On the Q(a)P(b)-Super Edge-Graceful (p,p+1)-Graphs

Sin-Min Lee
Department of Computer Science
San Jose State University
San Jose, California 95192 U.S.A.

Yung-Chin Wang
Department of Physical Therapy
Tzu-Hui Institute of Technology
Taiwan, Republic of China

Yihui Wen
Department of Mathematics
Suzhou Science and Technology College
Suzhou, Jiangsu 215009
People's Republic of China

ABSTRACT

Let a and b be two positive integers. For the graph G with vertex set V(G) and edge set E(G) with p=|V(G)| and q=|E(G)|, we define two sets Q(a) and P(b) as follows:

$Q(a) = \{ \pm a, \pm (a+1), \ldots, \pm(a + (q-2)/2) \}$ if q is even,

$Q(a) = \{0\} \cup \{ \pm a, \pm (a+1), \ldots, \pm(a + (q-3)/2) \}$ if q is odd,

$P(b) = \{ \pm b, \pm (b+1), \ldots, \pm(b + (p-2)/2) \}$ if p is even,

$P(b) = \{0\} \cup \{ \pm b, \pm (b+1), \ldots, \pm(b + (p-3)/2) \}$ if p is odd.

For the graph G with p=|V(G)| and q=|E(G)|, G is said to be

JCMCC **62** (2007), pp. 193-216

Q(a)P(b)-super edge-graceful (in short Q(a)P(b)-SEG), if there exists a function pair (f, f^+) which assigns integer labels to the vertices and edges; that is, $f^+: V(G) \rightarrow P(b)$, and $f: E(G) \rightarrow Q(a)$ such that f^+ is onto P(b) and f is onto Q(a), and $f^+(u) = \Sigma\{f(u,v): (u,v) \in E(G)\}$.

We investigate Q(a)P(b) super-edge-graceful labelings for three classes of (p,p+1)- graphs.

1. Introduction.

If G is a (p,q) graph in which the edges labeling $h: E(G) \rightarrow \{1,2,3,...q\}$ is a bijection so that the vertex sums defined by $h^+(u) = \Sigma\{h(u,v): (u,v) \text{ in } E\}$ (mod p) is distinct, then G is called edge-graceful. ([18]) Figure 1 shows a grid with 12 vertices and 17 edges with two different edge-graceful labelings·

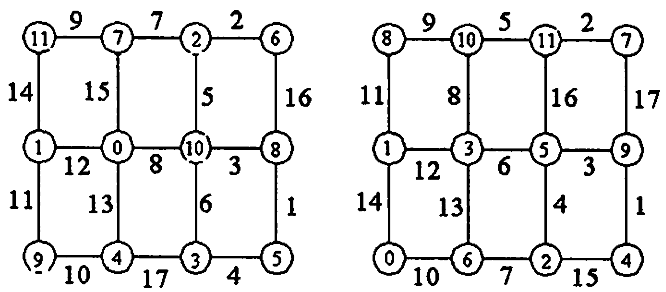


Figure 1.

The edge-graceful labeling of graph was introduced by S.P. Lo [18] in 1985. A necessary condition of edge-gracefulness is (Lo [18])

$$q(q+1) \equiv \frac{p(p-1)}{2} \pmod{p}$$

This latter condition may be more practically stated as $q(q+1) \equiv 0$ or $p/2$ (mod p) depending on whether p is odd or even.

The cartesian product of two paths is frequently called the grid graph . Some edge-graceful grid graphs were considered in [10]. The cartesian product of two cycles is called the torus graph. It was shown in [19,22,25] that the torus graph $C_m \times C_n$ is edge-graceful for all odd m,n >2.

Lee, Lee, Murthy [5] showed that if G is a (p,q)-graph with $p \equiv 2 \pmod 4$ then G is not edge-graceful. Schaffer and Lee [22] have shown that if G and H are both odd-order, regular, edge-graceful graphs, where G is d-regular and has m vertices, and H is k-regular and has n vertices, and furthermore GCD(d,n) = GCD (k,m) =1, then G x H is edge-graceful. In particular, they showed that the torus graph $C_{2i+1} \times C_{2j+1}$ is edge-graceful.

Dharam and Lee [1] recently introduced the following new graph labeling problem. Let a and b be two positive integers. For the graph G with vertex set V(G) and edge set E(G) with p=|V(G)| and q=|E(G)|, we define two sets Q(a) and P(b) as follows:

$Q(a) = \{\pm a, \pm(a+1), \ldots, \pm(a+(q-2)/2)\}$ if q is even,

$Q(a) = \{0\} \cup \{\pm a, \pm(a+1), \ldots, \pm(a+(q-3)/2)\}$ if q is odd,

$P(b) = \{\pm b, \pm(b+1), \ldots, \pm(b+(p-2)/2)\}$ if p is even,

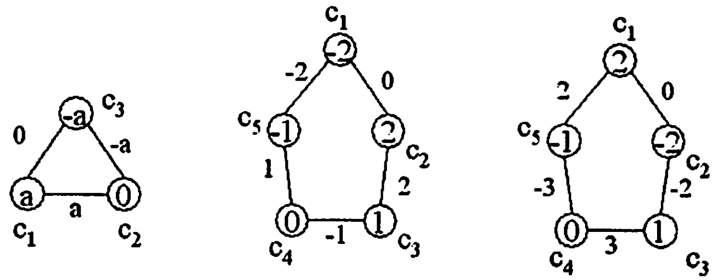
$P(b) = \{0\} \cup \{\pm b, \pm(b+1), \ldots, \pm(b+(p-3)/2)\}$ if p is odd.

Definition 1. A (p,q)-graph G is said to be a Q(a)P(b)-super edge-graceful graph if there exists *a function pair* (f, f^+) which assigns integer labels to the vertices and edges; that is, $f: E(G) \to Q(a)$ and $f^+: V(G) \to P(b)$, such that f^+ and f are bijections, and $f^+(u) = \Sigma\{f(u,v) : (u,v) \in E(G)\}$.

When a =b =1, the notion of Q(1)P(1)- super edge graceful graphs is identical to the concept of super edge-graceful graphs which was introduced by Mitchem and Simoson [20].

We illustrate the above concept with several examples

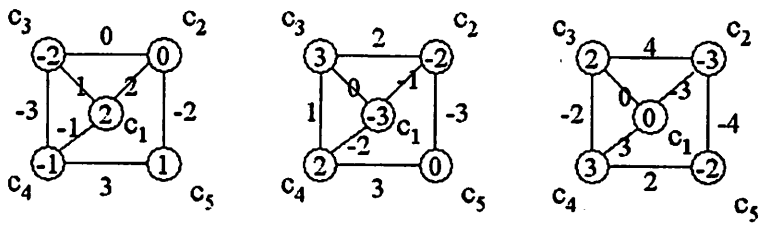
Example 1. The cycle C_3 is Q(a)P(a)-SEG for any $a \geq 1$. However, C_5 is Q(a)P(1)-SEG for a = 1,2.



