# Properties of Twisted Involutions in Signed Permutation Notation

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**Abstract**

In algebraic contexts Weyl group elements are usually represented in terms of generators and relations, where representation is not unique. For computational purposes, a more combinatorial representation for elements of classical Weyl groups as signed permutation vectors was introduced in [5]. This paper characterizes some special classes of Weyl group elements using this notation. These classes are especially useful for the study of symmetric spaces and their representations.

# 1 Introduction

Weyl groups are defined as reflection groups of root systems. That is, given a root system $\Phi$ in a Euclidian vector space $V = \mathbb{R}^n$, for each vector $\alpha \in \Phi$, let $s_\alpha$ be the reflection through $\alpha$. The Weyl group is the group generated by these reflections. In algebraic contexts, Weyl group elements are usually given in terms of generators and relations. While this is useful for many theoretical purposes, it is not efficient for computational problems since

*Partially supported by N.S.F. Grant DMS-0532140

†Supported by Susan M. Rambo Fund for undergraduate mathematics summer research at Smith College

representation of an element as a product of generators is not unique. For the classical Weyl groups the standard basis vectors $e_i$ in $\mathbb{R}^n$ will be mapped to $\pm e_j$ (where $i$ and $j$ may be equal). We use this to represent the Weyl group elements in a more combinatorial manner, as *signed permutations.*

In this paper we give several results characterizing special types of Weyl group elements. An involution $\hat{\theta}$ of the root system induces an involution $\theta$ on its Weyl group by $\theta(w) := \hat{\theta}w\hat{\theta}^{-1}$. If $\theta$ is an involution of $W$ then the set $\mathcal{I}_\theta = \{w \in W \mid \theta(w) = w^{-1}\}$ is called the set of $\theta$-twisted involutions in $W$. This set is important in the study of orbits of minimal parabolic $k$-subgroups acting on symmetric $k$-varieties, see [7, 8]. The geometry of these orbits and their closures induce a poset structure on the set $\mathcal{I}_\theta$. Understanding this poset structure is key to understanding the structure of the orbits. In order to describe the orbits we need to have a characterization of the twisted involutions. In this paper we give an explicit characterization of these elements, when written in signed permutation notation. For $\theta = \text{id}$, the identity map we get the set $\mathcal{I}_{\text{id}} = \{w \in W \mid w = w^{-1}\}$.

Many of these results are relatively easy to state and easy to prove. This is in contrast to similar results for Weyl groups using generators and relations. Thus these results provide additional evidence of the usefulness of this more combinatorial approach. This work will be used in a computational package for Weyl groups being developed by the first two authors. The signed permutation notation for the classical Weyl groups and the results in this paper can also be used to simplify the algorithms for symmetric spaces such as those found in [1, 2] and [3]. Those papers depend heavily on the presentation of the Weyl group in terms of generators and relations, but can be reframed.

# 2 Preliminaries and Notation

## 2.1 Classical Weyl Groups and Signed Permutations

In the Classical Weyl groups (those with root systems corresponding to Dynkin Diagrams of type A, B, C, and D,) each root is the sum or difference of at most two standard basis vectors $e_i$. Weyl groups of root systems of type $B$ and $C$ are the same, hence from now on we only discuss type $B$. An element $w \in W$ can be completely described by its action on the $e_i$. In particular, $w(e_i) = \pm e_j$ for all $i$, $j$ not necessarily different than $i$. For a nonzero real number $a$, define $\text{sgn}(a) = +\text{ if } a > 0$ and $\text{sgn}(a) = -\text{ if } a < 0$. We represent $w \in W$ by the vector $(a_1, a_2, \ldots a_n)$, where $w(e_i) = \text{sgn}(a_i)e_{|a_i|}$. That is, the vector corresponds to a signed permutation matrix where the ith column has its $\pm 1$ in the $a_i$th row, and the sign of $a_i$ equals the sign of that unique nonzero entry. Equivalently, starting with the permutation of the standard basis vectors in standard notation

$$\begin{pmatrix} 1 & 2 & \dots & n & -1 & -2 & \dots & -n \\ a_1 & a_2 & \dots & a_n & -a_1 & -a_2 & \dots & -a_n \end{pmatrix},$$

the signed permutation is just the first $n$ places in the bottom row. Hence the Weyl groups can be seen as subgroups of $S_{2n}$.

