On Partial Chromatic Ordinomials

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Abstract

We introduce the ring of ordinomials which will be utilized in defining the partial chromatic ordinomials of infinite graphs with certain properties - a generalization of chromatic polynomials of finite graphs.

1 The Ring of Ordinomials

Let $\mathbb{O}$ be the class of all ordinal numbers and $\mathbb{O}^* = \mathbb{O} - \{0\}$. For more information on ordinal numbers and their arithmetic see [4]. $\mathbb{N}$ denotes that set of all positive integers while $\mathbb{N}_0$ is the set of all natural numbers.

First, we will define a free multiplication on the set $\mathbb{B} = \{(\lambda - \zeta)^\alpha | \zeta \in \mathbb{C}, \alpha \in \mathbb{O}^*\} \cup \{1\}$ of formal objects by putting these building blocks next to each other, satisfying the following axioms:

$A_1)(\forall x_i \in \mathbb{B}) \left(x_1 x_2 = x_2 x_1\right)$;

$A_2)(\forall x_i \in \mathbb{B}) \left((x_1 x_2) x_3 = x_1 (x_2 x_3)\right)$;

$A_3)(\forall x \in \mathbb{B}) \left(x 1 = x\right)$;

$A_4)(\forall \zeta \in \mathbb{C})(\forall \alpha_i \in \mathbb{O}^*)$

$$\left(\alpha_1 \leq \alpha_2\right) \rightarrow \left((\lambda-\zeta)^{\alpha_1}(\lambda-\zeta)^{\alpha_2} = (\lambda-\zeta)^{\alpha_2}(\lambda-\zeta)^{\alpha_1} = (\lambda-\zeta)^{\alpha_1+\alpha_2}\right).$$

To clarify how A_4 works, we will give a few examples:

i) For $k, l \in \mathbb{N}$ such that $k \leq l$, $(\lambda-\zeta)^k(\lambda-\zeta)^l = (\lambda-\zeta)^l(\lambda-\zeta)^k = (\lambda-\zeta)^{k+l}$.

ii) For $k \in \mathbb{N}$, $(\lambda-\zeta)^\omega(\lambda-\zeta)^k = (\lambda-\zeta)^k(\lambda-\zeta)^\omega = (\lambda-\zeta)^{k+\omega} = (\lambda-\zeta)^\omega$.

iii) $(\lambda-\zeta)^\omega(\lambda-\zeta)^{\omega\cdot 2} = (\lambda-\zeta)^{\omega\cdot 2}(\lambda-\zeta)^\omega = (\lambda-\zeta)^{\omega+\omega\cdot 2} = (\lambda-\zeta)^{\omega\cdot 3}$.

iv) $(\lambda-\zeta)^\omega(\lambda-\zeta)^{\omega^2} = (\lambda-\zeta)^{\omega^2}(\lambda-\zeta)^\omega = (\lambda-\zeta)^{\omega+\omega^2} = (\lambda-\zeta)^{\omega^2}$.

Theorem 1.1 *If $\mathbb{B}[\lambda]$ denotes the set $\{\prod_{i=1}^{n} x_i \mid n \in \mathbb{N}, x_i \in \mathbb{B}\}$ defined using the above multiplication, then $\mathbb{B}[\lambda]$ is a free commutative monoid.*■

An element x of $\mathbb{B}[\lambda]$ is called a *reduced element*, when x is either 1 or $\prod_{i=1}^{n}(\lambda - \zeta_i)^{\alpha_i}$, for distinct ζ_i's. By *reduced form* of $y \in \mathbb{B}[\lambda]$, we mean a reduced element x, such that y is equal to, by applying axioms A_1, A_3, and A_4, finitely many times. Consequently, reduced elements $\prod_{i=1}^{n}(\lambda - \zeta_i)^{\alpha_i}$ and $\prod_{j=1}^{m}(\lambda - \xi_i)^{\beta_j}$ are equal, when $n = m$ and there exist $\sigma \in S_n$ such that $\zeta_i = \xi_{\sigma(i)}$ and $\alpha_i = \beta_{\sigma(i)}$, for $i \in \{1, 2, \ldots, n\}$. By convention, we will assume that 1 is not equal to any other reduced element.

Theorem 1.2 *Every element of $\mathbb{B}[\lambda]$ has a reduced form which is unique.*■

From now on, we will assume every $x \in \mathbb{B}[\lambda]$ is a reduced element. For every $x \in \mathbb{B}[\lambda]$, $1 \mid x$ and if $x = \prod_{i=1}^{n}(\lambda - \zeta_i)^{\alpha_i}$, then for $\beta \in \mathbb{O}^*$ and $\zeta \in \mathbb{C}$, $(\lambda - \zeta)^{\beta} \mid x$, when there exist $i \in \{1, 2, \ldots, n\}$, such that $\zeta = \zeta_i$ and $\beta \leq \alpha_i$. Let $y = \prod_{j=1}^{m} y_j$, then $y \mid x$, if $m \leq n$ and $y_j \mid x$, for every j. By convention, if $x \mid 1$, then $x = 1$. If $x, y \in \mathbb{B}[\lambda]$, then we will define the greatest common divisor of x and y, denoted by (x, y), as the element z such that $z \mid x$ and $z \mid y$, and if $w \mid x$ and $w \mid y$, then $w \mid z$.