Q(a)P(1)-SEG Q(1)P(1)-SEG Q(2)P(1)-SEG

Figure 2.

Example 2. The following graph is Q(1)P(1), Q(1)P(2) and Q(2)P(2)-SEG.

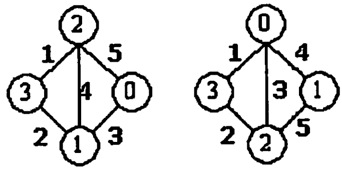


Q(1)P(1)-SEG Q(1)P(2)-SEG Q(2)P(2)-SEG

Figure 3.

Mitchem and Simoson [20] showed that if a tree of odd order is Q(1)P(1)-super edge-graceful then it is edge-graceful. In [8], we see that not all the (p,p+1)-graphs are edge-graceful.

Example 3. The following (4,5)-graph is the smallest order among all the (p,p+1)-graphs. It is edge-graceful (Figure 4). However, it is not Q(1)P(1)-super edge-graceful.



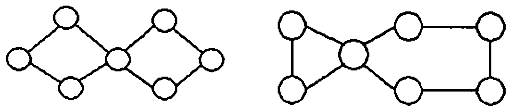
Two edge-graceful labelings

Figure 4.

In this paper we want to investigate the Q(a)P(b)-super edge-gracefulness of the three classes of (p,p+1)-graphs. Finding the Q(a)P(b) -super edge-graceful labelings of graphs are related to solving the systems of linear Diophantine equations. In general it is difficult to find them. Several classes of graphs had been shown to be edge-graceful ([2,3,4,5,6,7,8,10, 23,24,25]). For more conjectures and open problems on edge-graceful graphs the reader is referred to [7]. The reader should see the survey article of Gallian [2] for various labeling problems.

2.Q(a)P(b)-Super Edge Gracefulness of the Amalgamation of Two Cycles

Let G, H be two graphs with A, B are subsets of G and H respectively with |A| =|B|. The amalgamation of (G,A) with (H,B) is the graph obtained by forming the disjoint union of G and H and then identify A and B. If A and B each is one vertex, the construction is called the **one-point union.** We will use Amal(G,H,(A,B)) to denote the amalgamation of (G,A) and (H,B). The following graphs C(4,4) and C(3,5) are the Amal(C_4,C_4,(u,u)) and Amal(C_3,C_5,(u,v)) respectively (Figure 5). We will denote Amal(C_m, C_n,(u,v)) by DC(m,n) and called it **double cycle.**



C(4,4) C(3,5)

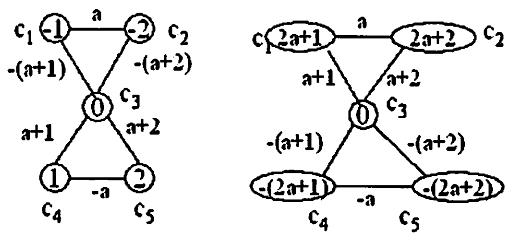
Figure 5.

Lee, Lee, Murthy [5] showed that if G is a (p,q)-graph with $p \equiv 2 \pmod 4$ then G is not edge-graceful. However, we have C(m,n) with $m+n \equiv 3 \pmod 4$ which is Q(1)P(1)-SEG.

We want to address in this section the following problem: for what $m,n \geq 3$, C(m,n) is Q(a)P(b)-SEG.

<u>**Theorem 2.1**</u>. C(3,3) is Q(a)P(1)-SEG and Q(a)P(2a+1)-SEG for $a \geq 1$.

Proof. Figure 6 illustates that C(3,3) is Q(a)P(1)-SEG and Q(a)P(2a+1)-SEG for $a \geq 1$.



Q(a)P(1)-SEG Q(a)P(2a+1)-SEG

Figure 6.

<u>**Theorem 2.2**</u>. C(3,4) is not edge-graceful but it is Q(a)P(1)-SEG for a =1,2,3 and Q(1)P(2)-SEG, Q(2)P(4)-SEG. Furthermore it is not Q(a)P(b)-SEG for $a \neq 1,2,3$.

Proof. Figure 7 shows that C(3,4) is Q(a)P(1)-SEG for a =1,2,3 and Q(1)P(2)-SEG.

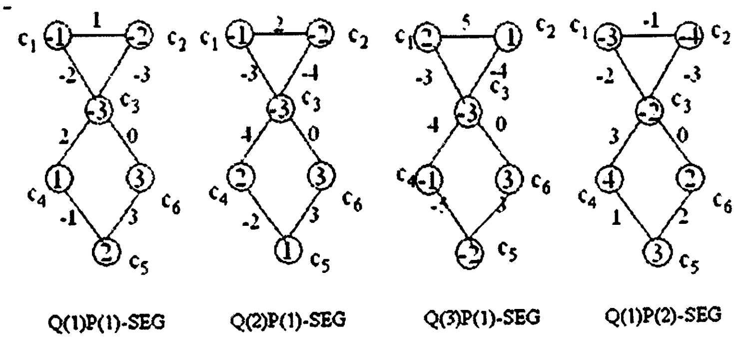


Figure 7.

To see that it is Q(2)P(4)-SEG we label the edges (c_1,c_2), (c_1,c_3), (c_2,c_3),(c_3,c_4), (c_3,c_6), (c_4,c_5),(c_5,c_6) by 2,3,4,0, -3,-4,-2 in C(3,4), then C(3,4) is Q(2)P(4)-SEG.

Now we want to show that C(3,4) is not Q(a)P(b)-SEG for a $\neq$1,2,3. Let a$\geq$4, note that deg $c_3 = 4$, deg c_i =2 in C(3,4), $i = 1,2,4,5,6$.

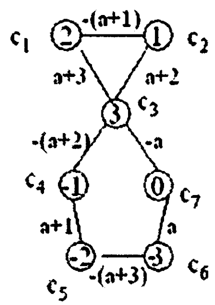
Case 1. If the edge with vertex c_3 is labeled by 0, we have only one $c_{i_0} \in \{c_1,c_2,c_4,c_6\}$, and $f^+(c_{i_0}) = \Sigma\{f(c_{i_0},v) :(c_{i_0}, v) \in E(C(3,4))\} \in \{\pm$ a、$\pm$(a+1)、$\pm$(a+2)}.

