On P_3-Degree of Graphs

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Abstract

It is known that there is not any non-trivial graph with vertices of distinct degrees, and any non-trivial graph must have at least two vertices of the same degree. In this article, we will consider the concept of P_3-degree of vertices and will introduce a class of connected graphs with exactly two vertices of the same P_3-degree. Also, the graphs with distinct P_3-degree vertices will be constructed and it will be proven that for any $n \geq 6$ there is at least one graph of order n, with distinct P_3-degree vertices.

Keywords: P_3-degree, regular, irregular, and pairlone graphs
AMS Subject Classification: 05C15.

1 Introduction

Unless otherwise stated all graphs considered are connected, finite, simple, undirected, and of order $n \geq 3$. It is known that there is not a graph with vertices of distinct degrees, and any non-trivial graph must have at least two vertices of the same degree. Behzad-Chartrand [2] and Nebesky [7] have studied the graphs containing exactly two vertices of the same degree. Also, Kac-Nesterova have investigated the graphs with exactly three vertices of the same degree. In this article, just for the sake of brevity, a graph G is said to be *pairlone* if there are precisely two vertices with the same degree. In Figure 1, three examples of pairlone graphs of different orders are illustrated, and each vertex is labeled by its degree. Even though the existence of pairlone graphs has been shown in [2, 7], for the sake of completeness and for later use, we give the proof of the following theorem.

Theorem 1.1. *For any $n \geq 2$ there is a unique pairlone graph of order n. Furthermore, the degree sequence is 1 through $n-1$ with $\lfloor n/2 \rfloor$ appearing twice.*

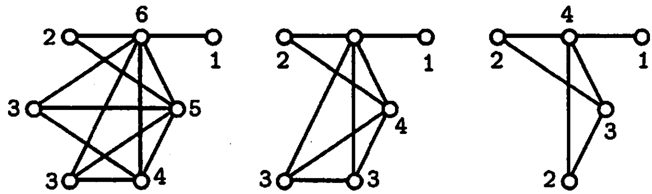

Figure 1: Three examples of pairlone graphs.

Proof. We proceed by induction on n, case of $n = 2$ is obvious. Now let G be the unique connected pairlone graph of order $n \geq 2$, that has degree sequence 1 through $n-1$ with $\lfloor n/2 \rfloor$ appearing twice. That is, $V(G) = \{u_1, u_2, \cdots, u_n\}$ and

$$\deg_G u_i = \begin{cases} i & \text{for} \quad i \leq \lfloor n/2 \rfloor \\ i-1 & \text{for} \quad i > \lfloor n/2 \rfloor \end{cases}$$

We use G to construct a graph H by adding a vertex u, $V(H) = V(G) \cup \{u\}$, and edges uu_k $\lfloor n/2 \rfloor < k \leq n$, as illustrated in Figure 2. With this construction, $\deg_H u_i = i$ $(1 \leq i \leq n)$, and $\deg_H u = n - \lfloor n/2 \rfloor = \lfloor \frac{n+1}{2} \rfloor$.

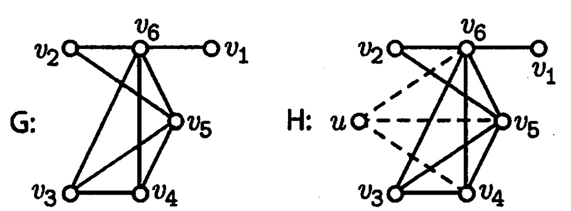

Figure 2: Construction of H from G, when $n = 6$.

Therefore, H is a connected graph of order $n + 1$, which has degree sequence 1 through n with $\lfloor \frac{n+1}{2} \rfloor$ repeated twice. The uniqueness of such graph follows from the fact that in any connected pairlone graph of order $n + 1$ removal of one of the vertices of order $\lfloor \frac{n+1}{2} \rfloor$ will result in the unique connected pairlone graph of order n. □

In what follows PL_n stands for the unique connected pairlone graph of order n. Let $v_1, v_2, \cdots, v_n$ be the vertices of PL_n labeled so that $\deg v_1 \leq \deg v_2 \leq \cdots \leq \deg v_n$.

