On super edge-magic graphs which are weak magic

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ABSTRACT

A (p,q) graph G is ***total edge-magic*** if there exits a bijection f: V∪ E → {1.2,…,p+q} such that for each e=(u,v) in E, we have f(u) + f(e) + f(v) as a constant. For a graph G, denote M(G) the set of all total edge-magic labelings. The magic strength of G is the minimum of all constants among all labelings in M(G), and denoted by emt(G). The maximum of all constants among M(G) is called the maximum magic strength of G and denoted by eMt(G). Hegde and Shetty classify a magic graph as **strong** if emt(G) = eMt(G), **ideal magic** if $1 \leq$ eMt(G)-emt(G) $\leq$ p and **weak magic**, if eMt(G)-emt(G) > p. A total edge-magic graph is called a *super edge-magic* if f(V(G))= {1,2,…,p}.The problem of identifying which kinds of super edge-magic graphs are weak-magic graphs is addressed in this paper.

1. Introduction. A (p,q)-graph G= (V, E) with p vertices and q edges is called ***total edge magic*** if there is a bijection f : V∪ E->{1,2,...,p+q} such that there exists a constant s for any (u,v) in E f(u) + f((u,v)) + f(v)=s. The original concept of a total edge-magic graph is credited to Kotzig and Rosa [12,13]. Originally, they termed it a magic graph. Motivated by the definition of total edge-magic labelings, Enomoto,Llado, Nakamigawa and Ringel [4] introduced the concept of super edge-magic graphs in 1998. A total edge-magic graph G is called ***super edge-magic*** if f(V(G))= {1,2,…,p}.

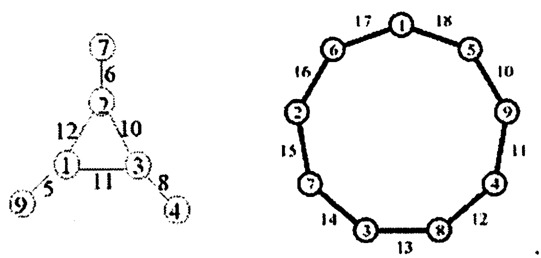


Figure 1.

The figure 1 shows a unicyclic graphs with 6 vertices which permits a total edge-magic labeling and the cycle C_9 with a super edge-magic labeling.
In fact, in [12] Kotzig and Rosa showed that every caterpillar and (2k+1) K_2 are super edge-magic. However, the edge-magic labeling of even cycles [12] is not super edge-magic. In [4], Enomoto et. al. gave a super edge-magic labeling for odd cycles. Independently, Craft and Tesar [3], Godbold and Slater [10] rediscovered that all cycles are total edge-magic.

A subset S of integers is called ***consecutive*** if S consists of consecutive integers. Chen [2] showed that a graph G is super edge-magic if and only if there exists a vertex labeling f such that the two sets f(V(G)) and $\{f(u)+f(v) : (u,v) \in E(G)\}$ are both consecutive. Independently, Figueroa-Centeno et. al. [6,8] have also obtained the same result. They have shown that if $f:V(G)\to\{1,2,\ldots,p\}$ is a bijection of a (p,q)-graph G and $S=\{f(u)+f(v): (u,v) \in E\}$ is consecutive with s=min(S), then f can be extended to a super edge-magic labeling of G defined by f((u,v))=p+q+s-f(u)-f(v) for all edge (u,v) of E(G). In light of this result, it suffices to exhibit the vertex labeling of a super edge-magic graph.

Numerous classes of graphs have been identified total edge-magic or super edge-magic. [2,4,7,9,10,14,15,16,17]

For a graph G, let M(G) denote the set of all total edge-magic labelings. If f is in M(G) with magic sum k, then its dual f* which is defined by
$f^*(x) = (p+q+1)- f(x)$ for each x in V(G) and $f^*((x,y)) = (p+q+1)-f((x,y))$ for each (x,y) in E(G) is also in M(G) with magic sum 3(p+q+1)-k.

The **magic strength** of G is the minimum of all constants among all labelings in M(G), and is denoted by emt(G). The maximum of all constants among M(G) is called the **maximum magic strength** of G and is denoted by eMt(G). Hegde and Shetty [11] classify a magic graph as **strong** if emt(G) = eMt(G), **ideal** magic if $1 \leq$ eMt(G)-emt(G) $\leq$ p and **weak magic** if eMi(G)-emt(G) > p.