Without loss of generality, we may assume that the $f^+(c_{i_0}) = \Sigma\{f(c_{i_0},v) :(c_{i_0}, v) \in E(C(3,4))\}$= a , Since a$\geq$4 ,for any x、y$\in$ {$\pm$ a ,$\pm$ (a+1),$\pm$ (a+2)}, x+y$\notin$ {$\pm$ (a+1), $\pm$ (a-1)}, no matter what the $f^+(c_3)$ may be, $\{f^+(c_i): c_i \in V(C(3,4))\}$ is not a subset of consecutive numbers. According to our definition, C(3,4) is not Q(a)P(b)-SEG for a $\neq$1,2,3

Case 2. If the edge with vertex c_3 is not labeled by 0, we have two vertices c_{i_1} and $c_{i_2} \in \{c_1, c_2, c_4, c_5, c_6\}$ and $f^+(c_{i_1})$、$f^+(c_{i_2}) \in \{\pm a, \pm(a+1), \pm(a+2)\}$. Assume $f^+(c_{i_1}) \neq -f^+(c_{i_2})$, since $a \geq 4$, for any x、y、s、t $\in \{\pm a, \pm(a+1), \pm(a+2)\}$, we have $\{x+y+s+t, s+t\} \neq \{-f^+(c_{i_1}), -f^+(c_{i_2})\}$, or $\{x+y, s+t\} \neq \{-f^+(c_{i_1}), -f^+(c_{i_2})\}$. If $f^+(c_{i_1}) = -f^+(c_{i_2})$, for any x、y、s、t $\in \{\pm a$、$\pm(a+1)$、$\pm(a+2)\}$, we also see $\{x+y+s+t, s+t\} \neq \{\pm(f^+(c_{i_1})+1)\}$, or $\{x+y, s+t\} \neq \{\pm(f^+(c_{i_1})+1)\}$, or $\{x+y+s+t, s+t\} \neq \{\pm(f^+(c_{i_1})-1)\}$, or $\{x+y, s+t\} \neq \{\pm(f^+(c_{i_1})-1)\}$. According to our definition, C(3,4) is not Q(a)P(b)-SEG for $a \neq 1,2,3$.

__Theorem 2.3.__ C(3,5) is Q(a)P(1)-SEG for $a \geq 1$ and Q(1)P(b)-SEG if and only if b =2,3,4.

Proof. Figure 8 shows that C(3,5) is Q(a)P(1)-SEG for $a \geq 1$



Q(a)P(1)-SEG

Figure 8.

Figure 9 shows that C(3,5) is Q(1)P(b)-SEG for b =2,3,4.

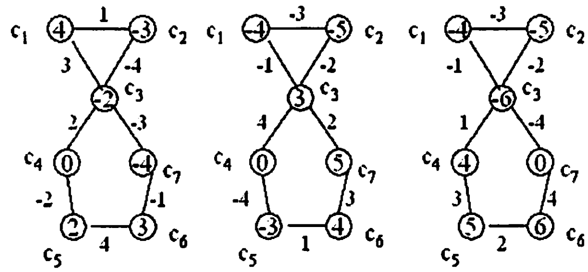


Figure 9.

Suppose that the graph C(3,5) is Q(1)P(b)-SEG , for b≥5. Since C(3,5) has 7 vertices and 8 edges, We label the edges of C(3,5) by ± 1, ± 2, ± 3, ± 4, and the induced vertex label set are {0,±b , ±(b+1) ±(b+2)},b≥5. Notice that we need to label the some edges of the C_5 in the C(3,5) with 1,2,3,4 (or with −1,−2,−3,−4), if not, there is a vertex in C_5 with a label value smaller than b, if not ,there is a label value of vertex in C_5 ,the value is smaller than b.

We consider the following case : Assume any 4–permutations of 1,2,3,4 (or −1,−2,−3,−4) is $x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}$ Now as { $x_{i_1}+x_{i_2}$, $x_{i_2}+x_{i_3}$, $x_{i_3}+x_{i_4}$ } ≠{ b,b+1,b+2 },b≥5 (or { $x_{i_1}+x_{i_2}$, $x_{i_2}+x_{i_3}$, $x_{i_3}+x_{i_4}$ }≠{−b,−(b+1),−(b+2)}, b≥5), we see that C(3,5) is not Q(1)P(b)-SEG, b≥5.

Theorem 2.4. C(3,6) is Q(a)P(1)-SEG for a =1,2,3,4 and Q(1)P(b)-SEG for b =2 and 3.

Proof. We see that C(3,6) is Q(a)P(1)-SEG for a =1,2,3,4, and Q(1)P(b)-SEG for b =2 and 3.

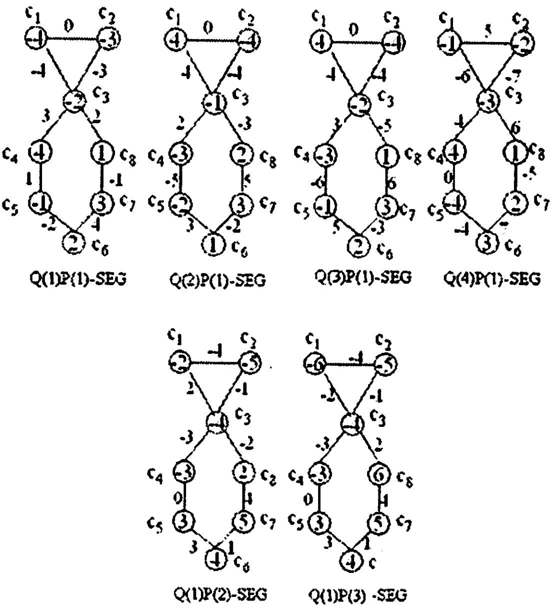


Figure 10.

Theorem 2.5. C(4,4), C(4,5), C(4,6), C(4,7) are Q(1)P(1)-SEG.

Proof. We see that C(4,4) and C(4,6) are Q(a)P(1)-SEG for $a \geq 1$, and that C(4,5) and C(4,7) are Q(1)P(1)-SEG.

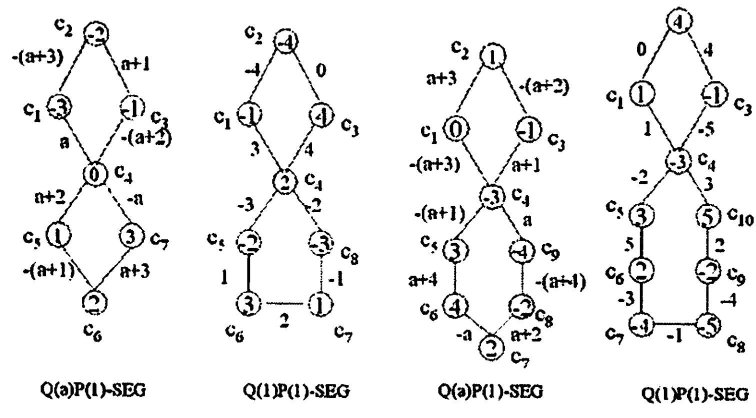


Figure 11.

