adjacent to a_i. Notice that $\{a_i\} \cup (A_i - (A_j \cup A_k))$ is complete and $\beta_v((A_i \cap A_j) - A_k) \leq 2$ for distinct i, j, and k. Furthermore, since S is in no induced $K_{3,3}$, $\beta_v(A_i \cap A_j \cap A_k) \leq 2$. By the Goddard et al result, $es((A_i \cap A_j) - A_k) \leq 3$ (three sets) and $es(A_i \cap A_j \cap A_k) \leq 3$. Adding one guard for each complete graph $\{a_i\} \cup (A_i - (A_j \cup A_k))$ yields the 15.

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