Assume that for $y = \prod_{j=1}^{m}(\lambda - \xi_j)^{\beta_j}$ and $x = \prod_{i=1}^{n}(\lambda - \zeta_i)^{\alpha_i}$, $y \mid x$. So, by definition $m \leq n$ and $(\lambda - \xi_j)^{\beta_j} \mid \prod_{i=1}^{n}(\lambda - \zeta_i)^{\alpha_i}$, for every j. Without loss of generality, assume that $\xi_j = \zeta_j$ and as a result, $\beta_j \leq \alpha_j$; Hence, there exist γ_j such that $\beta_j + \gamma_j = \alpha_j$. By using these facts, we have $x = y\,(\prod_{i=1}^{m}(\lambda - \zeta_i)^{\gamma_i})\,(\prod_{i=m+1}^{n}(\lambda - \zeta_i)^{\alpha_i})$. So, when $y \mid x$, there exist z such that $x = yz$.

Our next goal is to define a degree function, denoted by $deg(x)$, for every element x of $\mathbb{B}[\lambda]$. Conventionally, $deg(1) = 0$. For $x = \prod_{i=1}^{n}(\lambda - \zeta_i)^{\alpha_i}$, let $\beta_1, \beta_2, \ldots, \beta_n$ be the rearrangement of $\alpha_1, \alpha_2, \ldots, \alpha_n$, such that $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_n$. Then, $deg(x)$ is equal to the sum $\beta_1 + \beta_2 + \beta_3 + \ldots + \beta_n$. We will call x *purely infinite*, when β_1 is an infinite ordinal, which clearly implies $deg(x)$ is infinite. As $deg(\ )$ is now a well-defined mapping from $\mathbb{B}[\lambda]$ to $\mathbb{O}$, it is clear that for $x, y \in \mathbb{B}[\lambda]$, if $deg(x) \neq deg(y)$, then $x \neq y$.

If for $x \in \mathbb{B}[\lambda]$, $deg(x)$ is infinite, then it is easy to see that x can be uniquely decomposed into the form yz such that y is purely infinite, $deg(z) \in \mathbb{N}_0$, and $(y, z) = 1$. In the case that x is purely infinite, it is clear that the decomposition would be $x\,1$.

Now it is time to equip $\mathbb{B}[\lambda]$ with a complex scalar product, defined as $\tau \cdot x = (\tau, x) \in \mathbb{C} \times \mathbb{B}[\lambda]$ and having the following properties:

$P_1)(\forall \tau \in \mathbb{C})(\forall x \in \mathbb{B}[\lambda])((\,deg(x) \in \mathbb{N}_0\,) \rightarrow (\,\tau \cdot x = \tau x \in \mathbb{C}[\lambda]\,))$;

[Particularly, $\tau \cdot 1 = \tau$ and to be more specific, $0 = 0 \cdot 1$ and $1 = 1 \cdot 1$]
$P_2)(\forall x \in \mathbb{B}[\lambda])((0 \cdot x = 0) \wedge (1 \cdot x = x))$;
$P_3)(\forall \tau \in \mathbb{C})(\forall x \in \mathbb{B}[\lambda])((\tau \cdot x = 0) \rightarrow (\tau = 0))$;
$P_4)(\forall \tau_i \in \mathbb{C}^*)(\forall x_i \in \mathbb{B}[\lambda])((\tau_1 \cdot x_1 = \tau_2 \cdot x_2) \rightarrow ((\tau_1 = \tau_2) \wedge (x_1 = x_2)))$.