Observation 1.2. *The chromatic number of PL_n, the unique connected pairlone graph of order n, is $n - \lfloor \frac{n-1}{2} \rfloor$.*

Proof. We observe that

$$\deg v_i = \begin{cases} i & \text{for } i \leq \lfloor n/2 \rfloor \,; \\ i-1 & \text{for } i > \lfloor n/2 \rfloor \,, \end{cases} \tag{1}$$

and the edges of G consist of all $v_{n-i}v_j$ with $i = 0, 1, \cdots, \lfloor n/2 \rfloor - 1$, and $i+1 \leq j \leq n-i-1$. Adjacency of the vertices of the same degree $\lfloor n/2 \rfloor$ depends on the parity of n. When n is even, they are adjacent and if n is odd they are not. We also observe that the vertices $v_1, v_2, \cdots, v_{\lfloor \frac{n+1}{2} \rfloor}$ are not adjacent, while the vertices $v_n, v_{n-1}, \cdots, v_{\lfloor \frac{n+1}{2} \rfloor}$ are pairwise adjacent. Consequently, the subgraph of PL_n induced by $p = n - \lfloor \frac{n-1}{2} \rfloor$ vertices $v_n, v_{(n-1)}, \cdots, v_{\lfloor \frac{n+1}{2} \rfloor}$ is isomorphic to the complete graph K_p. Therefore, $\chi(PL_n) \geq p$. On the other hand, in any coloring $c : V(PL_n) \longrightarrow \mathbb{N}$ of PL_n, since $v_1, v_2, \cdots, v_{\lfloor \frac{n+1}{2} \rfloor}$ are not adjacent, they can be assigned the same color; Specifically,

$$c(v_i) = \begin{cases} n - \lfloor \frac{n-1}{2} \rfloor & \text{for} \quad 1 \leq i \leq \lfloor \frac{n+1}{2} \rfloor; \\ n+1-i & \text{for} \quad \lfloor \frac{n+1}{2} \rfloor \leq i \leq n, \end{cases}$$

which implies that $\chi(PL_n) \leq p$. Therefore, $\chi(PL_n) = p$. For the pairlone graphs PL_5 and PL_7, this coloring is illustrated in Figure 3. □

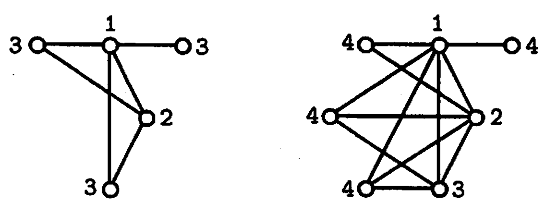

Figure 3: $\chi(PL_5) = 3$ and $\chi(PL_7) = 4$.

2 P_3-pairlone graphs

The notion of degree of a vertex in a graph has been generalized in [4] by introducing F-degree in the following way: For a given graph F, the *F-degree* of a vertex v in G, denoted by F-deg(v), is the number of subgraphs of G isomorphic to F that contain v. A concept related to the F-degree of a vertex was first introduced by Kocay [6], when reconstructing degree sequence on graphs. A graph G is said to be *F-regular* if the F-degrees of all the vertices of G are the same and it is called *F-irregular* if the F-degrees of the vertices of G are distinct. We observe that the ordinary degree of a vertex v is the P_2-degree ($F = P_2$) of v, and when we consider P_2-degrees there is not any irregular graph. In this section, we will focus on the P_3-degree of a graph and will show that, as opposed to ordinary degree, for $n \geq 6$ there is at least one graph of order n which is P_3-irregular. Also, the existence of P_3-pairlone graphs that are not (ordinary degree) pairlone will be proven.

Definition 2.1. *Given a graph G, the P_3-degree of a vertex v in G, denoted by P_3-deg(v), is the number of subgraphs of G isomorphic to P_3 that contain v. The graph G is said to be P_3-regular if the P_3-degrees of all the vertices are the same, and G is said to be P_3-irregular if the P_3-degrees of its vertices are distinct.*

To illustrate the definition of P_3-degree, consider the graph in Figure 4.

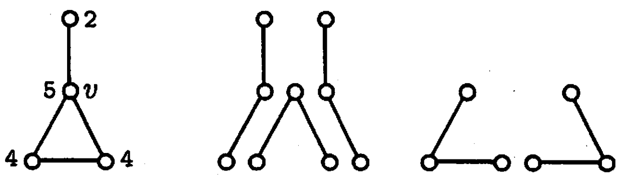

Figure 4: $P_3 - \deg(v) = 5$.