3. Q(a)P(b)-Super Edge Gracefulness of Cycle with a Chord

In this section, we consider the (p,p+1)-graphs that are cycles with a chord. Assume the vertices of cycle are$\{v_1,v_2,\ldots,v_p\}$ and the chord connect vertex v_1 with v_r, we denote this graph by $C_p(r)$.

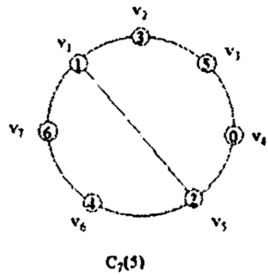


Figure 12.

In [8], Lee, Chen and Wang showed that

Theorem 3.1. If G is a cycle with one chord of odd order p, then G is edge-graceful.

Theorem 3.2. The graph $C_4(3)$ is not Q(a)P(b)-SEG for any a , $b \geq 1$.

Proof. Suppose that the graph $C_4(3)$ is Q(a)P(b)-SEG for some a , b. Since $C_4(3)$ has 4 vertices and 5 edges, there is one and only edge labeled by 0.

Consider the case that the edge (v_1, v_3) is not labeled by 0. Without loss of generality, the possible labeling is displayed in the following figure, where x should be a or a+1 and y should be the other one.

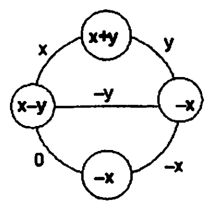


Figure 13

We find that there are two vertices labeled by the same value, -x. That is a contradiction.

So we consider the other case that the edge (v_1, v_3) is labeled by 0.

Remove the edge (v_1, v_3), we should have a Q(a)P(b)-SEG labeling on C_4 ,which is impossible. The reason is shown in Figure 14.

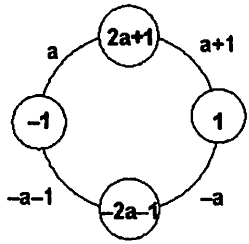


Figure 14.

<u>**Theorem 3.3**</u>. The graph $C_5(3)$ is Q(a)P(b)-SEG for

(1) a = 1 and $b \leq 3$

(2) a = 2 and $b \leq 1$.

Proof. We list here four Q(1)P(1)-SEG labelings for $C_5(3)$. We note here $l_3 \neq l_4$. However $l_3^+ = l_4^+$.

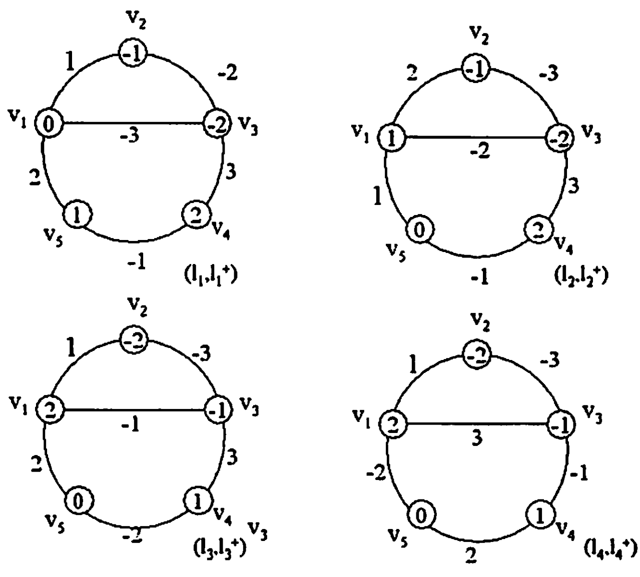


Figure 15.

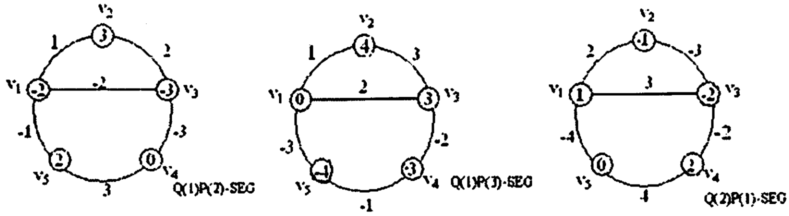


Figure 16.

Remark. C_6 is not Q(1)P(1)-SEG. However, we see that $C_6(3)$ (Figure 17) and $C_6(4)$ (Figure 18) are Q(1)P(1)-SEG.

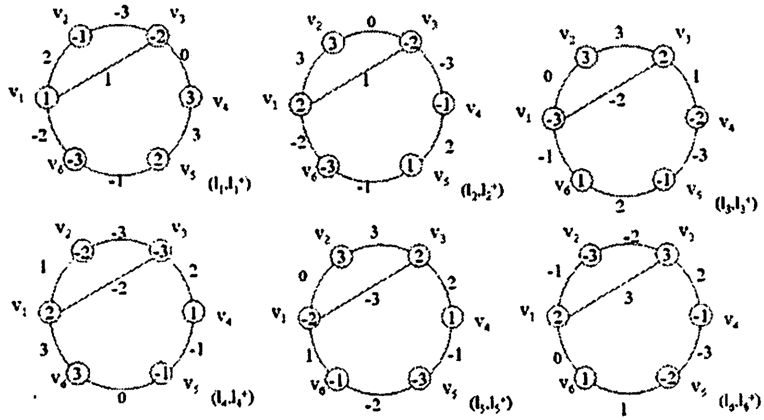


Figure 17.

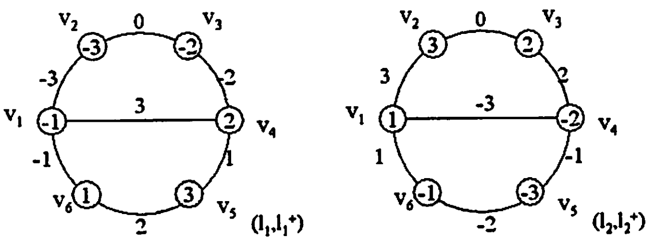


Figure 18.

Theorem 3.4. $C_{2n}(r)$ is Q(1)P(1)-SEG for any $n = 4k$ and any $r \geq 3$.

Proof. We give an Q(1)P(1)-SEG edge labeling of $C_{8k}(r)$ as follows:

. We label the edges $(v_{8k},v_1),(v_1, v_2), (v_2, v_3), \dots, (v_{2k-1},v_{2k})$ by $4k, -1, 4k-1, -2, 4k-2, -3, \dots, 4k-i+1, -i, \dots, 3k+1, -k$. We label the edges $(v_{2k}, v_{2k+1}), (v_{2k+1},v_{2k+2}), (v_{2k+2},v_{2k+3}), \dots, (v_{4k-1},v_{4k})$ by $-3k, k+1, -3k+1, k+2, -3k+2, k+3, \dots, -3k+i, k+i+1, \dots, -2k-1, 2k$. We label the edges, $(v_{4k},v_{4k+1}), (v_{4k+1}\ v_{4k+2}), (v_{4k+2},v_{4k+3}), \dots, (v_{6k-1}, v_{6k})$ by $-4k, 1, -4k+1, 2, -4k+2, 3, \dots, -4k+i-1, i, \dots, -3k-1, k$. We label the edges, $(v_{6k},v_{6k+1}),(v_{6k+1}, v_{6k+2})(v_{6k+2}, v_{6k+3}) \dots, (v_{8k-1}, v_{8k})$ by $3k, -k-1, 3k-1, -k-2, 3k-2, -k-3, \dots, 3k-i, -k-i-1, \dots, 2k+1, -2k$.