We will define a free addition and a multiplication on elements of $\mathbb{C} \times \mathbb{B}[\lambda]$ having the following properties in order to introduce the desired free ring.
Addition:
$P_1^+)(\forall \tau_i \in \mathbb{C})(\forall x_i \in \mathbb{B}[\lambda])\ (\tau_1 \cdot x_1 + \tau_2 \cdot x_2 = \tau_2 \cdot x_2 + \tau_1 \cdot x_1)$;
$P_2^+)(\forall \tau_i \in \mathbb{C})(\forall x_i \in \mathbb{B}[\lambda])\ (\tau_1 \cdot x_1 + (\tau_2 \cdot x_2 + \tau_3 \cdot x_3) = (\tau_1 \cdot x_1 + \tau_2 \cdot x_2) + \tau_3 \cdot x_3)$;
$P_3^+)(\forall \tau_i \in \mathbb{C})(\forall x \in \mathbb{B}[\lambda])\ (\tau_1 \cdot x + \tau_2 \cdot x = (\tau_1 + \tau_2) \cdot x)$;
$P_4^+)(\forall \tau \in \mathbb{C})(\forall x \in \mathbb{B}[\lambda])\ (\tau \cdot x + 0 = \tau \cdot x)$;
$P_5^+)(\forall \tau \in \mathbb{C})(\forall x \in \mathbb{B}[\lambda])\ (\tau \cdot x + (-\tau) \cdot x = 0)$.
Multiplication:
$P_1^\times)(\forall \tau_i \in \mathbb{C})(\forall x_i \in \mathbb{B}[\lambda])\ ((\tau_1 \cdot x_1) \times (\tau_2 \cdot x_2) = (\tau_1\tau_2) \cdot (x_1 x_2))$;
$P_2^\times)(\forall \tau_i \in \mathbb{C})(\forall x_i \in \mathbb{B}[\lambda])$
$\quad (\tau_1 \cdot x_1 \times (\tau_2 \cdot x_2 + \tau_3 \cdot x_3) = (\tau_1 \cdot x_1 \times \tau_2 \cdot x_2) + (\tau_1 \cdot x_1 \times \tau_3 \cdot x_3))$,
$\quad ((\tau_1 \cdot x_1 + \tau_2 \cdot x_2) \times \tau_3 \cdot x_3 = (\tau_1 \cdot x_1 \times \tau_3 \cdot x_3) + (\tau_2 \cdot x_2 \times \tau_3 \cdot x_3))$;
$P_3^\times)(\forall \tau \in \mathbb{C})(\forall x \in \mathbb{B}[\lambda])\ (\tau \cdot x \times 1 = 1 \times \tau \cdot x = \tau \cdot x)$;
$P_4^\times)(\forall \tau_i \in \mathbb{C})(\forall x_i \in \mathbb{B}[\lambda])\ (\tau_1 \cdot x_1 \times (\tau_2 \cdot x_2 \times \tau_3 \cdot x_3) = ((\tau_1 \cdot x_1 \times \tau_2 \cdot x_2) \times \tau_3 \cdot x_3))$;
$P_5^\times)(\forall \tau_i \in \mathbb{C})(\forall x_i \in \mathbb{B}[\lambda])\ (\tau_1 \cdot x_1 \times \tau_2 \cdot x_2 = \tau_2 \cdot x_2 \times \tau_1 \cdot x_1)$.
It is important to remark that one can deduce P_5^+ from P_1^+, and $P_4^\times$ and $P_5^\times$ from $P_1^\times$ and properties of $\mathbb{B}[\lambda]$ and $\mathbb{C}$. It is easy to check that for every $\tau \in \mathbb{C}$ and $x \in \mathbb{B}[\lambda]$, $0 \times (\tau \cdot x) = 0$. The free ring we are looking for is defined as

$$\mathbb{O}^*[\lambda] = \{ \sum_{i=1}^{n} \tau_i \cdot x_i \,|\, n \in \mathbb{N}, \tau_i \in \mathbb{C}, x_i \in \mathbb{B}[\lambda] \}.$$

Theorem 1.3 *$\mathbb{O}^*[\lambda]$ is a free commutative ring having a multiplicative identity.*■

From now on, an element of $\mathbb{O}^*[\lambda]$ will be called an *ordinomial* and the ring itself will be the *ring of ordinomials* as it is a free ring constructed in a similar fashion as the familiar polynomial rings were and it also utilizes ordinal arithmetic.

Theorem 1.4 *$\mathbb{C}[\lambda]$ is a subring of $\mathbb{O}^*[\lambda]$.*

Proof: Let $p(\lambda)$ be an element of $\mathbb{C}[\lambda]$. It is clear from P_1 and the fundamental theorem of algebra that $p(\lambda) \in \mathbb{O}^*[\lambda]$, and thus $\mathbb{C}[\lambda] \subset \mathbb{O}^*[\lambda]$. Now, define $g : \mathbb{C}[\lambda] \hookrightarrow \mathbb{O}^*[\lambda]$ that maps every element $p(\lambda) = \tau_n\lambda^n + \cdots + \tau_2\lambda^2 + \tau_1\lambda + \tau_0$ ($n \in \mathbb{N}$, $\tau_i \in \mathbb{C}$ and $\tau_n \neq 0$) of $\mathbb{C}[\lambda]$ to $\tau_n \cdot \lambda^n + \cdots + \tau_2 \cdot \lambda^2 + \tau_1 \cdot \lambda + \tau_0 \cdot 1$ of $\mathbb{O}^*[\lambda]$. It is easy to check that g is a well-defined mapping and for every $p(\lambda)$ and $q(\lambda)$, $g(p(\lambda) + q(\lambda)) = g(p(\lambda)) + g(q(\lambda))$ and $g(p(\lambda)q(\lambda)) = g(p(\lambda)) \times g(q(\lambda))$.■

Let's assume that for $x = \sum_{i=1}^{n} \tau_i \cdot x_i$, $F_x \subseteq \{1, 2, \ldots, n\}$ is the set of all i's such that $deg(x_i)$ is finite. If F_x is non-empty, then for $i \in F_x$, $\tau_i \cdot x_i$ is in $\mathbb{C}[\lambda]$ and so is $\sum_{i \in F_x} \tau_i \cdot x_i$. Without loss of generality, assume that, for $1 \leq k \leq n$, $F_x = \{k, k+1 \ldots, n\}$. As $\sum_{i \in F_x} \tau_i \cdot x_i$ is in $\mathbb{C}[\lambda]$, three cases may occur: it is equal to zero, a non-zero complex number, or a polynomial. If it is equal to zero, then we will omit it from x and rewrite x as $\sum_{i=1}^{k-1} \tau_i \cdot x_i$; If it is a non-zero complex number, say τ, we will rewrite x as $\sum_{i=1}^{k-1} \tau_i \cdot x_i + \tau \cdot 1$; Finally, if it is a polynomial $\tau \cdot y$ with $y \in \mathbb{B}[\lambda]$, we can rewrite x as $\sum_{i=1}^{n} \tau \cdot x_i = \sum_{i=1}^{k-1} \tau \cdot x_i + \tau \cdot y$.

