On $Q(a)P(b)$-Super Edge-Gracefulness of Hypercubes

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Abstract

Let a, b be two positive integers. A (p, q)-graph G is said to be $Q(a)P(b)$-super edge-graceful, or simply (a, b)-SEG, if there exist onto mappings $f : E(G) \to Q(a)$ and $f^* : V(G) \to P(b)$, where

$$Q(a) = \begin{cases} \{\pm a, \pm(a+1), \ldots, \pm(a+(q-2)/2)\} & \text{if } q \text{ is even,} \\ \{0, \pm a, \pm(a+1), \ldots, \pm(a+(q-3)/2)\} & \text{if } q \text{ is odd,} \end{cases}$$

$$P(b) = \begin{cases} \{\pm b, \pm(b+1), \ldots, \pm(b+(p-2)/2)\} & \text{if } p \text{ is even,} \\ \{0, \pm b, \pm(b+1), \ldots, \pm(b+(p-3)/2)\} & \text{if } p \text{ is odd,} \end{cases}$$

such that $f^*(v) = \sum_{vu \in E(G)} f(uv)$. We find the values of a and b for which the hypercube Q_n, $n \le 3$, is (a, b)-SEG.

1 Introduction

Around 1967, Rosa [8] introduced the notion of graceful labeling to study the cyclic decomposition of a complete graph into trees. A graph G with p vertices and q edges is said to be ***graceful*** if there is an one-to-one vertex labeling $f : V(G) \to \{0, 1, \ldots, q\}$ such that the induced edge labeling $f^* : E(G) \to \{1, 2, \ldots, q\}$ defined by

$$f^*(uv) = |f(u) - f(v)|$$

is onto. The famous conjecture of Ringel and Kotzig that states

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All trees are graceful

remains unsolved. Since the appearance of Rosa's article, graph labeling has become an active area in graph theory; see [2] for an extensive bibliography.

S.P. Lo [6] studied a dual form in 1985. He called G ***edge-graceful*** if there exists an one-to-one edge labeling $f : E(G) \to \{1, 2, \ldots, q\}$ so that the induced vertex labeling $f^* : V(G) \to \{0, 1, \ldots, p-1\}$ defined by

$$f^*(v) \equiv \sum_{uv \in E(G)} f(uv) \pmod p$$

is onto. Lo found $q(q+1) \equiv p(p-1)/2 \pmod p$ a necessary condition for a graph to be edge-graceful. S.M. Lee [3] conjectured that it is also a sufficient condition for a connected graph to be edge-graceful. A sub-conjecture resembles the famous conjecture of Ringel and Kotzig:

All trees of odd order are edge-graceful.

In their attempt to solve this conjecture, Mitchem and Simoson [7] noticed that modulo p, half of the integers in the vertex labels $0, 1, \ldots, p-1$ can be written as negative residues. This observation led them to define

$$Q = \begin{cases} \left\{\pm 1, \pm 2, \ldots, \pm \frac{q}{2}\right\} & \text{if } q \text{ is even,} \\ \left\{0, \pm 1, \pm 2, \ldots, \pm \frac{q-1}{2}\right\} & \text{if } q \text{ is odd,} \end{cases}$$

$$P = \begin{cases} \left\{\pm 1, \pm 2, \ldots, \pm \frac{p}{2}\right\} & \text{if } p \text{ is even,} \\ \left\{0, \pm 1, \pm 2, \ldots, \pm \frac{p-1}{2}\right\} & \text{if } p \text{ is odd.} \end{cases}$$

and call a graph ***super edge-graceful*** if there exists an one-to-one edge labeling $f : E(G) \to Q$ such that the induced vertex labeling $f^* : V(G) \to P$ defined by $f^*(v) = \sum_{uv \in E(G)} f(uv)$ is onto.

S.M. Lee further generalized this by replacing 1 with positive integers a and b in Q and P respectively. Let

$$Q(a) = \begin{cases} \left\{\pm a, \pm(a+1), \ldots, \pm\left(a + \frac{q-2}{2}\right)\right\} & \text{if } q \text{ is even,} \\ \left\{0, \pm a, \pm(a+1), \ldots, \pm\left(a + \frac{q-3}{2}\right)\right\} & \text{if } q \text{ is odd,} \end{cases}$$

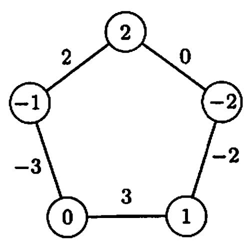
$$P(b) = \begin{cases} \left\{\pm b, \pm(b+1), \ldots, \pm\left(b + \frac{p-2}{2}\right)\right\} & \text{if } p \text{ is even,} \\ \left\{0, \pm b, \pm(b+1), \ldots, \pm\left(b + \frac{p-3}{2}\right)\right\} & \text{if } p \text{ is odd.} \end{cases}$$

A graph is said to be $Q(a)P(b)$***-super edge-graceful*** if there is an one-to-one edge labeling $f : E(G) \to Q(a)$ such that the induced vertex labeling $f^* : V(G) \to P(a)$ defined by

$$f^*(v) = \sum_{uv \in E(G)} f(uv)$$

is onto. For brevity, we will simply call the graph (a, b)-SEG.