The standard generators for Weyl groups correspond to reflections. The Weyl group of type $A_{n-1}$ is generated by the transpositions $s_1 = (1,2), s_2 = (2,3), s_3 = (3,4), \dots, s_{n-1} = (n-1, n)$. Note this is $S_n$, the usual group of permutations on $n$ elements. The Weyl group of type $B_n$ is generated by the transpositions $s_1 = (1,2), s_2 = (2,3), s_3 = (3,4), \dots, s_{n-1} = (n-1, n), s_n = (n, -n)$. Note that if $w \in W$ of type $B_n$ then $w(-i) = -w(i)$ for any $i = 1, \dots, n$, justifying the use of only the first $n$ positions in the signed permutation notation. The Weyl group of type $D_n$ is generated by the transpositions $s_1 = (1,2), s_2 = (2,3), s_3 = (3,4), \dots, s_{n-1} = (n-1, n)$, and the element $s_n = (n, -(n-1))\,(n-1, -n)$ which is a product of transpositions. In this case as well, $w(-i) = -w(i)$ for any $i = 1, \dots, n$. To differentiate between the elements $s_n$ for $B_n$ and $D_n$ we will denote the signed permutation $(n, -(n-1))\,(n-1, -n)$ in $D_n$ by $s_{\hat{n}}$.

The following proposition from [6] describes multiplication by basis elements in signed permutation notation.

**Proposition 2.2.** *For* $1 \le i < n$,

*(i)* $(a_1, a_2, \dots, a_n)s_i = (a_1, a_2, \dots, a_{i+1}, a_i, \dots, a_n)$.

*(ii) If* $a_k = i$, *and* $a_l = i+1$ *then* $s_i(a_1, a_2, \dots, a_n) =$ $(a_1, a_2, \dots, a_{k-1}, \operatorname{sgn}(a_k)|a_l|, a_{k+1}, \dots, a_{l-1}, \operatorname{sgn}(a_l)|a_k|, a_{l+1}, \dots, a_n)$,

*(iii)* $(a_1, a_2, \dots, a_n)s_n = (a_1, a_2, \dots, a_{(n-1)}, -a_n)$.

*(iv)* $(a_1, a_2, \dots, a_n)s_{\hat{n}} = (a_1, a_2, \dots, -a_n, -a_{(n-1)})$.

*(v) If* $a_l = n$ *then* $s_n(a_1, a_2, \dots, a_n) = (a_1, a_2, \dots, -a_l, \dots, a_n)$.

*(vi) If* $a_k = n-1$, *and* $a_l = n$ *then* $s_{\hat{n}}(a_1, a_2, \dots, a_k, \dots a_l, \dots, a_n) =$ $(a_1, a_2, \dots, -\operatorname{sgn}(a_k)|a_l|, \dots, -\operatorname{sgn}(a_l)|a_k|, \dots, a_n)$.

## 2.3 Involutions of Weyl groups

As shown in [4] it is sufficient to consider only those involutions $\hat{\theta}$ that fix the basis of the root system, and hence involutions $\theta$ that fixes the set of generators of $W$. For type $A_{n-1}$ the only non-trivial involution that fixes the basis of root system is defined by $\hat{\theta}(i) = (n+1-i)$, for $1 \le i \le n$. For type $B_n$ there are no non-trivial involutions that fixes the basis of root system. For type $D_n$ the only non-trivial involution that fixes the basis of root system is defined by $\hat{\theta}(i) = i$, for $1 \le i \le n-1$, and $\hat{\theta}(n) = -n$.

# 3 Results

We characterize several special types of elements in the signed permutation notation. We begin by characterizing inverses.

**Proposition 3.1.** *$(a_1, a_2, \dots a_n)^{-1} = (b_1, b_2, \dots b_n)$, if $|b_{|a_i|}| = i$, and the* $\operatorname{sgn}(b_i) = \operatorname{sgn}(a_{|b_i|})$.

For example, $(4, -1, 2, -3)^{-1} = (-2, 3, -4, 1)$. Consider $(a_1, a_2, \dots a_n)$ as denoting the position and sign of the unique $\pm 1$ in each column of the matrix $A$, then the inverse element is easily read off as the position and sign of the unique $\pm 1$ in each row of $A$.