With regard to the above observation, we call an ordinomial x *polynomially reduced*, when the set F_x defined above is either empty or a singleton. From definition of $0 = 0 \cdot 1$, $F_0 = \{1\}$. By *polynomially reduced form* of y, we mean a polynomially reduced ordinomial x which y is equal to using the above process. Thus,

Theorem 1.5 *Every element of $\mathbb{O}^*[\lambda]$ is equal to a polynomially reduced ordinomial.*■

Now, let $x = \sum_{i=1}^{n} \tau_i \cdot x_i$ be a polynomially reduced ordinomial such that either F_x is empty, or if not, $n > 1$. So, either $deg(x_i)$ is infinite for all i, or for all except one index which we assume it is equal to n. If the former happens, we know that every x_i can be decomposed into $y_i z_i$ such that $deg(y_i)$ is infinite, $deg(z_i) \in \mathbb{N}_0$, and $(y_i, z_i) = 1$. If the latter happens, we will do the same except for x_n. If all y_i's are distinct, we will call x a *reduced ordinomial.* Now, assume they are not all distinct. We will partition the set of indices into $S_1, S_2, \ldots, S_m$ while S_i contains all the indices i such that y_i's are equal; In latter case, we will assume that $S_m = \{n\}$. Then, we have $x = \sum_{i \in S_1} \tau_i \cdot x_i + \sum_{i \in S_2} \tau_i \cdot x_i + \cdots + \sum_{i \in S_m} \tau_i \cdot x_i$. We assume for $1 \leq j \leq m$, $y'_j = y_i$ such that $i \in S_i$; In latter case, we will assume that $y'_m = 1$. So, we can rewrite x as $y'_1 \times (\sum_{i \in S_1} \tau_i \cdot z_i) + y'_2 \times (\sum_{i \in S_2} \tau_i \cdot z_i) + \cdots + y'_m \times (\sum_{i \in S_m} \tau_i \cdot z_i)$. For $1 \leq j \leq m$, $\sum_{i \in S_j} \tau_i \cdot z_i$ is a member of $\mathbb{C}[\lambda]$: If it is zero, then $y_j \times (\sum_{i \in S_j} \tau_i \cdot z_i) = 0$; If it is a non-zero complex ϕ_j, then $y'_j \times (\sum_{i \in S_j} \tau_i \cdot z_i) = y'_j \times \phi_j = \phi_j \cdot y'_j$; Finally, it is equal to a polynomial $\phi_j \cdot z'_j$, then $y'_j \times (\sum_{i \in S_j} \tau_i \cdot z_i) = y'_j \times \phi_j \cdot z'_j = \phi_j \cdot y'_j z'_j = \phi_j \cdot y'_j z''_j$ in which z_j'' is an element of $\mathbb{B}[\lambda]$ such that $y_j'z_j''$ is the reduced form of $y_j'z_j'$ and $(y_j', z_j'') = 1$. By rewriting x as described, x will either be equal to zero or a reduced ordinomial. When $n = 1$ and F_x is not empty, then x is a polynomially reduced element of $\mathbb{C}[\lambda]$ which would be a reduced ordinomial by definition; Hence, 0 is a reduced ordinomial.

By the *reduced form* of an ordinomial y, we mean a reduced ordinomial x such that y is equal to x by going through the processes described above. If y becomes zero, we will call y a *degenerate ordinomial*; Otherwise, it is called a *non-degenerate ordinomial.*

Theorem 1.6 *Every element of* $\mathbb{O}^*[\lambda]$ *is equal to a reduced ordinomial.*■

Let $x = \sum_{i=1}^{n} \tau_i \cdot x_i$ be a non-degenerate ordinomial such that F_x is empty and for every i, x_i is a purely infinite element of $\mathbb{B}[\lambda]$; Then, we will call x a *purely infinite ordinomial.* When, only F_x is empty, x is just called an *infinite ordinomial.*

If $\sum_{i=1}^{n} \tau_i \cdot x_i$ is a non-degenerate ordinomial, then so is $\sum_{i=1}^{n}(-\tau_i) \cdot x_i$. Moreover, It is clear that when one is purely infinite (or just infinite), so would be the other.

Theorem 1.7 *Every element of* $\mathbb{O}^*[\lambda]$ *is either equal to 0 or a unique non-degenerate ordinomial up to permutation of indices.*

Proof: We only have to prove that if non-degenerate ordinomials $x = \sum_{i=1}^{n} \tau_i \cdot x_i$ and $y = \sum_{j=1}^{m} \phi_j \cdot y_j$ are equal, then $n = m$ and there exist $\sigma \in S_n$ such that $\tau_i = \phi_{\sigma(i)}$ and $x_i = y_{\sigma(i)}$ for all i. Three cases may occur: 1) F_x and F_y are both empty which is the case when they are both infinite ordinomials; 2) F_x and F_y are non-empty sets and we assume $F_x = \{n\}$ and $F_y = \{m\}$; 3) Just one of F_x or F_y is empty. We start with the first case:

Let $\sum_{i=1}^{n} \tau_i \cdot x_i$ and $\sum_{j=1}^{m} \phi_j \cdot y_j$ be infinite non-degenerate ordinomials such that $x_i = x_i'x_i''$ and $y_j = y_j'y_j''$ are the unique decomposition of x_i and y_j; Clearly, $(x_i', x_i'') = 1$ and $(y_j', y_j'') = 1$. Now assume that x_i''s and y_j''s are all distinct elements of $\mathbb{B}[\lambda]$. It is not hard to see that the sum $\sum_{i=1}^{n} \tau_i \cdot x_i + \sum_{j=1}^{m} \phi_j \cdot y_j = \sum_{i=1}^{n} \tau_i \cdot (x_i'x_i'') + \sum_{j=1}^{m} \phi_j \cdot (y_j'y_j'')$ is also an infinite non-degenerate ordinomial.