Here is a $(2,1)$-SEG labeling of C_5, in which $Q(2) = \{0, \pm 2, \pm 3\}$ and $P(1) = \{0, \pm 1, \pm 2\}$:



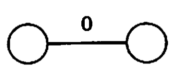
Read [1] for other results in (a, b)-SEG labelings, and [4, 5] for a dual version. In this article, we determine the values of a and b for which the hypercube Q_n, $n \leq 3$, is (a, b)-SEG.

2 Labeling of Q_1 and Q_2

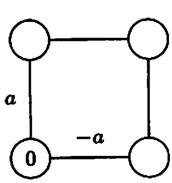
The cases of Q_1 and Q_2 are easy to settle.

Theorem 2.1 *The hypercubes Q_1 and Q_2 are not (a, b)-SEG for any positive integers a and b.*

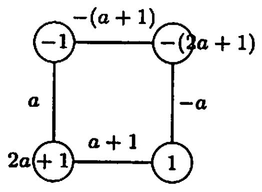
Proof. The hypercube $Q_1 = P_2$ is not (a, b)-SEG for any a and b, because $Q(a) = \{0\}$, which makes both vertex labels 0. See the diagram below.



The hypercube $Q_2 = C_4$ is also not (a, b)-SEG for any a and b. Here is the reason. Note that $Q(a) = \{\pm a, \pm(a+1)\}$, $P(b) = \{\pm b, \pm(b+1)\}$. The two edges labeled a and $-a$ cannot be incident at any vertex, for otherwise, that vertex will be labeled 0.



Hence the edges labeled a and $-a$ must be disjoint from each other. This forces the following labeling.



Since the positive vertex labels are not consecutive, Q_2 cannot be (a, b)-SEG. □

3 Labeling of Q_3 When $a \geq b$

We shall prove that Q_3 is (a, b)-SEG only for a few values of a if $a \geq b$.

Lemma 3.1 *The hypercube Q_3 cannot be (a, b)-SEG if $a > b + 4$.*

Proof. Assume $a - 4 > b$. Recall that $Q(a) = \{\pm a, \pm(a+1), \ldots, \pm(a+5)\}$ and $P(b) = \{\pm b, \pm(b+1), \pm(b+2), \pm(b+3)\}$. The smallest positive label is $a + (a+1) - (a+5) = a - 4$. Since $a - 4 > b$, no vertex in Q_3 would be labeled b. Hence Q_3 cannot be (a, b)-SEG. □

Lemma 3.2 *The hypercube Q_3 cannot be $(b + 4, b)$-SEG.*

Proof. Inspecting $Q(b + 4) = \{\pm(b + 4), \ldots, \pm(b + 9)\}$, we find only one way to label a vertex b. Symbolically, we write

$$b = (b+4) + (b+5) - (b+9) \tag{1}$$

to indicate that the vertex labeled b is incident to the edges labeled $b + 4$, $b + 5$ and $-(b + 9)$. In a similar fashion, there are only two ways to obtain the vertex label $b + 1$:

$$b + 1 = (b+4) + (b+5) - (b+8), \tag{2}$$
$$b + 1 = (b+4) + (b+6) - (b+9). \tag{3}$$

Note that (1) and (2) cannot happen at the same time, for it would require the edges labeled $b+4$ and $b+5$ to be incident to the same pair of vertices. Likewise, (1) and (3) cannot occur simultaneously either. Therefore we cannot obtain both vertex labels b and $b+1$ at the same time, proving that Q_3 cannot be $(b + 4, b)$-SEG. □

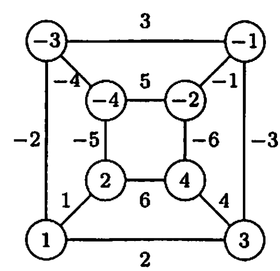
Lemma 3.3 *The hypercube Q_3 cannot be $(b + 3, b)$-SEG.*

Proof. The proof is similar to that of $a = b+4$. Comparing how the vertex labels b, $b+1$, $b+2$ could be obtained, we always find a contradiction due to one or more of the following reasons:

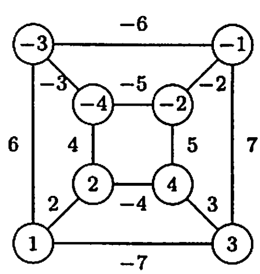
- More than one edge is assigned the same label.
- Multiple edges are formed between two vertices.
- A triangle is formed.