Vertices of G are labeled by their P_3-degrees; For example, there are 5 subgraphs of G isomorphic to P_3 that contain the vertex v. Therefore, $P_3 - \deg(v) = 5$. As we will see later 2.5, there are infinitely many P_3-degree irregular graphs. Figure 5 illustrate one of them for which the P_3-degrees of vertices are identified.

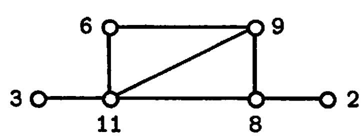

Figure 5: P_3-degree irregular graph of order 6.

We note that a vertex can occur either as the central vertex or as an end-vertex of P_3. Now let $v \in V(G)$ and let $N(v)$ be the set of all vertices of G adjacent to v. Then v is the central vertex of $\binom{\deg v}{2}$ subgraphs of G isomorphic to P_3 and is an end-vertex of $\sum_{u \in N(v)} (\deg u - 1)$ such subgraphs. Therefore

$$P_3 - \deg v = \binom{\deg v}{2} + \sum_{u \in N(v)} (\deg u - 1). \tag{2}$$

For regular graphs we have the following theorem [4]:

Theorem 2.2. *A graph G is P_3-regular of degree $k \geq 2$ if and only if G is regular of degree $r \geq 2$, where $k = 3\binom{r}{2}$.*

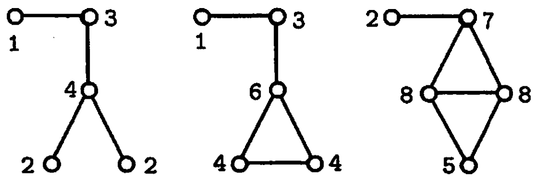

Figure 6: Three P_3-degree pairlone graphs.

Theorem 2.3. *For any $n \geq 4$, the graph PL_n is P_3-degree pairlone.*

Proof. Let PL_n be the unique connected pairlone graph of order n. Using (1) and (2), we notice that

$$P_3 - degv_i = \begin{cases} i(n-2) & \text{for } i \leq \lfloor n/2 \rfloor; \\ (i-1)(n-2) + \lfloor n/2 \rfloor + 1 - i & \text{for } i > \lfloor n/2 \rfloor. \end{cases} \tag{3}$$

Thus vertices of PL_n have distinct P_3-degrees, except for exactly one pair having each vertex the same ordinary degree $\lfloor n/2 \rfloor$. □

Theorem 2.4. *For any $n \geq 5$ there is at least one P_3-pairlone graph of order n, that is not (ordinary degree) pairlone.*

Proof. Let $G = PL_n$ be the unique connected pairlone graph of order n. We construct H from PL_n by adding a vertex v_0 to $V(G)$ and the edge v_0v_1 to $E(G)$, as illustrated in Figure 7, for $n = 4, 5, 6$. Then, as a result of this modification, H will have another subgraph with vertices v_0, v_1, and v_n, isomorphic to P_3 (with v_1 as its central vertex, and v_0, v_n as its end-vertices). Thus $P_3 - \deg v_0 = 1$, and

$$P_3 - \deg_H v_i = \begin{cases} P_3 - \deg_G v_i & \text{for } i \neq 1, n; \\ 1 + P_3 - \deg_G v_i & \text{for } i = 1, n. \end{cases} \tag{4}$$

Also, $P_3 - \deg_H v_1 = n - 1 < P_3 - \deg_H v_2 = 2(n-2)$. Therefore, H is a P_3-pairlone graph of order $n+1$. But H is not (ordinary degree) pairlone, because $\deg_H v_1 = \deg_H v_2 = 2$, and H has two pairs of vertices of the same degree. □

Theorem 2.5. *For any $n \geq 6$ there is at least one P_3-irregular graph of order n.*

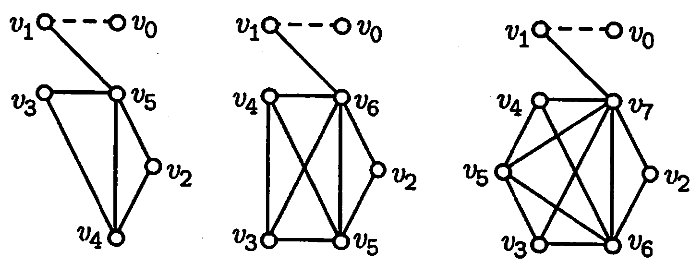

Figure 7: P_3-degree pairlone graphs that are not pairlone.