Now, assume that infinite non-degenerate ordinomials $\sum_{i=1}^{n} \tau_i \cdot x_i$ and $\sum_{j=1}^{m} \phi_j \cdot y_j$ are equal to each other. Let x_i', x_i'', y_j' and y_j'' be defined as was defined in the previous paragraph. We can write $\sum_{i=1}^{n} \tau_i \cdot x_i + \sum_{j=1}^{m}(-\phi_j) \cdot y_j = \sum_{i=1}^{n} \tau_i \cdot (x_i'x_i'') + \sum_{j=1}^{m}(-\phi_j) \cdot (y_j'y_j'') = 0$, and clearly there exist i_1 and j_1, such that $x_{i_1}' = y_{j_1}'$. As x_i''s and y_j''s are all distinct, x_{i_1}' is only equal to y'_{j_1}, and vise versa. Without loss of generality, assume $i_1 = n$ and $j_1 = m$ and $z = x'_n = y'_m$. So, $\sum_{i=1}^{n-1} \tau_i \cdot (x'_i x''_i) + \sum_{j=1}^{m-1} (-\phi_j) \cdot (y'_j y''_j) + z \times (\tau_n \cdot x''_n + (-\phi_m) \cdot y''_n) = 0$. If $\tau_n \cdot x''_n + (-\phi_m) \cdot y''_m$ is not equal to zero, because z is distinct from the rest of x'_i's and y'_j's, the sum cannot be degenerated to 0. So, $\tau_n \cdot x''_n = \phi_m \cdot y''_m$ and as a result $\tau_n = \phi_m$ and $x''_n = y''_m$; Hence, $x_n = y_m$. $\sum_{i=1}^{n-1} \tau_i \cdot (x'_i x''_i) + \sum_{j=1}^{m-1} (-\phi_j) \cdot (y'_j y''_j) = 0$. By repeating the argument given above we can prove that $n = m$ and there exist $\sigma \in S_n$ such that $y_i = x_{\sigma(i)}$ and $\phi_i = \tau_{\sigma(i)}$.

Now, assume for $x = \sum_{i=1}^{n} \tau_i \cdot x_i$ and $\sum_{j=1}^{m} \phi_j \cdot y_j$, we have $x = y$, $F_x = \{n\}$, and $F_y = \{m\}$. So, we can write $\sum_{i=1}^{n-1} \tau_i \cdot x_i + \sum_{j=1}^{m-1} (-\phi_j) \cdot y_j + \tau_n \cdot x_n + (-\phi_m) \cdot y_m = 0$. Assume $\tau_n \cdot x_n$ and $(-\phi_m) \cdot y_m$ are not equal. Then, on one side of the above equation we have a non-degenerate ordinomial while the other side is zero which is impossible. It follows that $\tau_n \cdot x_n = \phi_m \cdot y_m$ and as a result $\tau_n = \phi_m$ and $x_n = y_m$. So, we can rewrite the above equation as $\sum_{i=1}^{n-1} \tau_i \cdot x_i + \sum_{j=1}^{m-1} (-\phi_j) \cdot y_j = 0$. If n and m are both greater than 1, then we can argue as we did for the first case; If one was 1 while the other one was greater than 1, we would have a non-degenerate ordinomial equal to 0 which surely is not possible; Finally, if they are both equal to one we are not left with anything to argue about.

Finally, for non-degenerate ordinomials x and y, assume that $x = y$ and $F_x = \{n\}$ while F_y is empty. Clearly, this case can not happen.■

From now on by an ordinomial we mean either 0 or a non-degenerate ordinomial. The degree of an ordinomial $x = \sum_{i=1}^{n} \tau_i \cdot x_i$, denoted by $deg(x)$, is defined as the largest ordinal occurring in the set $\{deg(x_i) | \, i = 1, 2, \ldots, n\}$ and as $0 = 0 \cdot 1$, have $deg(0) = 0$. Our previous theorem asserts that $deg(\ )$ is a well-defined mapping from $\mathbb{O}^*[\lambda]$ to $\mathbb{O}$. So, for x and y ordinomials, if $deg(x) \neq deg(y)$, then $x \neq y$. An ordinomial x is called a *monic ordinomial*, when for all i, $\tau_i = 1$ and $deg(x) > 0$.