**Theorem 3.2.** *(i) For $w \in W$ a Weyl group of type $A_{n-1}$, $\theta(w) = w$, if and only if $w = (a_1, a_2, ..., a_n)$ satisfies $\hat{\theta}(a_i) = a_{n-i+1}$ for all $i$.*

*(ii) For $w \in W$ a Weyl group of type $D_n$, $\theta(w) = w$, if and only if $w = (a_1, a_2, ..., a_n)$ satisfies $a_n = \pm n$.*

*Proof.* Recall that $\theta(w) = \hat{\theta} w \hat{\theta}$. For type $A_{n-1}$, $\theta((a_1, a_2, \dots, a_n)) = (\hat{\theta}(a_n), \hat{\theta}(a_{n-1}), \cdots, \hat{\theta}(a_1))$ and result (i) follows. For type $D_n$, if $a_k = \pm n$ then $\theta((a_1, a_2, \dots, a_n)) = (a_1, a_2, \dots, a_{k-1}, -a_k, a_{k+1}, \dots, a_{n-1}, -a_n)$ and result (ii) follows. □

It is difficult to tell whether an element given by generators is in $\mathcal{I}_{\mathrm{Id}}$ or $\mathcal{I}_\theta$. On the other hand, we can quickly see if an element given as a signed permutation vector is in $\mathcal{I}_{\mathrm{Id}}$.

**Lemma 3.3.** *An element $w = (a_1, a_2, \dots a_n)$ is in $\mathcal{I}_{\mathrm{Id}}$ if and only if $|a_{|a_i|}| = i$ for all $i$, and* $\operatorname{sgn}(a_i) = \operatorname{sgn}(a_{|a_i|})$.

*Proof.* This follows immediately from the characterization of inverses in Proposition 3.1 above. □

Next we give a characterization of the signed permutation vectors that are contained in $\mathcal{I}_\theta$.

**Theorem 3.4.** *Let $W$ be Weyl group of type $A_n$, and $\theta$ the nontrivial diagram involution. An element $w = (a_1, a_2, \dots a_n) \in W$ is in $\mathcal{I}_\theta$ if and only if $a_{(n+1)-a_i} = (n+1) - i$ for all $i$.*

*Proof.* First, if $w^{-1} = (b_1, \dots, b_n)$ then $b_{|a_i|} = i$ for all $i$. Additionally, if $\theta(w) = (c_1, \dots, c_n)$ then $c_i = (n+1) - a_{n+1-i}$, or equivalently, $c_{n+1-i} = n + 1 - a_i$, for all $i$. Now, $a_{n+1-a_i} = n + 1 - i$ if and only if $b_{n+1-i} = c_{n+1-i} = n + 1 - a_i$. □

**Theorem 3.5.** *Let $W$ be Weyl group of type $D_n$, and $\theta$ the nontrivial diagram involution. An element $w = (a_1, a_2, \ldots a_n)$ is in $\mathcal{I}_\theta$ if and only if each of the following conditions holds:*

*(i) For $i \neq n$, if $a_i = n$ then $a_n = -i$.*

*(ii) For $i \neq n$, if $a_i = -n$ then $a_n = i$.*

*(iii) For $i \neq n$, and $j \neq n$, if $a_i = j$ then $a_j = i$.*

*(iv) For $i \neq n$ and $j \neq n$, if $a_i = -j$ then $a_j = -i$.*

*(v) No restrictions if $a_i = \pm i$ (even if $i = n$)*

*Proof.* First, note that if $w^{-1} = (b_1, \ldots, b_n)$ then $b_{|a_i|} = i \operatorname{sgn} a_i$ for all $i$. Additionally, $\theta(w) = (c_1, \ldots, c_n)$ is completely determined by

$$c_i = a_i \begin{cases} \operatorname{sgn}(a_i) \text{ if } i \neq n \text{ and } |a_i| \neq n \\ -\operatorname{sgn}(a_i) \text{ if one of } i = n \text{ or } |a_i| = n \\ \operatorname{sgn}(a_i) \text{ if } i = n \text{ and } |a_i| = n \end{cases}$$

Suppose $c_i = b_i$. This implies $|a_{|a_i|}| = i$ for all $i$, that is $|a_i| = j$ if and only if $|a_j| = i$. As well,

$$\operatorname{sgn}(a_i) = \begin{cases} \operatorname{sgn}(a_{|a_i|}) \text{ if } i \neq n \text{ and } |a_i| \neq n \\ -\operatorname{sgn}(a_i) \text{ if one of } i = n \text{ or } |a_i| = n \\ \operatorname{sgn}(a_i) \text{ if } i = n \text{ and } |a_i| = n \end{cases}$$