<u>**Example 1**</u>. P_2 is strong magic, for we have $M(P_2) = \{f_1,f_2\}$ where $f_1(v_1)= 1$, $f_1(v_2) = 2$ and $f_2(v_1)=2$, $f_2(v_2)=1$.

<u>**Example 2**</u>. P_3 is ideal magic.

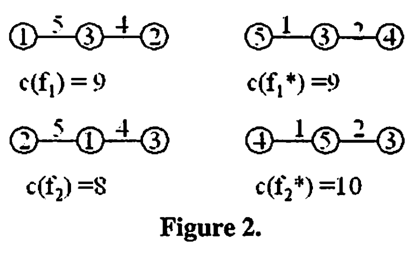


Figure 2.

<u>**Example 3**</u>. Godbold and Slater [10] have shown that $eMt(C_{2n+1}) = 7n+5$, $emt(C_{2n+1}) = 5n+4$. As $eMt(C_{2n+1}) - emt(C_{2n+1}) = 2n+1 = p(C_{2n+1})$. Thus all odd cycles are ideal magic.

Example 4. $P_3 + K_1$ is weak magic. To see that the graph is weak magic, it suffices to have a total edge-magic labeling f, such that its dual f* has the property that c(f*)-c(f) > p. Figure 3 illustrates such a labeling exits for $P_3 + K_1$.

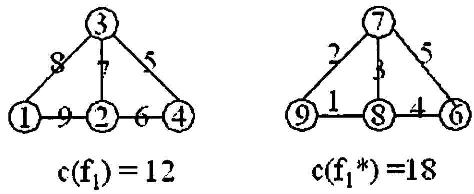


Figure 3.

The problem of identifying which kinds of super edge-magic graphs are weak-magic graphs is addressed in this paper. We present here eight families, out of many, super edge-magic graphs which are weak magic.

2. Super edge-magic graphs which are weak magic.

Family 1. Graph $\Xi(n)$, $n \geq 1$

For $n \geq 1$, let $\Xi(n)$ be the graph with $V(\Xi(n)) = \{c_1, c_2, \ldots, c_6\} \cup \{v_1, v_2, \ldots, v_n\}$ and $E(\Xi(n)) = \{(c_1, c_2), (c_2, c_3), (c_3, c_4), (c_4, c_5), (c_5, c_6), (c_1, c_6), (c_1, c_4)\} \cup \{(c_4, v_t) : t = 1, 2, \ldots, n\}$. We see that p=n+6 and q = n+7. (see Figure 4).

Theorem 1. The graph $\Xi(n)$ is super edge-magic and weak magic for all $n \geq 1$.

Proof. Let us define f: $V(\Xi(n)) \cup E(\Xi(n)) \rightarrow \{1, 2, 3, \ldots, 2n+13\}$ as follows:

$f(c_1) = 3$, $f(c_2) = 2$, $f(c_3) = 6$, $f(c_4) = 4$, $f(c_5) = 5$, $f(c_6) = 1$,

$f(v_t) = t+6$, $t = 1, \ldots, n$, and $f((c_4,v_t)) = 2n + 7 - t$, $t = 1, \ldots, n$.

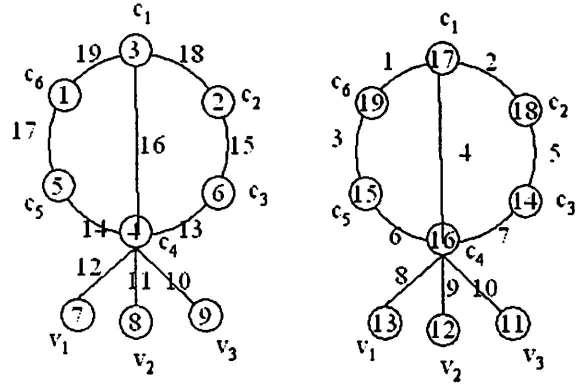
$f((c_1,c_2)) = 2n + 12$, $f((c_2,c_3)) = 2n + 9$, $f((c_3,c_4)) = 2n + 7$, $f((c_4,c_5)) = 2n + 8$, $f((c_5,c_6)) = 2n + 11$, $f((c_6,c_1)) = 2n + 13$, $f((c_1,c_4)) = 2n + 10$.