The subset $\{1 \cdot x \,|\, x \in \mathbb{B}[\lambda]\}$ of $\mathbb{O}^*[\lambda]$ denoted by $\overline{\mathbb{B}[\lambda]}$ and armed with $\times$, is a monoid isomorphic to $\mathbb{B}[\lambda]$. From P_3 it is clear that if $1 \cdot x$ and $1 \cdot y$ are elements of $\overline{\mathbb{B}[\lambda]}$, then $1 \cdot x \times 1 \cdot y = 1 \cdot xy$ is not equal to zero. It is clear from definition that every element of $\overline{\mathbb{B}[\lambda]}$ is a monic ordinomial. On the other hand, if $\mathbb{M}[\lambda]$ is the set all monic polynomials in $\mathbb{C}[\lambda]$, we know that with polynomial multiplication it is a submonoid of $\mathbb{C}[\lambda]$ homomorphic to $\overline{\mathbb{B}[\lambda]}$. This observation will be used in our next section in which we talk about chromatic ordinomials. But in general, as one may see in the following example, $\mathbb{O}^*[\lambda]$ is not an integral domain: $(\lambda - \zeta)^{\omega^3} \times \left((\lambda - \zeta)^{\omega+1} + (-1) \cdot (\lambda - \zeta)^{\omega^2}\right) = (\lambda - \zeta)^{\omega^3} + (-1) \cdot (\lambda - \zeta)^{\omega^3} = 0$.

At the end of this section, we will now develop a limit operation which would be the foundation of how chromatic ordinomials will be defined for infinite graphs in section 2. Let $f : \mathbb{N} \longrightarrow \mathbb{N}$ ($f : \omega \longrightarrow \omega$) be an order preserving mapping, meaning that if $k_1 < k_2$, then $f(k_1) \leq f(k_2)$. We will define $\lim_{k\to\omega}(\lambda - \zeta)^{f(k)}$ as follows:

$$\lim_{k\to\omega}(\lambda-\zeta)^{f(k)} = \lim_{k<\omega}(\lambda-\zeta)^{f(k)} = (\lambda-\zeta)^{\left(\bigcup_{k<\omega} f(k)\right)}.$$

In addition to that this limit operation has the properties

$$\lim_{k\to\omega}\big(\tau\cdot\prod_{i=1}^{n}(\lambda-\zeta_i)^{f_i(k)}\big) = \tau\cdot\big(\prod_{i=1}^{n}\lim_{k\to\omega}(\lambda-\zeta_i)^{f_i(k)}\big)$$

for distinct complex numbers ζ_i and order preserving mappings $f_i : \mathbb{N} \longrightarrow \mathbb{N}$, and

$$\lim_{k\to\omega}\big(\sum_{i=1}^{n}(\tau_i\cdot\prod_{j=1}^{m_i}(\lambda-\zeta_{ij})^{f_{ij}(k)})\big) = \sum_{i=1}^{n}\big(\lim_{k\to\omega}(\tau_i\cdot\prod_{j=1}^{m_j}(\lambda-\zeta_{ij})^{f_{ij}(k)})\big)$$

such that for every i, ζ_{ij} are distinct complex numbers and $f_{ij} : \mathbb{N} \longrightarrow \mathbb{N}$ are order preserving mappings. One may clearly see that

i) $\lim_{k\to\omega}(\lambda-\zeta)^{f(k)} = (\lambda-\zeta)^{k_0}$, when there exist $M \in \mathbb{N}$ such that for $k > M$ and $k_0 \in \mathbb{N}$, $f(k) = k_0$.

ii) $\lim_{k\to\omega}(\lambda-\zeta)^{f(k)} = (\lambda-\zeta)^{\omega}$, otherwise.

If $f(k)$ can be decomposed into the form $f_1(k) + f_2(k)$ in which $f_1, f_2 : \mathbb{N} \longrightarrow \mathbb{N}$ are two other order preserving mappings, then following is not necessarily valid

$$\lim_{k\to\omega}(\lambda-\zeta)^{f(k)} = \big(\lim_{k\to\omega}(\lambda-\zeta)^{f_1(k)}\big)\big(\lim_{n\to\omega}(\lambda-\zeta)^{f_2(k)}\big).$$

For example, we know $\lim_{k\to\omega}(\lambda-\zeta)^{2k} = (\lambda-\zeta)^{\omega}$. On the other hand, we can decompose $2k$ into the form $k + k$; So, we have $(\lim_{k\to\omega}(\lambda-\zeta)^{k})(\lim_{n\to\omega}(\lambda-\zeta)^{k}) = (\lambda-\zeta)^{\omega}(\lambda-\zeta)^{\omega} = (\lambda-\zeta)^{\omega\cdot 2}$ which is not equal to $(\lambda-\zeta)^{\omega}$.

2 Partially Defined Chromatic Ordinomial

The chromatic polynomial of a finite graph Γ, $C(\Gamma;\lambda)$, is the polynomial in λ which counts the number of distinct proper vertex λ-colorings of Γ, given λ colors. For more information on chromatic polynomials see [1], [2], and [3].