□

In [5] we developed an algorithm for computing the twisted (and non-twisted) involutions. The algorithms generates the $\mathcal{I}_\theta$ and $\mathcal{I}_{\text{id}}$ posets discussed in the introduction. Figure 1 gives the poset of the elements of $\mathcal{I}_\theta$ for the Weyl group of type $A_4$. From each element there is an edge for each group generator. The edges will correspond to one of two possible operations. The following lemmas give the two possibilities. In Figure 1, the operation in Lemma 3.6 is designated by a solid line while the operation in Lemma 3.7 is designated by a dashed line.

**Lemma 3.6.** *Given a Weyl group $W$, $w \in W$, an involution $\theta$ and a basis element $s_i$.*

*(i) If $w \in \mathcal{I}_{id}$ then $s_i w s_i \in \mathcal{I}_{id}$.*

*(ii) If $w \in \mathcal{I}_\theta$ then $s_i w \theta(s_i) \in \mathcal{I}_\theta$.*

**Lemma 3.7.** *Given a Weyl group $W$, $w \in W$, an involution $\theta$ and a basis element $s_i$.*

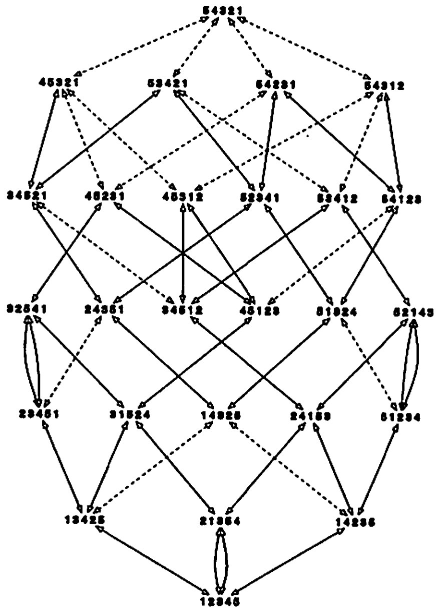

Figure 1: Poset of $\mathcal{I}_\theta$ for type $A_4$

*(i) If $w \in \mathcal{I}_{id}$ and $s_i w s_i = w$, then $s_i w \in \mathcal{I}_{id}$*

*(ii) If $w \in \mathcal{I}_\theta$ and $s_i w \theta(s_i) = w$, then $s_i w \in \mathcal{I}_\theta$*

The next theorems characterizes when each of the above operations will be used. These lead to improved algorithms for computing the involution posets.

**Theorem 3.8.** *Let $W$ be a Weyl group of type $A_{n-1}, B_n$ or $D_n$. For $i < n$, if $w \in W$ then $s_i w s_i = w$ if and only if $w = (a_1, \dots a_n)$ satisfies $\{|a_i|, |a_{i+1}|\} = \{i, i+1\}$ and $\text{sgn}(a_i) = \text{sgn}(a_{i+1})$.*

*For Weyl group of type $B_n$, then $s_n w s_n = w$ if and only if $w = (a_1, \dots a_n)$ satisfies $a_n = \pm n$. For Weyl group of type $D_n$, then $s_{\hat{n}} w s_{\hat{n}} = w$ if and only if $w = (a_1, \dots a_n)$ satisfies $\{|a_{n-1}|, |a_n|\} = \{n-1, n\}$, and $\text{sgn}(a_{n-1}) = \text{sgn}(a_n)$.*

*Proof.* From Proposition 2.2 if $i < n$ then $s_i w$ acts on $w$ by switching the elements which are equal to $\pm i$ and $\pm(i+1)$ but leaving their signs in the old positions, while $w s_i$ switches the elements in the $i$th and $(i+1)$st positions and their signs as well. Hence $s_i w s_i = w$ if and only if $\{a_i, a_{i+1}\} = \{i, i+1\}$ and $\text{sgn}(a_i) = \text{sgn}(a_{i+1})$. Similarly, multiplying on the right and left by

