Completely independent critical cliques

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ABSTRACT. If K is an r-clique of G and $\chi(G)$ decreases by r upon the removal of all of the vertices in K, then K is called a critical r-clique. Two critical cliques are completely independent provided that no vertex in one clique is adjacent to a vertex from the other. An infinite family of graphs is constructed which demonstrates that for every $s, t \in \mathbb{N}$, there exists a vertex critical graph which admits a critical s-clique and a critical t-clique that are completely independent.

1. INTRODUCTION AND NOTATION

In the late 1940's, G. A. Dirac defined critical graphs and the main objective for defining them was to simplify the central problems that arose in the theory of graph coloring. There are several results on critical graphs containing few edges; e.g., [1],[2],[5], and [6], and on critical graphs containing many edges; e.g., [8] and [4]. The objective here is not to establish bounds on the number of edges in critical graphs but rather to investigate relations between particular sets of critical vertices assuming certain properties are satisfied. It seems equally important to determine and understand these potential relations. For example, under what condition(s) is the subgraph induced by the set of all critical vertices of a graph connected? Such a condition is given in [7].

This paper explores the lack of a relationship in the sense that particular critical vertices fail to be adjacent. The graphs constructed in this and the next section exhibit a property that is believed to be undesirable. It is then natural to ask what are the appropriate condition(s) that can be imposed on a graph so as not to feature this and perhaps other undesirable properties?

The graphs considered in this paper are finite, undirected, and simple. For a given graph G, the vertex and edge sets of G are denoted by $V(G)$ and $E(G)$, respectively. The order of G is the cardinality of $V(G)$ and is denoted by $|V(G)|$. If K is a complete subgraph having order r, then K will be called an r-clique. A set $I \subseteq V(G)$ is independent provided that no two distinct vertices in I are adjacent. The maximum cardinality of an independent subset of $V(G)$ is denoted by $\alpha(G)$. For $X \subseteq V(G)$, the subgraph of G induced by X is denoted $G[X]$. Lastly, G is vertex k-critical whenever $\chi(G) = k$ and $\chi(G - v) = k-1$ for every vertex $v \in V(G)$, where $\chi(G)$ is the chromatic number of G.

Before proceeding with the constructions, it is necessary to define additional terminology that is not common in the literature. Recall that a vertex v is critical whenever $\chi(G - v) = \chi(G) - 1$. As $G[\{v\}] \cong K_1$, the following definition generalizes the notion of a critical vertex.

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Definition 1. *Let K be an r-clique of G. Then K is a critical r-clique of G, written K_r^c, provided that $\chi(G-K) = \chi(G) - r$.*

The parameter $\omega_c(G)$, defined as the maximum order of a critical clique of G, is introduced in [7] and some elementary properties of $\omega_c(G)$ and critical cliques are given. Let K_r^c and K_s^c be two critical cliques of G. We would like to know what relations, if any, hold between these two critical cliques. And if no relations hold, under what conditions would certain relations hold? For instance, if K_r^c and K_s^c are maximal critical cliques in a vertex critical graph, then does it necessarily follow that $r = s$? The answer to this question is no and it is an immediate consequence of Proposition 1 below. The next definition is the focus of this paper and describes in a sense a type of separation between two critical cliques.

Definition 2. *Let K_r^c and K_s^c be two critical cliques. Then K_r^c and K_s^c are completely independent whenever $N(v) \cap V(K_s^c) = \emptyset$ for every vertex $v \in V(K_r^c)$.*

The motivation for this definition arises in connection with a conjecture of Lovász [3] that the only vertex double-critical graph is the complete graph. It was originally believed that if v was a critical vertex of G and K_s^c was a critical s-clique of G, then $|N(v) \cap V(K_s^c)| \geq s-1$. In the present language, this means that there do not exist completely independent critical cliques having orders 1 and s, $s \geq 2$. If this last inequality was true, then it is not difficult to prove in the affirmative the double-critical conjecture of Lovász. However, the fact that this inequality does not hold is Proposition 1 below.