Proof. Let PL_n be the unique connected pairlone graph of order n. This time we construct H from PL_n by adding vertex v_0 to $V(G)$ and joining v_0 to the vertex $v_{\mu+1}$, where $\mu = \lfloor n/2 \rfloor$. (See Figure 8, for $n = 5, 6, 7$.) Now, note that v_0 is only adjacent to $v_{\mu+1}$, that has degree $\mu = \lfloor n/2 \rfloor$, therefore $P_3\text{-deg}\, v_0 = \mu - 1$. Also, this modification will not effect the P_3-degrees of vertices $v_1, v_2, \cdots, v_{\mu-1}$, because they are not adjacent to $v_{\mu+1}$. Moreover, if v_i, $(i \neq 0)$ is adjacent to $v_{\mu+1}$, its P_3-degree will be increased by one (for the new subgraph $v_i, v_{\mu+1}, v_0$, which is isomorphic to P_3). Finally, $P_3 - \deg_H v_\mu = P_3 - \deg_G v_\mu + \dfrac{1+(-1)^n}{2}$, because, the parity of n will determine whether the two vertices v_μ and $v_{\mu+1}$ are adjacent or not. In summary,

$$P_3\text{-}\deg_H v_i = \begin{cases} \lfloor n/2 \rfloor - 1 & \text{for} \quad i = 0 \\ i(n-2) & \text{for} \quad 0 < i < \lfloor n/2 \rfloor \\ \lfloor n/2 \rfloor (n-2) + \frac{1+(-1)^n}{2} & \text{for} \quad i = \lfloor n/2 \rfloor \\ \lfloor n/2 \rfloor (n-2) + \lfloor n/2 \rfloor & \text{for} \quad i = \lfloor n/2 \rfloor + 1 \\ 2 + (i-1)(n-2) + \lfloor n/2 \rfloor - i & \text{otherwise.} \end{cases}$$

Clearly, P_3-degrees of $v_0, v_1, \cdots, v_{\mu+1}$ are different, as are P_3-degrees of $v_{\mu+2}, \cdots, v_n$. It only remains to show that

$$P_3 - \deg_H v_{\mu+1} < P_3 - \deg_H v_{\mu+2},$$

or $\lfloor n/2 \rfloor (n-2) + \lfloor n/2 \rfloor < 1 + P_3 - \deg_G v_{\mu+2}$.
By (3), this is equivalent to $\lfloor n/2 \rfloor (n-2) + \lfloor n/2 \rfloor < (\lfloor n/2 \rfloor + 1)(n-2)$, or $\lfloor n/2 \rfloor < n - 2$, which is correct, since $n \geq 6$.
Next, we consider when $n \leq 5$. For $n = 3$, there are only two connected graphs of order three, namely P_3 and K_3. These two graphs are neither P_3-pairlone nor P_3-irregular; in fact, both are P_3-regular. For $n = 4$, the unique connected pairlone graph of order four is the only P_3-pairlone graph, and there is no P_3-irregular graph of order four. Also, it can easily be verified that there is no P_3-irregular graph of order five. □

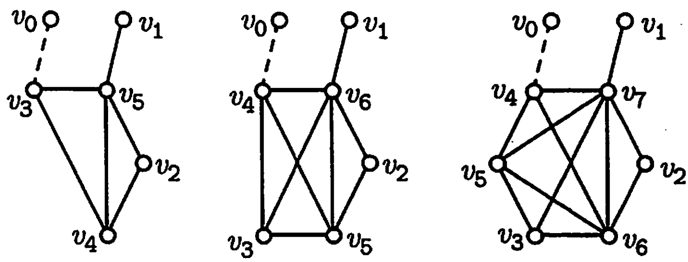

Figure 8: Three P_3-irregular graphs.

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