It can be easily checked that f is a super edge-magic with magic sum c(f) = 2n+17.

Now let f* be the dual of f and its magic sum is c(f*) =3(p+q+1)- (2n+17) = 3(2n+14)-(2n+17)= 4n+25.

We see that eMt(G)-emt(G)=c(f*)-c(f) =2n+8> n+6 =$p(\Xi(n))$. Thus $\Xi(n)$ is weak magic.

Example 5. Super edge-magic labelings of $\Xi(3)$ in Figure 4.



c(f) = 23　　　c(f*) = 37

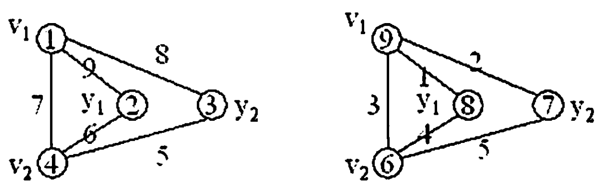
Ξ(3)　　　$c(f^*)-c(f) = 14 > 9$

Figure 4.

Family 2. Let $P_{2n}(+)N_m$ be the graph with $p=2n+m$ and $q = 2(m + n) - 1$. $V(P_{2n}(+)N_m) = \{v_1, v_2, \ldots, v_{2n}, y_1, y_2, \ldots, y_m\}$ where $V(P_{2n}) = \{v_1, v_2, \ldots, v_{2n}\}$ and $V(N_m) = \{y_1, y_2, \ldots, y_m\}$ and $E(P_{2n}(+)N_m) = E(P_{2n}) \cup \{(v_1, y_1), (v_1, y_2), \ldots (v_1, y_m), \ldots, (v_{2n}, y_1), (v_{2n}, y_2), \ldots, (v_{2n}, y_m)\}$ (see Figure 5).

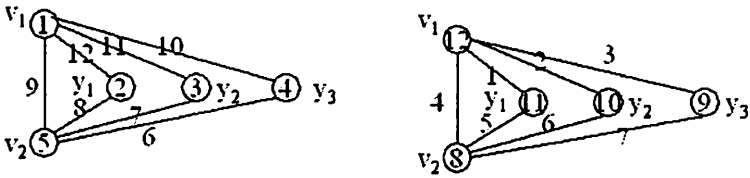
Theorem 2. $P_{2n}(+)N_m$ is super edge-magic and weak magic for all $n, m > 1$.
Proof. Let us define $f: V(P_{2n}(+)N_m) \to \{1, 2, 3, \ldots, 2n+m\}$ as follows:
$f(v_{1+2t}) = 1 + t$, $t = 0, 1, \ldots, n - 1$
$f(v_{2+2t}) = m + n + 1 + t$, $t = 0, 1, \ldots, n - 1$
$f(y_k) = k + n$, $k = 1, 2, \ldots, m$
It can be easily checked that $f^*(E(P_{2n}(+)N_m))$ is a consecutive set with the magic sum $c(f) = 5n+3m+1$ and $c(f^*) = 6m + 7n - 1$.
We see that $eMt(G)-emt(G) \geq c(f^*)-c(f) = 2n + 3m - 2 > 2n+m = p(P_{2n}(+)N_m)$.
Thus $P_{2n}(+)N_m$ is weak magic.

Example 6. Super edge-magic labeling of $P_2(+)N_m$ for $m = 2, 3$ and $P_4(+)N_2$ in Figure 5.



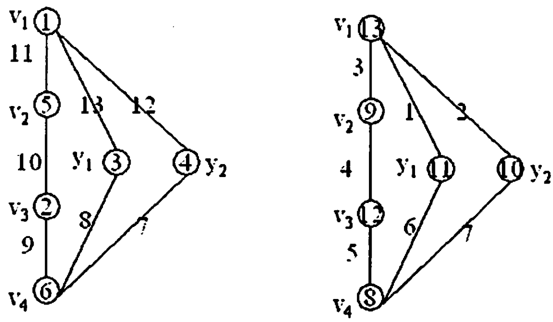
c(f) = 12　　　c(f*) = 18

$P_2(+)N_2$　　　$c(f^*) - c(f) = 6 > 4$



c(f) =15 c(f*) =24

$P_2 (+) N_3$ c(f*) −c(f)= 9 > 5



c(f) =17 c(f*) =25

$P_2 (+) N_2$ c(f*) −c(f)= 8 > 6

Figure 5.