For the case $r = s = 1$, graphs which admit completely independent critical 1-cliques abound. Among the simplest examples are the non-complete odd cycles C_{2n+3}, $n \in \mathbb{N}$. By generalizing the construction of the cycle C_5, we can prove the existence of graphs which admit completely independent critical cliques having orders 1 and s for any $s \geq 1$. Let $s \in \mathbb{N}$ and define $X = \{u_1, u_2, \ldots, u_{s+1}\}$, $Y = \{v_1, v_2, \ldots, v_{s+1}\}$, and $Z = \{w\}$. Require $\{X, Y, Z\}$ to be a pairwise disjoint collection so that $X \cup Y \cup Z$ consists of $2s+3$ distinct points. Then define a graph G_s by setting $V(G_s) = X \cup Y \cup Z$ and letting $E(G_s)$ consist of all edges determined by the following three categories:

$$\begin{array}{ll} (a) & u_i u_j \in E(G_s) \text{ for all } i \neq j,\ 1 \leq i, j \leq s+1. \\ (b) & u_i v_j \in E(G_s) \text{ if and only if } i \neq j,\ 1 \leq i, j \leq s+1. \\ (c) & v_i w \in E(G_s) \text{ for all } i,\ i = 1, 2, \ldots, s+1. \end{array} \qquad (1)$$

It is not difficult to confirm that $G_1 \cong C_5$. More generally, we have:

Proposition 1. *The set $\mathcal{G} = \{G_s : s \in \mathbb{N}\}$ is a family of vertex $(s+2)$-critical graphs in which $G_s[Z]$ and $G[X - \{u_i\}]$, for fixed i, $1 \leq i \leq s+1$, are completely independent critical cliques having orders 1 and s, respectively.*

Proof. By 1(a), $G_s[X] \cong K_{s+1}$ and so $\chi(G_s) \geq s+1$. Observe now that $G_s - w$ is uniquely $(s+1)$-chromatic. Moreover, vertex w is adjacent to exactly one vertex from each of the $s+1$ color classes in the unique $(s+1)$-coloring of $G_s - w$. Hence, w is a critical vertex of G_s; i.e., $G_s[Z]$ is a critical 1-clique and $\chi(G_s) = s+2$. Next, it is shown that for any i, $1 \le i \le s+1$, $G[X - \{u_i\}]$ is a critical s-clique. First, color vertex $u_i \in X$ with color c_i, for $i = 1, 2, \ldots, s+1$. Then, because Y is an independent subset of $V(G_s)$ by 1(b,c), each vertex $v_j \in Y$, $1 \le j \le s+1$, can be colored with the single color c_{s+2}. As w is not adjacent to any of the vertices in X by 1(c), it follows that w can be colored with any one of the colors, say c_i, from among $c_1, c_2, \ldots, c_{s+1}$. Consequently, $G[X - \{u_i\}]$ is a critical s-clique and each vertex in X is critical. Lastly, it is shown that every vertex in Y is critical. Again, color vertex $u_i \in X$ with color c_i, for $i = 1, 2, \ldots, s+1$. Then, for any fixed j, $1 \le j \le s+1$, color vertex $v_j \in Y$ with color c_{s+2} and vertices $v_i \in Y$, $i \ne j$, with color c_i. Finally, color vertex w can with color c_j. Therefore, each vertex in Y is critical. Since w is not adjacent to any vertex in X, it follows that $G_s[Z]$ and $G[X - \{u_i\}]$, for fixed i, $1 \le i \le s+1$, are completely independent critical cliques having orders 1 and s, respectively. ∎

2. The Main Construction

In this section, it is demonstrated that for every $s, t \in \mathbb{N}$, there exists a vertex critical graph admitting completely independent critical cliques having orders r and s. When $s = 1$ and $t \in \mathbb{N}$, the problem is not too difficult and the construction used in the previous section was straightforward. However, it does not appear that the construction extends to this more general result. Hence, a more technical and interesting construction seems necessary. Proceed by letting S be an infinite set of distinct points. Let $S^{m \times n}$ denote the set of all $m \times n$ matrices whose entries are members of S. Neither S nor $S^{m \times n}$ are required to have any algebraic structure; standard matrix notation is used for convenience only. For a matrix $A \in S^{m \times n}$, $A = [a_{ij}]$, denote the rows of A, considered as elements of $S^{1 \times n}$, by $A_1, A_2, \ldots, A_m$ and the columns of A, considered as members of $S^{m \times 1}$, by $B_1, B_2, \ldots, B_n$. For further convenience, define the set

$$A^* = \{a_{ij} : 1 \le i \le m \text{ and } 1 \le j \le n\}.$$

Thus, A^* is the range of the matrix A when viewed as a function defined by the rule $(i,j) \longmapsto a_{ij}$. Lastly, the submatrix of A obtained by deleting the i^{th} row and j^{th} column of A will be denoted by $A(i|j)$.