Hence Q_3 is not $(b+3, b)$-SEG. □

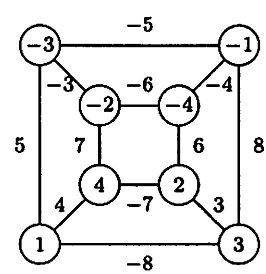
We are left with $a = b, b+1, b+2$. We find the following $(a, 1)$-SEG labelings of Q_3, where $a = 1, 2, 3$.



$Q(1) = \{\pm 1, \ldots, \pm 6\}$
$P(1) = \{\pm 1, \ldots, \pm 4\}$



$Q(2) = \{\pm 2, \ldots, \pm 7\}$
$P(1) = \{\pm 1, \ldots, \pm 4\}$



$Q(3) = \{\pm 3, \ldots, \pm 8\}$
$P(1) = \{\pm 1, \ldots, \pm 4\}$

Call an (a, b)-SEG labeling ***balanced*** if the the numbers of positive and negative edges incident at any vertex always differ by ± 1; and ***unbalanced*** otherwise. In other words, an (a, b)-SEG labeling is unbalanced if there is a vertex incident to three edges all with the same sign.

Lemma 3.4 *If a cubic graph admits a balanced (a, b)-SEG labeling, then it is $(a+k, b+k)$-SEG for any nonnegative integer k.*

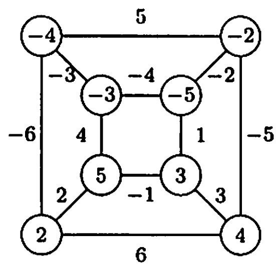
Proof. Replacing each positive edge label x with $x+k$, and each negative edge label $-x$ with $-x-k$ yields an $(a+k, b+k)$-SEG labeling. □

Since all $(a,1)$-SEG labelings shown above are balanced, we see that Q_3 is (a,b)-SEG whenever $a = b, b+1, b+2$. Combining this with Lemmas 3.1, 3.2 and 3.3, we obtain the following result.

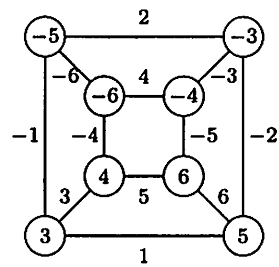
Theorem 3.5 *For $a \geq b$, the hypercube Q_3 is (a,b)-SEG if and only if $a = b, b+1, b+2$.*

4 Labeling of Q_3 When $a < b$

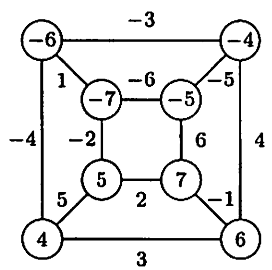
Balanced $(1,b)$-SEG labelings of Q_3 for $b = 2,3,4,5$ are depicted below.



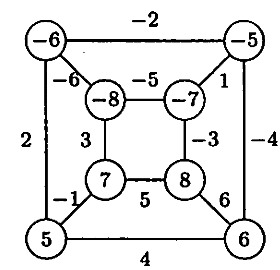
$Q(1) = \{\pm1, \ldots, \pm6\}$
$P(2) = \{\pm2, \ldots, \pm5\}$



$Q(1) = \{\pm1, \ldots, \pm6\}$
$P(3) = \{\pm3, \ldots, \pm6\}$



$Q(1) = \{\pm1, \ldots, \pm6\}$
$P(4) = \{\pm4, \ldots, \pm7\}$



$Q(1) = \{\pm1, \ldots, \pm6\}$
$P(5) = \{\pm5, \ldots, \pm8\}$

The next result follows directly from Lemma 3.4.

Theorem 4.1 *The hypercube Q_3 is (a,b)-SEG for $b - a = 1,2,3,4$.*

It remains to find the values of $b \geq a+5$ for which Q_3 admits an (a,b)-SEG labeling. It turns out that we only need to consider a small range of values for both a and b. To understand why, we first take a closer look at unbalanced labelings.