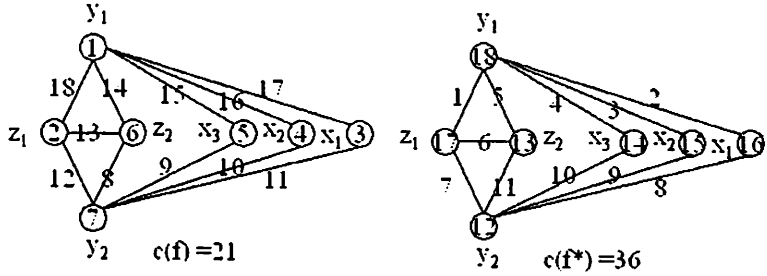
Family 3. The planar graph $(P_2 \cup k K_1) + N_2$

Theorem 3. For $k > 1$, the planar graph $(P_2 \cup k K_1) + N_2$ is weak magic.
Proof. Let the vertex set of $P_2 \cup kK_1$ be $\{z_1, z_2, x_1, \ldots, x_k\}$ and $V(N_2) = \{y_1, y_2\}$. We have $p = k + 4$ and $q = 2k + 5$.

Define a labeling $f: V((P_2 \cup kK_1) + N_2) \rightarrow \{1, 2, \ldots, k + 4\}$ by $f(y_1) = 1$, $f(y_2) = k + 4$, $f(z_1) = 2$, $f(z_2) = k + 3$ and $f(x_s) = s + 2$ for $s = 1, 2, \ldots, k$.

It is clear that f induces a consecutive labeling on the edges. Therefore $(P_2 \cap kK_1) + N_2$ is super edge-magic with magic constant $c(f) = 3k + 12$. As $c(f^*) = 6k + 18$ and $c(f^*) - c(f) = 3k + 6 > p((P_2 \cup kK_1) + N_2))$. We conclude that $(P_2 \cup kK_1) + N_2$ is weak magic.

Example 6. We give a super edge-magic labeling of $(P_2 \cup 3 K_1) + N_2$ which shows that it is weak magic.

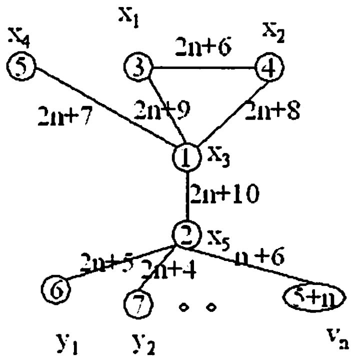


$(P_2 \cup 3K_1) + N_2$ $\qquad$ $c(f^*) - c(f) = 15 > 7$

Figure 6.

<u>Family 4</u>. The unicyclic graph W(n)

For any integer $n \geq 1$, the uncyclic graph W(n) is the graph with vertex set $V(W(n))=\{x_1, x_2\, x_3, x_4 x_5, y_1, y_2, \ldots, y_n\}$ and the edge set $E(W(n))= \{(x_1, x_2), (x_2, x_3), (x_3, x_1), (x_3, x_4), (x_3, x_5)\} \cup \{(x_5, y_i): i = 1, 2, \ldots, n\}$ (see Figure 7).



W(n) $\qquad$ c(f) =2n+13

Figure 7.

<u>Theorem 4.</u> For any integer $n \geq 1$, the graph W(n) is super edge-magic and weak edge-magic.

Proof. The graph W(n) has 5 + n vertices and 5+n edges.

Define a labeling $f:V(W(n)) \to \{1,2, \ldots, 5 + n\}$ by

$f(x_1) = 3$, $f(x_2) = 4$, $f(x_3) = 1$, $f(x_4) = 5$,$f(x_5) = 2$ and $f(y_i) = 5 + i$, for $i = 1, \ldots, n$.

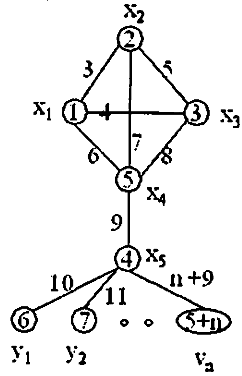
It is clear that f induces a consecutive labeling on the edges. Thus f is a super edge-magic labeling with

$$c(f) = 2n+13.$$

Now $c(f^*) - c(f) = 2n + 7 > n + 5 = p(W(n))$. Therefore W(n) is weak magic.