Let $r \in \mathbb{N}$ and choose any three matrices $A \in S^{(r+1)\times(r+1)}$, $X \in S^{r \times 1}$, and $Y \in S^{1 \times r}$ such that the collection $\{A^*, X^*, Y^*\}$ is pairwise disjoint. Thus, $A^* \cup X^* \cup Y^*$ consists of $(r+1)^2 + 2r$ distinct points of S. Define a graph G_r by setting $V(G_r) = A^* \cup X^* \cup Y^*$ and letting $E(G_s)$ consist of all edges determined by the following five categories:

$$\begin{array}{ll} (a) & a_{ij}a_{pq} \in E(G_r) \text{ if and only if } a_{pq} \in [A(i|j)]^*. \\ (b) & x_ix_j \in E(G_r) \text{ for all } i \ne j. \\ (c) & y_iy_j \in E(G_r) \text{ for all } i \ne j. \\ (d) & a_{ij}x_m \in E(G_r) \text{ if and only if } i \ne m. \\ (e) & a_{ij}y_n \in E(G_r) \text{ if and only if } j \ne n. \end{array} \qquad (2)$$

The next theorem generalizes the result of Proposition 1. In order to make the proof of the theorem less cumbersome, the following two lemmas concerning the structure of various subgraphs of G_r are useful. Extensions of these two lemmas are mentioned briefly in the last section.

Lemma 1. *For every $r \geq 2$, G_r contains a subgraph isomorphic to G_{r-1}.*

Proof. Let $r \geq 2$ and choose any i and j, $1 \leq i, j \leq r$. Let $X(i)$ denote the $(r-1) \times 1$ submatrix of X obtained by deleting the i^{th} entry of X. Similarly, let $Y(j)$ denote the $1 \times (r-1)$ submatrix of Y obtained by deleting the j^{th} entry of Y. Define $G_r(i,j) = G_r\left[[A(i|j)]^* \cup [X(i)]^* \cup [Y(j)]^*\right]$. Then $G_r(i,j) \cong G_{r-1}$. To see this, first observe that $A(i|j) \in S^{r \times r}$, $X(i) \in S^{(r-1)\times 1}$, and $Y(j) \in S^{1\times(r-1)}$. Hence, the collection

$$\left\{[A(i|j)]^*, [X(i)]^*, [Y(j)]^*\right\}$$

is a pairwise disjoint collection consisting of $r^2 + 2(r-1)$ distinct points of S. By the definition of $E(G_r)$, the sets $A_i^* \cup \{x_i\}$ and $B_j^* \cup \{y_j\}$ are independent subsets of $V(G_r)$ with $(A_i^* \cup \{x_i\}) \cap (B_j^* \cup \{y_j\}) = \{a_{ij}\}$. Hence, $G_r(i,j)$ satisfies all defining properties of the graph G_{r-1}. ■

Lemma 2. *For every $r \geq 2$, $G_r[A^*]$ is $(r+1)$-chromatic.*

Proof. Let I be an arbitrary independent subset of A^* containing the vertex a_{ij}. By the definition of $E(G_r)$, it must be that $I \subseteq A_i^*$ or $I \subseteq B_j^*$. Thus, $|I| \leq r+1$ and so $\alpha(G_r[A^*]) = r+1$ as the rows (or columns) of A determine independent subsets of A^* having cardinality $r+1$. Because $\{A_s^* : 1 \leq s \leq r+1\}$ is a partition of A^* into $r+1$ independent subsets and

$$\chi(G_r[A^*]) \geq \frac{|G_r[A^*]|}{\alpha(G_r[A^*])} = \frac{(r+1)^2}{r+1} = r+1,$$

it follows that $\chi(G_r[A^*]) = r+1$. ■

Theorem 1. *The set $\mathcal{G} = \{G_r : r \in \mathbb{N}\}$ is a family of $(2r+1)$-chromatic graphs in which $G_r[X^*]$ and $G_r[Y^*]$ are completely independent critical r-cliques.*

Proof. The proof proceeds by induction on r. For $r = 1$ it is easy to see that G_1 is 3-chromatic and that both x_1 and y_1 are critical vertices of G_1. Thus, $G_1[\{x_1\}]$ and $G_1[\{y_1\}]$ are completely independent critical 1-cliques. Inductively assume that $G_{r'}$ is a $(2r'+1)$-chromatic graph in which $G_{r'}[X^*]$ and $G_{r'}[Y^*]$ are completely independent critical r'-cliques for $1 \leq r' < r$. Now consider the graph G_r and an arbitrary partition $\mathcal{P}$ of $V(G_r)$ into independent subsets. Choose an arbitrary $x_m \in X^*$. Then, there exists a $P = P(m) \in \mathcal{P}$ such that $x_m \in P$. Observe that P can contain at most one member of Y^*. Consequently, for any $x_m \in X^*$, there exists a $y_n = y_n(m) \in Y^*$ such that $y_n \notin P$. From this, it follows that there exist $P, Q = Q(m,n) \in \mathcal{P}$ such that $x_m \in P$ and $y_n \in Q$. Furthermore,