By a *sequence of graphs*, we mean the family $\{\Gamma_n\}_{n\in\mathbb{N}}$ in which Γ_n is a finite graph. We call a sequence, a *chain of graphs* provided that $\Gamma_1 \subset \Gamma_2 \subset \cdots \subset \Gamma_n \subset \cdots$ and for all $n \in \mathbb{N}$, $|V(\Gamma_n)| = f(n)$ in which $f(n)$ is a strictly increasing function from $\mathbb{N}$ to $\mathbb{N}$. When for all $n \in \mathbb{N}$, Γ_n is connected, the sequence is called a *sequence of connected graphs.* On the other hand, if there exists $M \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $\Delta(\Gamma_n) < M$, then we call $\{\Gamma_n\}_{n\in\mathbb{N}}$ a *finite-degree sequence of graphs.* It is obvious from our definition that $\{N_n\}_{n\in\mathbb{N}}$ is a finite-degree chain of (disconnected) graphs while $\{K_n\}_{n\in\mathbb{N}}$ is a (non-finite-degree) chain of connected graphs. Finally, a chain $\{\Gamma_n\}_{n\in\mathbb{N}}$ is called *non-oscillating* provided that for all $n \in \mathbb{N}$ and for all $v \in V(\Gamma_n)\backslash V(\Gamma_{n-1})$, for every $m > n$, $d_{\Gamma_m}(v) = d_{\Gamma_{n+1}}(v)$.

A chain is called *chromatically conformal*, when for $m \in \mathbb{N}$, there exist distinct $\zeta_1, \zeta_2, \ldots, \zeta_m$ in $\mathbb{C}$ such that for all $n \in \mathbb{N}$, $C(\Gamma_n;\lambda) = (\lambda - \zeta_1)^{f_1(n)}(\lambda-\zeta_2)^{f_2(n)}\cdots(\lambda-\zeta_m)^{f_m(n)}$, in which for $1 \le i \le m$, $f_i(n) : \mathbb{N} \to \mathbb{N}$ is order preserving and $f(n) = f_1(n) + f_2(n) + \cdots + f_m(n)$. Furthermore, for $r \in \mathbb{N}$ a chain is called *chromatically recursive of degree* r, when for all $n \in \mathbb{N}$, $C(\Gamma_{n+r};\lambda) = p_1(\lambda)\,C(\Gamma_{n+r-1};\lambda) + p_2(\lambda)\,C(\Gamma_{n+r-2};\lambda) + \cdots + p_r(\lambda)\,C(\Gamma_n;\lambda)$, in which for $1 \le i \le r$, $p_i(n)$ is an element of $\mathbb{C}[\lambda]$ which are not dependent on n, and r the smallest natural number such a recursion holds. Moreover, we will assume that $p_1(\lambda) + p_2(\lambda) + \cdots + p_r(\lambda)$ is a non-constant element of $\mathbb{C}[\lambda]$. As one may see, $\{L_n \simeq K_2 \times P_n\}_{n\in\mathbb{N}}$ is a finite-degree chain of connected graphs which is chromatically conformal and chromatically recursive:

$$C(L_n;\lambda) = \lambda(\lambda-1)(\lambda - (\tfrac{3+i\sqrt{3}}{2}))^n(\lambda - (\tfrac{3-i\sqrt{3}}{2}))^n,$$
$$C(L_{n+1};\lambda) = (\lambda^2 - 3\lambda + 3)C(L_n;\lambda).$$

In general, a sequence is not necessarily convergent to a graph. A sequence converges to a graph Γ, if for every $n \in \mathbb{N}$, $\Gamma_n \subseteq \Gamma$ and if there exists a graph Γ' such that $\Gamma_n \subseteq \Gamma'$ for every $n \in \mathbb{N}$, then $\Gamma \subseteq \Gamma'$.

Theorem 2.1 *When a sequence $\{\Gamma_n\}_{n\in\mathbb{N}}$ is a chain, it is convergent to $\Gamma = \biguplus_{n\in\mathbb{N}} \Gamma_n$. Moreover, when there exist $m \in \mathbb{N}$ such that for every $n \ge m$, Γ_n is connected, then Γ is connected.*■

From now on, whenever a sequence $\{\Gamma_n\}_{n\in\mathbb{N}}$ converges to a limit Γ, we will use the notation $\Gamma = \lim_{n\to\omega}\Gamma_n$ (interchangeable with $\lim_{n\in\mathbb{N}}\Gamma_n$, $\lim_{n\in\omega}\Gamma_n$, or $\lim_{n<\omega}\Gamma_n$). Moreover, as Theorem 2.1 guarantees that there exists a unique limit for a chain, our attention would be more focused on such families of graphs.

Corollary 2.2 *If $\{\Gamma_n\}_{n\in\mathbb{N}}$ is a chain of connected graphs. Then so is $\Gamma = \lim_{n\to\omega}\Gamma_n$.*

Theorem 2.3 *Suppose $\{\Gamma_n\}_{n\in\mathbb{N}}$ is a finite degree chain. Then Γ is a graph of finite degree, where $\Gamma = \lim_{n\to\omega}\Gamma_n$.* ■

Let Γ be an infinite graph and $\{\Gamma_n\}_{n\in\mathbb{N}}$ be a finite-degree chain of connected graphs such that $\Gamma = \lim_{n\to\omega}\Gamma_n$. All the infinite graphs under investigation are countable graphs of finite-degree. For more information on infinite graphs see [5]. Throughout this section, by a chain of graphs, a non-oscillating finite-degree chain of connected graphs is meant, unless stated otherwise. If the limit $\lim_{n\to\omega} C(\Gamma_n;\lambda)$ exists, then *chromatic ordinomial* of Γ is *partially* defined and we have $C_p(\Gamma;\lambda) = \lim_{n\to\omega} C(\Gamma_n;\lambda)$. Clearly, when $\{\Gamma_n\}_{n\in\mathbb{N}}$ is a chromatically conformal, $C_p(\Gamma;\lambda)$ is defined. As chromatic polynomial of Γ_n is monic, partial chromatic ordinomial of Γ is also monic and consequently in $\overline{\mathbb{B}[\lambda]}$, due to properties of the limit operation we introduced at the end of our first section.