Then the vertices, $v_1, v_2, v_3, \dots, v_{2k-1}$ are labeled by $4k-1, 4k-2, 4k-3, \dots, 2k+1$, the vertices $v_{2k+1}, v_{2k+2}, v_{2k+3}, \dots, v_{4k-1}$ by $-2k+1, -2k+2, -2k+3, \dots, -1$, the vertices $v_{4k+2}, v_{4k+3}, v_{4k+4}, \dots, v_{6k-1}$ by $-4k+1, -4k+2, -4k+3, \dots, -2k-1$, the vertices $v_{6k+1}, v_{6k+2}, v_{6k+3} \dots, v_{8k-1}$ by $2k-1, 2k-2, 2k-3, \dots, 1$, and the four vertices $v_{2k}, v_{4k}, v_{6k}, v_{8k}$ by $-4k, -2k, 4k, 2k$, respectively.

Now we extend the edge label of (v_1,v_r) by 0.We see the induced vertex labels is unchanged.

Example 4. We illustrate a Q(1)P(1)-SEG labeling for $C_8(5)$ in Figure 19.

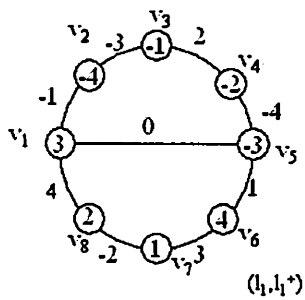


Figure 19.

Theorem 3.5. $C_{2n}(n+1)$ is Q(1)P(1)-SEG if and only if $n \geq 3$.

Proof. For n =3, we see $C_6(4)$ (Figure 18) is Q(1)P(1)-SEG.

For n =4, we see $C_8(5)$ (Figure 20) is Q(1)P(1)-SEG.

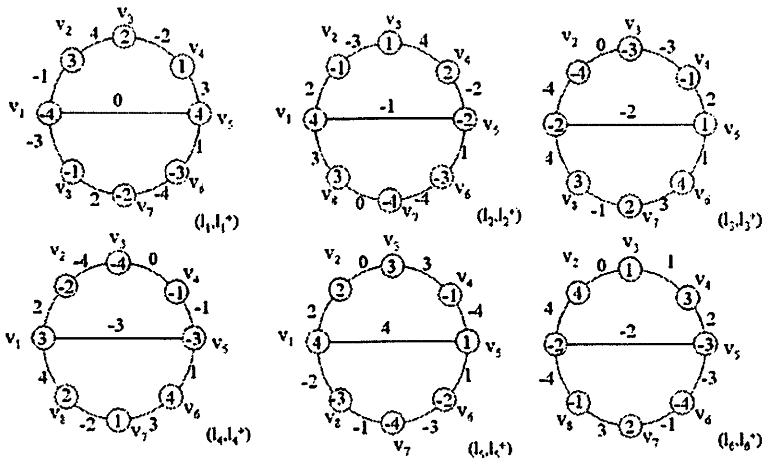


Figure 20.

For n =5 , we see $C_{10}(6)$ (Figure 21) is Q(1)P(1)-SEG.

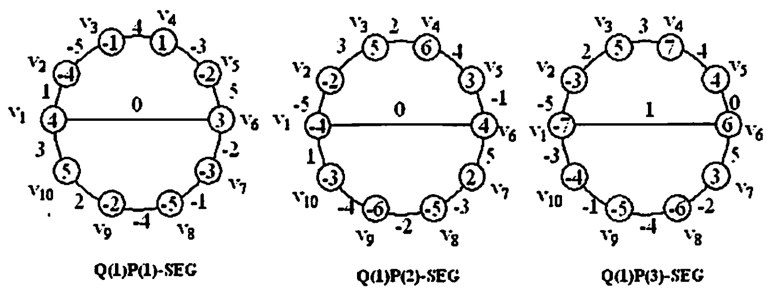


Figure 21.

4. Q(a)P(b)-Super Edge Gracefulness of Dumbbell Graphs.

The dumbbell graph D(a,b) is formed by join two disconnected cycles C_a and C_b by an edge.(Figure 22.)

Theorem 4.1. The dumbbell graph D(3,3) is Q(a)P(b) SEG for

(1) a =1 and b =1,2,3

(2) a = 2, 3 and b =1.

(3) a$\geq$4, b = 2a+1.

The graph D(3,3) is not Q(a)P(1) SEG for a$\geq$4.

Proof. (1) and (2) The dumbbell graph D(3,3) is Q(a)P(b) SEG for a =1 and b =1,2,3 and a = 2, 3 and b =1.

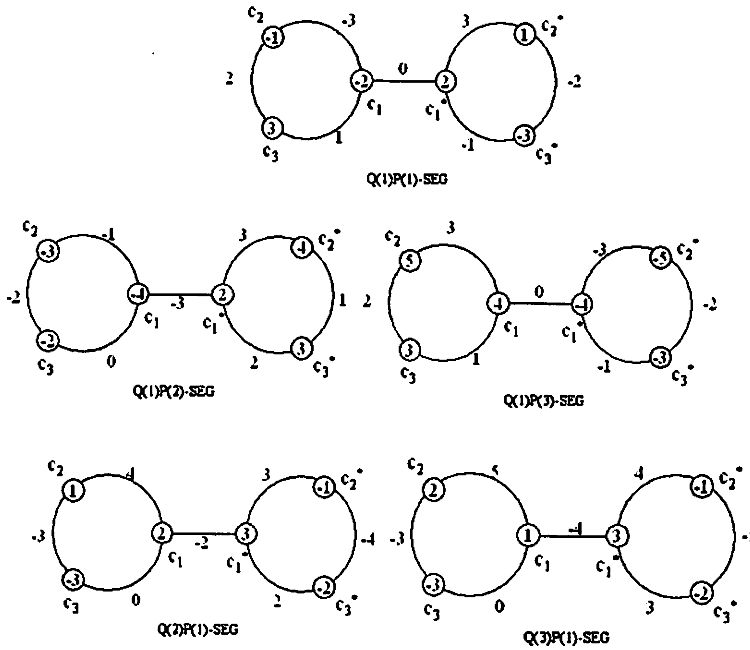


Figure 22 .