Lemma 4.2 *For $a+5 \leq b$, any (a,b)-SEG labeling must be unbalanced.*

Proof. Assume Q_3 admits a balanced (a,b)-SEG labeling. Since the largest balanced positive vertex label is $(a+5)+(a+4)-a = a+9$, we need $b+3 \leq a+9$, or equivalently, $b \leq a+6$. It remains to show that any (a,b)-SEG labeling cannot be balanced if $b = a+5, a+6$.

When $b = a+6$, the only way to label a vertex $a+9$ is

$$a+9 = (a+5)+(a+4)-a,$$

and the only ways to obtain the vertex label $a+8$ are

$$a+8 = (a+5)+(a+4)-(a+1),$$
$$a+8 = (a+5)+(a+3)-a.$$

Any combination of these solutions would require two edges connecting the two vertices labeled $a+9$ and $a+8$, hence no $(a,a+6)$-SEG labeling can be balanced.

When $b = a+5$, it is easy to verify that there are only three possible combinations of edge labelings that would produce legitimate vertex labels $a+8$ and $a+7$:

$$\begin{cases} a+8 = (a+5)+(a+4)-(a+1) \\ a+7 = (a+5)+(a+2)-a \end{cases}$$

$$\begin{cases} a+8 = (a+5)+(a+4)-(a+1) \\ a+7 = (a+4)+(a+3)-a \end{cases}$$

$$\begin{cases} a+8 = (a+5)+(a+3)-a \\ a+7 = (a+5)+(a+4)-(a+2) \end{cases}$$

None of them can be combined with the vertex labels for $a+6$:

$$a+6 = (a+5)+(a+4)-(a+3),$$
$$a+6 = (a+5)+(a+3)-(a+2),$$
$$a+6 = (a+5)+(a+2)-(a+1),$$
$$a+6 = (a+5)+(a+1)-a,$$
$$a+6 = (a+4)+(a+3)-(a+1),$$
$$a+6 = (a+4)+(a+2)-a.$$

Hence no balanced $(a,a+5)$-SEG labeling of Q_3 could exist. □

Call a vertex ***unbalanced*** if it is incident to three edges all with the same sign.

Lemma 4.3 *An (a, b)-SEG labeling of Q_3 contains at most two unbalanced positive vertices.*

Proof. Suppose the positive vertices u, v and w are unbalanced, with

$$f^*(u) = x_1 + x_2 + x_3, \quad x_i > 0,$$
$$f^*(v) = y_1 + y_2 + y_3, \quad y_i > 0.$$

Recall that there are only six positive edges. If $x_i \neq y_j$ for all i, j, then $f^*(w)$ has to share two edge labels with either $f^*(u)$ or $f^*(v)$. Since this is not allowed, we may assume $x_1 = y_1$, and $x_i \neq y_j$ if $i, j \geq 2$. In order to avoid multiple edges, we must have

$$f^*(w) = z + x_i + y_j,$$

where $i, j \geq 2$, and z is the edge label not used in $f^*(u)$ or $f^*(v)$. But a triangle would be formed on u, v and w. Thus any (a, b)-SEG labeling of Q_3 cannot contain more than two unbalanced positive vertices. □

Lemma 4.4 *If $|f^*(u)| \geq a + 10$, then u must be unbalanced.*

Proof. Without loss of generality, we may assume $f^*(u) > 0$. Since the largest positive *balanced* vertex label is $(a + 5) + (a + 4) - a = a + 9$, the labeling of u must be unbalanced. □

Theorem 4.5 *For $a + 5 \leq b$, the hypercube Q_3 is (a, b)-SEG if and only if* $\max(a + 5, 3a) \leq b \leq a + 8$, *and $a \leq 3$.*

Proof. Lemmas 4.3 and 4.4 restrict $P(b)$ to contain, at best, $a + 10$ and $a + 11$. Hence $b + 3 \leq a + 11$. In other words, $b \leq a + 8$.

Since $a + 5 \leq b$, at least one vertex is unbalanced, which we may assume is incident to three positive edges. Since the smallest positive unbalanced vertex label $Q(a)$ admits is $a + (a + 1) + (a + 2) = 3a + 3$, we also need $3a + 3 \leq b + 3$. Consequently, $\max(a + 5, 3a) \leq b \leq a + 8$, which has no solution when $a \geq 5$.