<u>Family 5</u>. The graph H(n).

For any integer $n \geq 1$, the graph H(n) is the graph with vertex set $V(H(n))= \{x_1, x_2, x_3, x_4, x_5, y_1, y_2, \ldots, y_n\}$ and the edge set $E(H(n))= \{(x_1, x_2), (x_1, x_3), (x_1, x_4), (x_2, x_3), (x_2, x_4), (x_3, x_4), (x_4, x_5)\} \cup \{(x_5, y_i): i = 1, 2, \ldots, n\}$ (see Figure 8).



H(n) c(f) =2n+15

Figure 8.

Theorem 5. For any integer $n \geq 1$, the graph H(n) is super edge-magic and weak edge-magic.

Proof. The graph H(n) has 5 + n vertices and 7+n edges.

Define a labeling $f:V(H(n))\rightarrow\{1,2, \ldots, 5 + n\}$ by

$f(x_1) = 1$, $f(x_2) = 2$, $f(x_3) = 3$, $f(x_4) = 5$, $f(x_5) = 4$ and $f(y_i) = 5 + i$, for $i = 1, \ldots, n$.

It is clear that f induces a consecutive labeling on the edges. Thus f is a super edge-magic labeling with

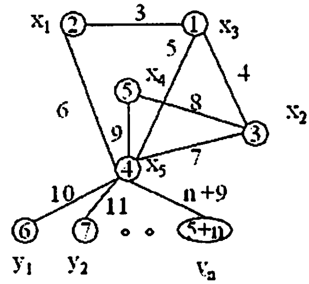
$$c(f) = 2n+15.$$

Now $c(f^*) - c(f) = 3(p+q+1)-2c(f) =2n + 9 > n +5 = p(H(n))$.

Therefore H(n) is weak magic.

Family 6. The graph X(n).

For any integer $n \geq 1$, the graph X(n) is the graph with vertex set $V(X(n))=\{x_1, x_2, x_3, x_4, x_5, y_1, y_2, \ldots, y_n\}$ and the edge set $E(X(n))= \{(x_1, x_3), (x_1, x_5), (x_2, x_3), (x_2, x_4), (x_2, x_5), (x_3, x_5), (x_4, x_5)\} \cup \{(x_5, y_i): i = 1, 2, \ldots, n\}$ (see Figure 9).



X(n) c(f) =2n+15

Figure 9.

Theorem 6. For any integer $n \geq 1$, the graph X(n) is super edge-magic and weak edge-magic.

Proof. The graph X(n) has 5 + n vertices and 12+n edges.

Define a labeling $f: V(X(n)) \rightarrow \{1, 2, \ldots, 5 + n\}$ by

$f(x_1) = 2$, $f(x_2) = 3$, $f(x_3) = 1$, $f(x_4) = 5$, $f(x_5) = 4$ and $f(y_i) = 5 + i$, for $i = 1, \ldots, n$.

It is clear that f induces a consecutive labeling on the edges. Thus f is a super edge-magic labeling with

$$c(f) = 2n+15.$$

Now $c(f^*) - c(f) = 3(p+q+1) - 2c(f) = 2n + 9 > n + 5 = p(X(n))$.

Therefore X(n) is weak magic.

Family 7. The braid graph B(n).

We consider a family of graphs which we will call **braid graphs.** For each n > 2, the braid graph

B(n) is defined as follows:

$V(B(n)) = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$ and

$E(B(n)) = \{(x_i, x_{i+1}): i = 1, 2, \ldots, n-1\} \cup \{(y_i, y_{i+1}): i = 1, 2, \ldots, n-1\} \cup \{(x_i, y_{i+1}): i = 1, 2, \ldots, n-1\} \cup \{(y_i, x_{i+2}): i = 1, 2, \ldots, n-2\}$.

Theorem 7. The braid graph B(n) is weak magic for all $n \geq 3$.