$P \subseteq A_m^* \cup \{x_m\}$ and $Q \subseteq B_n^* \cup \{y_n\}$. Therefore, every partition $\mathcal{P}$ of $V(G_r)$, for $r \geq 2$, contains at least two members, in particular P and Q, neither one of which contains a vertex of $G_r(m,n)$. Hence, $\chi(G_r) \geq (2r-1)+2 = 2r+1$ since by Lemma 1 and the inductive hypothesis $\chi(G_r(m,n)) = \chi(G_{r-1}) = 2r-1$. However, the sets $A_i^* \cup \{x_i\}$ for $i = 1,2,\ldots,r$, the set A_{r+1}^*, and the sets $\{y_j\}$ for $j = 1,2,\ldots,r$ determine a partition of $V(G_r)$ into $2r+1$ independent subsets. As a result, $\chi(G_r) = 2r+1$. It remains to confirm that $G_r[X^*]$ and $G_r[Y^*]$ are completely independent critical r-cliques. To this end, consider $G_r[A^* \cup Y^*]$. It must be shown that $\chi(G_r[A^* \cup Y^*]) = r+1$. By Lemma 2, $\chi(G_r[A^*]) = r+1$. Thus, $\chi(G_r[A^* \cup Y^*]) \geq r+1$. Note that this also follows from the facts that $G_r - X^* \cong G_r[A^* \cup Y^*]$ and $\chi(G_r - X^*) \geq \chi(G_r) - r$. Next, observe that the sets $B_j^* \cup \{y_j\}$ for $j = 1,2,\ldots,r$ and the set B_{r+1}^* determine a partition $A^* \cup Y^*$ into $r+1$ independent subsets. We conclude that $\chi(G_r[A^* \cup Y^*]) = r+1$ and that $G_r[X^*]$ is a critical r-clique. Similarly, $G_r[Y^*]$ is a critical r-clique. Finally, because $x_i y_j \notin E(G_r)$ for every $1 \leq i,j \leq r$, $G_r[X^*]$ and $G_r[Y^*]$ are completely independent critical r-cliques. ■

Corollary 1. *For every $s,t \in \mathbb{N}$, there exists a vertex critical graph which admits a critical s-clique and a critical t-clique that are completely independent.*

3. Concluding Remarks

We showed the existence of graphs which admit completely independent critical cliques and it was mentioned that graphs exhibiting such a property were in some sense undesirable. This is because of the nonexistence of edges between the critical cliques. After investigating conditions on $G - K_s^c - K_t^c$ necessary for the existence of completely independent critical cliques, the constructions were obtainable. If K_s^c and K_t^c are to be completely independent critical cliques of a graph G, then is it possible to determine necessary and sufficient conditions on $G - K_s^c - K_t^c$, $G - K_s^c$, and $G - K_t^c$ in order to guarantee the existence of the completely independent critical cliques? For instance, in the graph G_r, the number of edges in $G_r[A^*]$ is $\frac{1}{2}r^2(r+1)^2$. So, at least asymptotically, $G_r[A^*]$ and G_r contains almost all of its edges. So is it true, at least in the case $s,t \geq 2$, that if there exist completely independent critical cliques having orders r and s, then G must contain nearly all of it edges? Do there exist graphs having few edges admitting completely independent critical cliques, for $s,t \geq 2$? If K_s and K_t are two cliques, then what are necessary and sufficient conditions on $G - K_s^c - K_t^c$ to force the existence of edges between K_s^c and K_t^c?

Lastly, does there exist a graph which admits an arbitrary number of completely independent critical cliques having orders at least two? The construction that was used in the theorem does not appear to generalize further. By defining $I_t = \{1,2,\ldots,t\}$ and an n-dimensional matrix A as a function $A : I_{t_1} \times \cdots \times I_{t_n} \longrightarrow S$, various generalizations of Lemma 1 and Lemma 2 were obtainable and are interesting in their own right. However, it did not seem that 2(d,e) could be generalized in any way.

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