For instance, let $P = \lim_{n\to\omega} P_n$ be the one-way infinite path. We know that $\{P_n\}_{n\in\mathbb{N}}$ is chromatically conformal, $C(P_n;\lambda) = \lambda(\lambda-1)^n$, and chromatic ordinomial of P is partially defined: $C_p(P;\lambda) = \lim_{n\to\omega} C(P_n;\lambda) = \lim_{n\to\omega}\lambda(\lambda-1)^n = \lambda(\lambda-1)^\omega$. As another example, let L be $\lim_{n\to\omega} L_n$. This example is also chromatically conformal and we have

$C_p(L;\lambda) = \lim_{n\to\omega} C(L_n;\lambda) = \lim_{n\to\omega}\lambda(\lambda-1)(\lambda^2-3\lambda+3)^n = \lambda(\lambda-1)(\lambda^2-3\lambda+3)^\omega$.

If for every chain of graphs $\{\Gamma_n\}_{n\in\mathbb{N}}$ such that $\Gamma = \lim_{n\to\omega}\Gamma_n$, partial chromatic ordinomial of Γ, $C_p(\Gamma;\lambda)$, exist and for every two chain having the aforementioned properties this ordinomial is equal, then *chromatic ordinomial* of Γ is *universally* defined and we have $C(\Gamma;\lambda) = \lim_{n\to\omega} C(\Gamma_n;\lambda)$. Because this definition for universal chromatic ordinomial is still in is early stages and proper refinements are needed, we will focus our attention on partial chromatic ordinomial. By convention, chromatic ordinomial of finite graphs are partially (and universally) defined and it is equal to their chromatic polynomial.

If Γ is an infinite graph with k components $\Gamma^1, \Gamma^2, \ldots, \Gamma^k$ such that for $1 \le l \le k$, $\Gamma^1, \Gamma^2, \ldots, \Gamma^l$ are infinite graphs and the rest of the components are finite graphs. If chromatic ordinomial of $\Gamma^1, \Gamma^2, \ldots, \Gamma^l$ are partially defined, the chromatic ordinomial of Γ is partially defined and we have

$$C_p(\Gamma;\lambda) = C_p(\Gamma^1;\lambda)\,C_p(\Gamma^2;\lambda)\cdots C_p(\Gamma^l;\lambda)\,C(\Gamma^{l+1};\lambda)\cdots C(\Gamma^k;\lambda).$$

From the remark we made in Section 1, as all the factors on the right-hand side of the expressions we have for $C_p(\Gamma;\lambda)$ are elements of $\overline{\mathbb{B}[\lambda]}$, this ordinomial is also a non-zero element of $\overline{\mathbb{B}[\lambda]}$.

Now, let's assume $\{\Gamma_n\}_{n\in\mathbb{N}}$ is a chromatically recursive chain of graphs of degree r. By definition, for all $n \in \mathbb{N}$,

$$C(\Gamma_{n+r};\lambda) = p_1(\lambda)\,C(\Gamma_{n+r-1};\lambda) + p_2(\lambda)\,C(\Gamma_{n+r-2};\lambda) + \cdots + p_r(\lambda)\,C(\Gamma_n;\lambda).$$

Provided that $C_p(\Gamma;\lambda)$ exists, we have

$$C_p(\Gamma;\lambda) = p_1(\lambda)\,C_p(\Gamma;\lambda) + p_2(\lambda)\,C_p(\Gamma;\lambda) + \cdots + p_r(\lambda)\,C_p(\Gamma;\lambda) = p(\lambda)\,C_p(\Gamma;\lambda).$$

We know $p(\lambda)$ is a non-constant element of $\mathbb{C}[\lambda]$ and as a result a finite degree ordinomial. Let's assume that $p(\lambda) = \tau(\lambda-\zeta_1)^{n_1}(\lambda-\zeta_2)^{n_2}\cdots(\lambda-\zeta_m)^{n_m}$ in which $m, n_1, n_2, \ldots, n_m \in \mathbb{N}$ and ζ_i's are distinct complex numbers. As $C_p(\Gamma;\lambda)$ is monic, $\tau = 1$. Furthermore, from uniqueness of elements in $\overline{\mathbb{B}[\lambda]}$ one can proof that $(\lambda-\zeta_1)^{\omega}(\lambda-\zeta_2)^{\omega}\cdots(\lambda-\zeta_m)^{\omega} \mid C_p(\Gamma;\lambda)$. So, if $C_p(\Gamma;\lambda) = x_1x_2$ is the unique decomposition of $C_p(\Gamma;\lambda)$ such that x_1 is purely infinite, $deg(x_2) \in \mathbb{N}$, and $(x_1, x_2) = 1$, then $(\lambda-\zeta_1)^{\omega}(\lambda-\zeta_2)^{\omega}\cdots(\lambda-\zeta_m)^{\omega} \mid x_1$.

Conjecture 2.4 $x_1 = (\lambda-\zeta_1)^{\omega}(\lambda-\zeta_2)^{\omega}\cdots(\lambda-\zeta_m)^{\omega}$.

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