(3)D(3,3) is Q(a)P(2a+1)-SEG, a≥4.

Let a≥4, If we label the edges (c_1, c_2) ,(c_1, c_3), (c_2, c_3), (c_1, c_1^*), (c_1^*, c_2^*), (c_1^*, c_3^*), (c_2^*, c_3^*) by a,a+1,a+2,0,-a,-(a+1),-(a+2) in D(3,3), then D(3,3) is Q(a)P(2a+1)-SEG.

D(3,3) is not Q(a)P(1)-SEG, a≥4. Let a≥4, note the dumbbell graph D(3,3) is formed by joining two disconnected cycles C_3 and C_3 by an edge. If we label the edge (c_1, c_1^*) by 0, for any x,y,z∈ {± a,± (a+1),± (a+2)}, we observe all the x,y,z are not positive integers at the same time, or all the x,y ,z are not negative integers at the same time. Without loss of generality, we may assume that the x,y are positive integers(or negative integers), z is negative integer (or positive integer), we have min{ $|x+y|$ }=2a+1, max{ $|x+z|,|y+z|$ }≤2, then we see {$f^+(c_i)$: $c_i \in$ V (D(3,3))} is not a subset of consecutive natural number and the negative natural number. According to our definition, D(3,3) is not Q(a)P(1)-SEG for a≥4.

If we label the edge $(c_1, c_1^*)$by x ,x∈{± a,± (a+1),± (a+2)}, we have at least one vertex $c_{i_0} \in$ V (D(3,3)) ,and $f^+(c_{i_0})$∈{± a , ± (a+1), ± (a+2)}, D(3,3) is not Q(a)P(1)-SEG due to a≥4.

Theorem 4.2. For any n ≥ 3, the dumbbell graph DB(n,n) is Q(1)P(1)-SEG.

Proof. Let C_a and C_b are cycles with vertices $u_1,u_2,\ldots,u_n$.and $v_1,v_2,\ldots,v_n$ respectively.

Consider two cases.

Case 1. n is odd. Assume that $n=2k-1, k\geq 2$. We label the edges $(u_1,u_2), (u_3,u_4), (u_5,u_6) \ldots, (u_{2k-3},u_{2k-2}), (u_{2k-1},u_1)$ by 0,-1,-2,...,-(k-1) and label the edges (u_2,u_3) , (u_4,u_5) , (u_6,u_7) ⋯ , $(u_{2k-4},u_{2k-3}), (u_{2k-2},u_{2k-1})$ by 2k-1,2k-2,2k-3,...,k+1.

We label the edges (v_1,v_2) , (v_3,v_4) , (v_5,v_6) ⋯, (v_{2k-3},v_{2k-2}) , (v_{2k-1},v_1) by $-(2k-1)$,-(2k-2),-(2k-3),...,-k, and label the edges (v_2,v_3) , (v_4,v_5) , (v_6,v_7) ⋯ , (v_{2k-4},v_{2k-3}) , (v_{2k-2},v_{2k-1}) by 1,2,3,...,(k-1).We label the edge (u_1,v_1) by k.

Then the vertices $u_1,u_2,\ldots,u_{2k-1}$ are labeled by 1,(2k-1),(2k-2),(2k-3),...,3,2, and the vertices $v_1,v_2,\ldots,v_{2k-1}$ are labeled by -(2k-1),-(2k-2),...,-3,-2,-1.

Case 2. n is even. Assume that $n=2k, k\geq 2$. We label the edges $(u_1,u_2), (u_3,u_4)$,
(u_5,u_6) ⋯, $(u_{2k-3},u_{2k-2}), (u_{2k-1},u_{2k})$ by 0,-1,-2,...,-(k-1) and label the edges (u_2,u_3),

$(u_4,u_5),(u_6,u_7)\cdots,\ (u_{2k-2},u_{2k-1}),(u_{2k},u_1)$ by 2k,2k-1,2k-2,...,k+1.

We label the edges (v_1,v_2) , (v_3,v_4) , (v_5,v_6) $\cdots$, (v_{2k-3},v_{2k-2}) , (v_{2k-1},v_{2k}) by $-2k$, -(2k-1),-(2k-2),...,-(k+1), and label the edges $(v_2,v_3),(v_4,v_5),(v_6,v_7)\cdots,\ (v_{2k-2},v_{2k-1}),\ (v_{2k},v_1)$ by 1,2,3,...,k.We label the edge (u_1,v_1) by $-k$.

Then the vertices $u_1,u_2,\cdots,u_{2k}$ are labeled by 1,2k,(2k-1),(2k-2),....,3,2, and the vertices $v_1,v_2,\ldots,v_{2k}$ are labeled by –(2k-1),-(2k-2), -(2k-3), ..., -3, -2, -1.

Thus for any $n=2k$, $k\geq 2$, the dumbbell graph DB(n,n) is Q(1)P(1)-SEG.

Example 5. . We illustrate a Q(1)P(1)-SEG labeling for DB(7,7), DB(8,6) and Q(1)P(1)-SEG labelings for DB(9,9), DB(10,10) respectively, in Figure 23 .

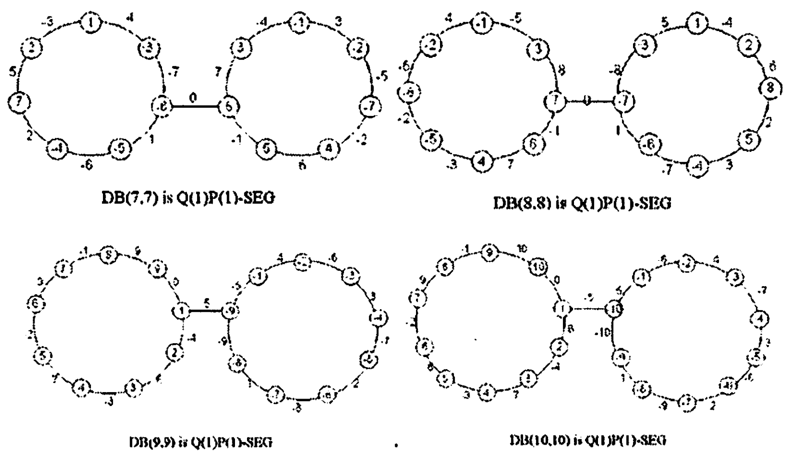


Figure 23 .

Theorem 4.3. DB(4,4) is Q(a)P(b)-SEG for

(1) a =1 and b = 1,2,3.

(2) a = 2,3,4 and b =1.

Proof.