Proof. The graph B(n) has 2n vertices and 4n-5 edges. We define a vertex labeling f on B(n) as follows:

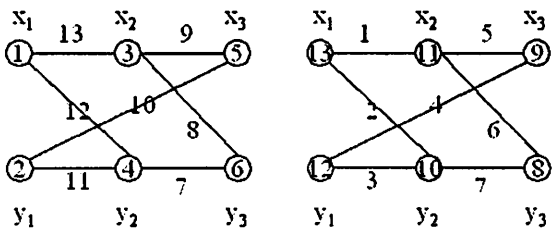
$$f(x_i) = 2i-1 \text{ for } i = 1, \ldots, n$$
$$f(y_i) = 2i \text{ for } i = 1, \ldots, n.$$

We see that f induces a consecutive labeling on the edge set (see Figure 10).

$$c(f) = 6n-1 \text{ and } c(f^*) = 12n-11$$

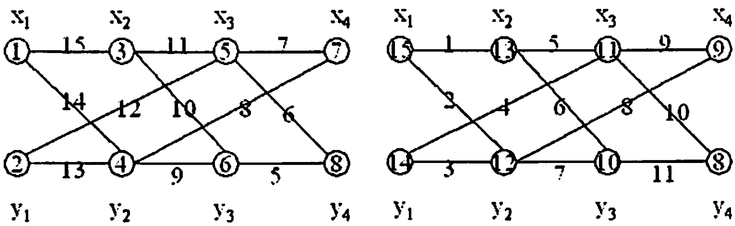
Therefore $eMt(B(n)) - emt(B(n)) \geq c(f) - c(f^*) = 6n - 10 > 2n = p(B(n))$. Thus B(n) is weak magic.

Example 8.



B(3) c(f) = 17 c(f*) = 25

$c(f^*) - c(f) = 8 > 6$



B(4) c(f) = 19 c(f*) = 29

c(f*) –c(f)= 10 ≥ 8

Figure 10.

<u>**Family 8.**</u> Jellyfish graph J(m,n).

For integers m, n > 0, we consider the graph J(m, n) with vertex set V(J(m, n)) = {u, v, x, y} ∪{x_1,x_2,..., x_m} ∪ {y_1, y_2,..., y_n} and edge set E(J(m, n)) = {(u, x), (u, v), (u, y), (v, x), (v, y)} ∪{(x_i, x): i = 1, 2, ..., m}∩{(y_i, y): i = 1, 2, ..., n}.

We will refer to J(m,n) as a Jellyfish graph. We see that p = m +n + 4 and q = m +n + 5.

<u>**Theorem 8.**</u> The Jellyfish graph J(m, n) is weak magic for all m, n > 0.

Proof. Without lost of generality we assume m ≤ n. We define a labeling f: V(J(m, n))→ {1, 2, ..., m+n+4} as follows:

$$f(x_i) = i \text{ for } i = 1, 2, \ldots, m$$

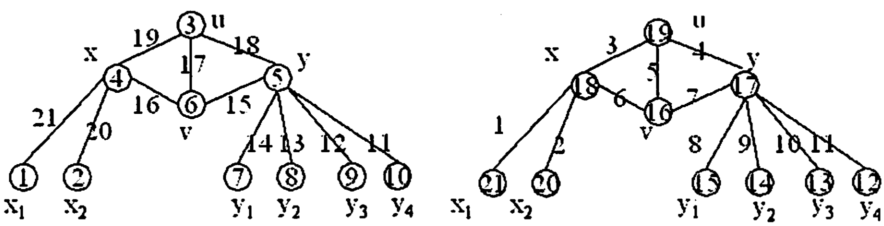
$$f(u) = m + 1,\ f(x) = m + 2,\ f(y) = m + 3,\ f(v) = m + 4,$$

$$f(y_i) = m + 4 + i \text{ for } i = 1, \ldots, n.$$

We see that the labeling f induces a consecutive labeling on E(J(m, n)). Thus, we conclude that J(m, n) is super edge-magic.

As c(f) = 3m +2n + 12 and c(f*) = 3m +4n +18. We have eMt(J(m, n)) – emt(J(m, n))≥ c(f*)-c(f) = 2n +6 > m+n+4 = p (J(m, n)). Thus J(m, n) is weak magic.

<u>**Example 9**</u>. A super edge-magic labeling of Jellyfish graph J(2, 4) is shown as follows:



c(f) = 26 c(f1*) =40

J(2,4) c(f*)-c(f) = 14 >10

Figure 11.

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