When $a = 4$, b must be 12. The smallest positive unbalanced vertex label is 15, and the largest positive balanced vertex label is 13. That leaves 14, which belongs to $P(12)$, neither balanced nor unbalanced, which is absurd. Therefore $a \leq 3$. □

Theorem 4.6 *The hypercube Q_3 is not $(3, b)$-SEG if $b \geq 8$.*

Proof. We only need to study $b = 9, 10, 11$. The possible ways of labeling a vertex 9 through 11 are listed below.

$$
\begin{array}{lll}
13 = 6 + 4 + 3, & 10 = 8 + 7 - 5, & 9 = 8 + 7 - 6, \\
12 = 8 + 7 - 3, & 10 = 8 + 6 - 4, & 9 = 8 + 6 - 5, \\
12 = 5 + 4 + 3, & 10 = 8 + 5 - 3, & 9 = 8 + 5 - 4, \\
 & 10 = 7 + 6 - 3, & 9 = 8 + 4 - 3, \\
11 = 8 + 7 - 4, & & 9 = 7 + 6 - 4, \\
11 = 8 + 6 - 3, & & 9 = 7 + 5 - 3,
\end{array}
$$

Only the combination

$$\begin{cases} 13 = 6 + 4 + 3, \\ 12 = 8 + 7 - 3, \end{cases}$$

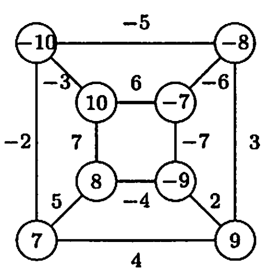
could contain both vertex labels 13 and 12 at the same time, but it cannot be combined with 11. Thus the vertex labels 11, 12 and 13 cannot coexist, which eliminates $b = 10, 11$. In a similar fashion, we find only three possible combinations of the vertex labels 12, 11 and 10:

$$\begin{cases} 12 = 5 + 4 + 3, \\ 11 = 8 + 7 - 4, \\ 10 = 8 + 5 - 3, \end{cases} \qquad \begin{cases} 12 = 5 + 4 + 3, \\ 11 = 8 + 7 - 4, \\ 10 = 7 + 6 - 3, \end{cases} \qquad \begin{cases} 12 = 5 + 4 + 3, \\ 11 = 8 + 6 - 3, \\ 10 = 8 + 7 - 5. \end{cases}$$

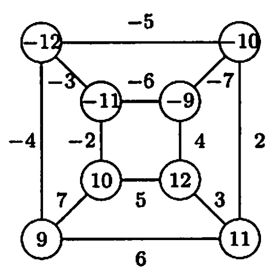
Comparing against how the vertex label 9 could be obtained, we determine that Q_3 cannot be $(3, 9)$-SEG either. □

Theorem 4.7 *For $b \geq 7$, Q_3 is $(2, b)$-SEG if and only if $b = 7, 9$.*

Proof. We only have to consider $b = 7, 8, 9, 10$. An analysis similar to that used above reveals that $b \neq 8, 10$. It remains to find a $(2, b)$-SEG labelings for $b = 7, 9$. They are given below. □



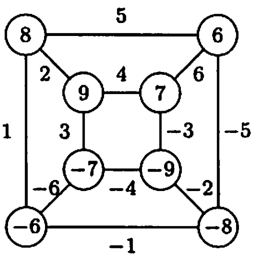
$Q(2) = \{\pm 2, \ldots, \pm 7\}$
$P(7) = \{\pm 7, \ldots, \pm 10\}$



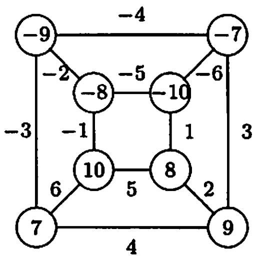
$Q(2) = \{\pm 2, \ldots, \pm 7\}$
$P(9) = \{\pm 9, \ldots, \pm 12\}$

Theorem 4.8 *For $b \geq 6$, Q_3 is $(1, b)$-SEG if and only if $b = 6, 7$.*

Proof. Displayed below are the labelings of Q_3 when $b = 6, 7$. The rest of the proof is straightforward, hence is omitted here. □



$Q(1) = \{\pm 1, \ldots, \pm 6\}$
$P(6) = \{\pm 6, \ldots, \pm 9\}$



$Q(1) = \{\pm 1, \ldots, \pm 6\}$
$P(7) = \{\pm 7, \ldots, \pm 10\}$

5 Conclusion

The hypercube Q_3 is (a, b)-SEG if and only if

$$\begin{cases} 1 \le b \le 7 & \text{if } a = 1, \\ 1 \le b \le 7 \text{ and } b = 9 & \text{if } a = 2, \\ a - 2 \le b \le a + 4 & \text{if } a \ge 3. \end{cases}$$

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