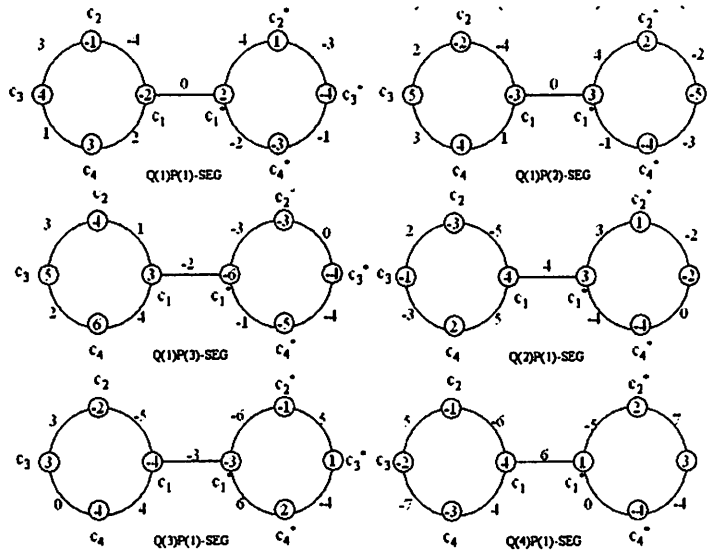


Figure 24.

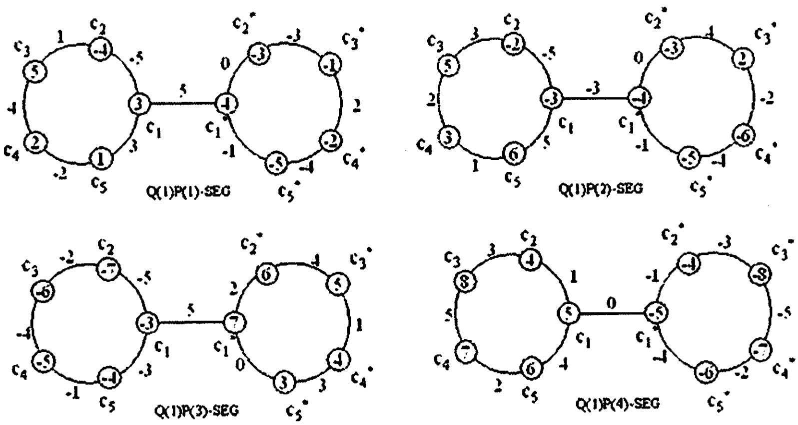
Theorem 4.4. DB(5,5) is Q(a)P(b)-SEG for

(1) a =1 and b = 1,2,3,4.

(2) a =2 and b = 1,2,3,4,5,6.

(3) a =3,4,5 and b =1.

Proof.



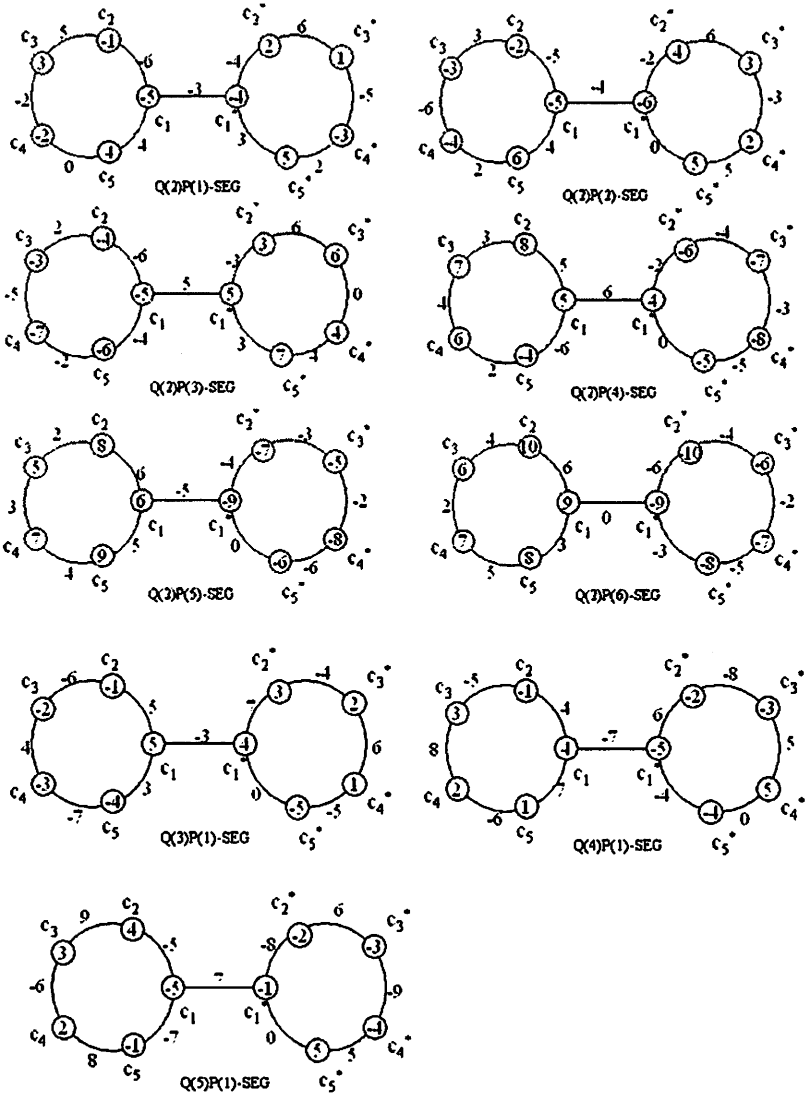


Figure 25.

Theorem 4.6. DB(6,6) is Q(a)P(b)-SEG for

(1) a = 1 and b = 1,2,3,4.

(2) a = 2 and b = 1,2,3,4,5,6.

Proof.