$s_n$ changes the sign of the element in the $n$th position, and the sign of the element that is $\pm n$. Finally, multiplying on the left and right by $s_{\hat{n}}$ switches signs of the elements in the $(n-1)$th and $n$th positions and that of whose absolute values are $n-1$ and $n$ so $s_{\hat{n}} w s_{\hat{n}} = w$ if and only if $w = (a_1, \ldots a_n)$ satisfies $\{|a_{n-1}|, |a_n|\} = \{n-1, n\}$, with perhaps some further condition on the signs. To see that signs must be the same, compute according to the rules of proposition 2.2.

$$\begin{aligned} s_{\hat{n}}(a_1, a_2, \ldots, a_n)s_{\hat{n}} &= s_{\hat{n}}(a_1, a_2, \ldots, -a_n, -a_{(n-1)}) \\ &= (a_1, a_2, \ldots, \operatorname{sgn}(a_n)|a_{n-1}|, \operatorname{sgn}(a_{n-1})|a_n|) \end{aligned}$$

□

The next theorem gives the twisted case.

**Theorem 3.9.** *Let $W$ be a Weyl group of type $A_{n-1}$. For $i < n$, if $w \in W$ then $s_i w\theta(s_i) = w$ if and only if $w = (a_1, \ldots a_n)$ satisfies $\{a_{(n-i)}, a_{(n-i+1)}\} = \{i, i+1\}$.*

*For $W$ of type $D_n$, $i < n-1$, $s_i w\theta(s_i) = w$ if and only if $w = (a_1, \ldots a_n)$ satisfies $\{|a_i|, |a_{i+1}|\} = \{i, i+1\}$ and $\operatorname{sgn}(a_i) = \operatorname{sgn}(a_{i+1})$. For $W$ of type $D_n$, and $i = n-1$ or $i = n$, then $s_i w\theta(s_i) = w$ if and only if $w = (a_1, \ldots a_n)$ satisfies $\{|a_{n-1}|, |a_n|\} = \{n-1, n\}$, and $\operatorname{sgn}(a_{n-1}) = -\operatorname{sgn}(a_n)$.*

*Proof.* For the $A_{n-1}$ case, observe $\theta(s_i) = s_{n-i}$. Proceed similarly to the proof of theorem 3.8. From Proposition 2.2 if $i < n$ then $s_i w$ acts on $w$ by switching the elements which are equal to $i$ and $(i+1)$, while $w\theta(s_i) = ws_{n-i}$ switches the elements in the $(n-i)$th and $(n-i+1)$st positions. Hence $s_i w\theta(s_i) = w$ if and only if $\{a_{(n-i)}, a_{(n-i+1)}\} = \{i, i+1\}$.

For the $D_n$ case, note first that $\theta(s_i) = s_i$ if $i < n-1$ so in these cases $s_i w\theta(s_i) = s_i w s_i$ and we get the same result as in theorem 3.8. Now $\theta(s_{n-1}) = s_{\hat{n}}$. Both $s_{n-1}$ and $s_{\hat{n}}$, operate on the elements $(n-1)$ and $n$ and the $(n-1)$st and $n$th positions. So that $s_{\hat{n}} w s_{n-1} = w$ and $s_{n-1} w s_{\hat{n}} = w$ if and only if $w = (a_1, \ldots a_n)$ satisfies $\{|a_{n-1}|, |a_n|\} = \{n-1, n\}$, with perhaps some further condition on the signs. To see that signs must be opposite, compute according to the rules of proposition 2.2.

$$\begin{aligned} s_{n-1}(a_1, a_2, \ldots, a_n)s_{\hat{n}} &= s_{n-1}(a_1, a_2, \ldots, -a_n, -a_{(n-1)}) \\ &= (a_1, a_2, \ldots, -\operatorname{sgn}(a_n)|a_{n-1}|, -\operatorname{sgn}(a_{n-1})|a_n|). \end{aligned}$$

A similar calculation shows the $s_{\hat{n}} w s_{n-1} = w$ also requires that $\operatorname{sgn}(a_{n-1}) = -\operatorname{sgn}(a_n)$. □

While both of the above theorems hold for all $w \in W$, we are in fact most interested in the case when $w \in \mathcal{I}_{\text{id}}$, or $w \in \mathcal{I}_\theta$. It is interesting to note that one of the conditions for $w$ in Theorem 3.9 is that $\operatorname{sgn}(a_{n-1}) = -\operatorname{sgn}(a_n)$ which matches the criterion in Theorem 3.5, namely, if $w \in \mathcal{I}_\theta$ then $\operatorname{sgn}(a_{n-1}) = -\operatorname{sgn}(a_n)$.

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