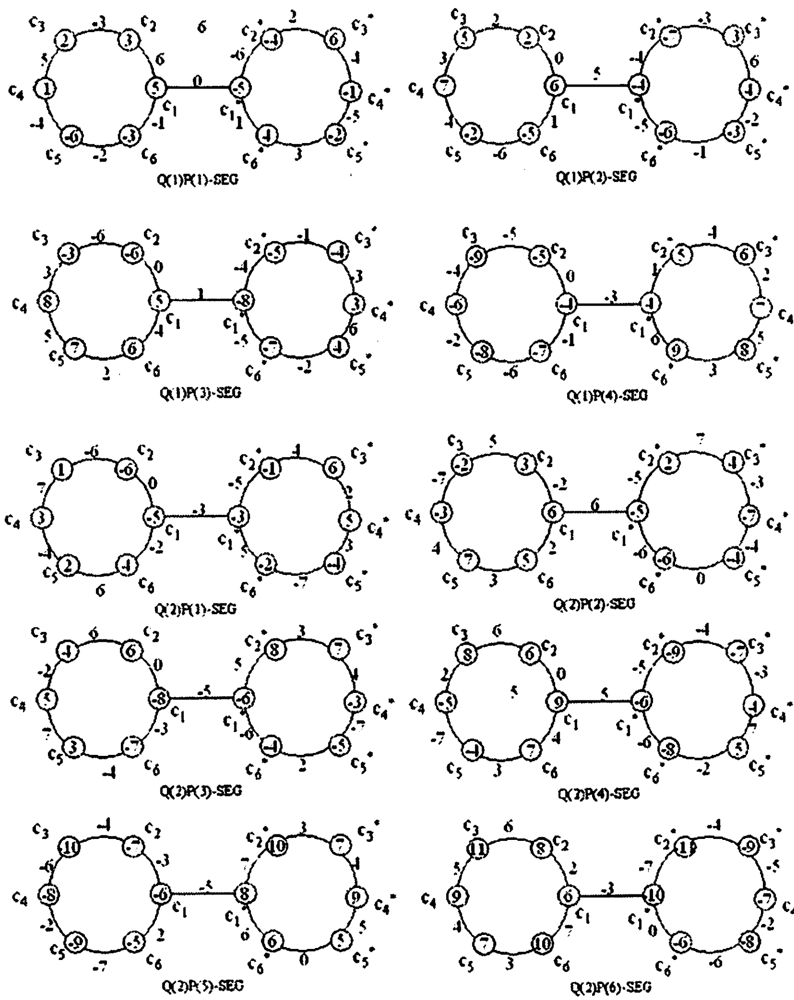


Figure 26.

5. Conclusion.

In this paper we try to address the following problem: "For what (a,b) we have Q(a)P(b)-SEG (p,p+1)-graphs?". At present we have only touched the surface of this problem, a lot of problems remain unsolved. We invite readers to consider the following three conjectures.

Conjecture 1. $C_n(3)$ is Q(1)P(1)-SEG for all $n \geq 8$

Conjecture 2. C(m,n) is Q(a)P(1)-SEG if m+n is even.

Conjecture 3. For any $r \geq 3$, $C_{2n}(r)$ is Q(1)P(1)-SEG for all $n \geq 3$.

References

[1] D. Chopra and Sin-Min Lee, On Q(a)P(b)-super edge-graceful graphs, *The Journal of Combinatoric Mathematics and Combinatoric Computing* **58**, 135-152,2006.

[2] J.A. Gallian, A dynamic survey of graph labeling, *The Electronic J. of Combin.* # DS6, 1-79, 2001.

[3] Jonathan Keene and Andrew Simoson, Balanced strands for asymmetric, edge-graceful spiders, *Ars Combinatoria* **42**,49-64,1996

[4] Q. Kuan, Sin-Min Lee, J. Mitchem, and A.K. Wang, On edge-graceful unicyclic graphs, *Congressus Numerantium* **61**, 65-74, 1988.

[5]Li Min Lee, Sin Min Lee, and G. Murty, On edge-graceful labelings of complete graphs - solutions of Lo's conjecture, *Congressum Numerantum* **62**, 225-233, 1988.

[6] Sin-Min Lee, A conjecture on edge-graceful trees, *Scientia*, Ser. A, vol.3, 45-57, 1989.

[7] Sin-Min Lee, New directions in the theory of edge-graceful graphs, *Proc. 6th Caribbean Conference. Combinatoric and Computing* 216-231, 1991.

[8] Sin-Min Lee, E. Chen , E.R.Yera and Ling Wang, On super edge-graceful (p,p+1)- graphs. *Congressum Numerantum* **171**, . 51-65, 2004.

[9] Sin-Min Lee, K.J.Chen and Yung-Chin Wang, On the Edge-graceful

spectra of the cycles with one chord and dumbbell graphs, *Congressum Numerantum* **170**, 171-183, 2004.

[10] Sin-Min Lee, Peining Ma, Linda Valdes, and Siu-Ming Tong, On the edge-graceful grids, *Congressus Numerantium* **154**,61-77, 2002.

[11]Sin-Min Lee and Eric Seah, Edge-graceful labelings of regular complete k-partite graphs, *Congressum Numerantum* **75**, 41-50, 1990.

[12] Sin-Min Lee and Eric Seah, On edge-gracefulnes of the composition of step graphs with null graphs, *Combinatorics, Algorithms, and Applications in Society for Industrial and Applied Mathematics,*, 326-330, 1991.

[13]Sin-Min Lee and Eric Seah, On the edge-graceful (n,kn)-multigraphs conjecture, *Journal of Combinatorial Mathematics and Combinatorial Computing*, Vol. 9, 141-147, 1991.

[14]Sin-Min Lee, E. Seah and S.P. Lo , On edge-graceful 2-regular graphs, *The Journal of Combinatoric Mathematics and Combinatoric Computing*, **12**, 109-117,1992.

[15] Sin-Min Lee, E. Seah, Siu-Ming Tong, On the edge-magic and edge-graceful total graphs conjecture, *Congressus Numerantium* **141**, 37-48 ,1999

[16] Sin-Min Lee, E. Seah and P.C. Wang , On edge-gracefulness of the kth power graphs, *Bulletin of the Institute of Math, Academia Sinica* **18**, No. 1, 1-11, 1990.

[17] Sin-Min Lee, E. Seah, Siu-Ming Tong, On the edge-magic and edge-graceful total graphs conjecture, *Congressus Numerantium* **141**,37-48, 1999.

[18] S.P. Lo, On edge-graceful labelings of graphs, *Congressus Numerantium*, **50**, 231-241, 1985.

[19] Peng Jin. and Li W., Edge-gracefulness of C_mxC_n, in *Proceedings of the Sixth Conference of Operations Research Society of China*, (Hong Kong: Global-Link Publishing Company), Changsha, October 10-15, p.942-948, 2000.

[20] J. Mitchem and A. Simoson, On edge-graceful and super-edge-graceful graphs. *Ars Combin.* **37**, 97-111, 1994.

[21] A. Riskin and S. Wilson, Edge graceful labellings of disjoint unions of cycles. *Bulletin of the Institute of Combinatorics and its Applications* **22**: 53-58,1998.

[22] Karl Schaffer and Sin-Min Lee, Edge-graceful and edge-magic labelings of Cartesian products of graphs, *Congressus Numerantium* **141**, 119-134, 1999.

[23] W.C. Shiu, Sin-Min Lee and K. Schaffer, Some k-fold edge-graceful labelings of (p,p-1)-graphs, *The Journal of Combinatoric Mathematics and Combinatoric Computing* **38**, 81-95, 2001.

[24] Yihui Wen, Sin-Min Lee and Siu-Ming Tong, On the edge-graceful edge-splitting extension of stars to appear in *JCMCC*.

[25] S. Wilson and A. Riskin, Edge-graceful labellings of odd cycles and their products, *Bulletin of the ICA*, **24**, 57-64, 1998.