A Search For Solvable Weighing Matrices

Paul E. Becker
Penn State — Erie
peb8@psu.edu

Sheridan Houghten*
Brock University
houghten@brocku.ca

Wolfgang Haas
Brock University

Abstract

A weighing matrix $W(n,k)$ of order n with weight k is an $n \times n$ matrix with entries from $\{0,1,-1\}$ which satisfies $WW^T = kI_n$. Such a matrix is group-developed if its rows and columns can be indexed by elements of a finite group G so that $w_{g,h} = w_{gf,hf}$ for all g, h, and f in G. Group-developed weighing matrices are a natural generalization of perfect ternary arrays and Hadamard matrices. They are closely related to difference sets.

We describe a search for weighing matrices with order 60 and weight 25, developed over solvable groups. There is one known example of a $W(60,25)$ developed over a nonsolvable group; no solvable examples are known.

We use techniques from representation theory, including a new viewpoint on complementary quotient images, to restrict solvable examples. We describe a computer search strategy which has eliminated two of twelve possible cases. We summarize plans to complete the search.

1 Introduction

A variety of combinatorial objects are defined as solutions to the matrix equation: $MM^T = \alpha I + \beta J$, where J denotes the matrix with all entries

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equal to one. Some of these objects can be described as group-developed; that is, their internal structures are most easily described via multiplication in a finite group.

Definition 1 (Group Developed Weighing Matrix). *A weighing matrix $W(n, k)$ of order n with weight k is an $n \times n$ matrix with entries from $\{0, 1, -1\}$ such that $WW^T = kI_n$. Such a matrix is group-developed under a group G if the rows and columns can be indexed so that $w_{g,h} = w_{gf,hf}$ for all $g, h, f \in G$.*

A variety of constructions are known for weighing matrices; a summary can be found in [11]. A great deal of the research concentrates on the special case of Hadamard matrices. A recent article by Ang and Ma concentrates on the special case of symmetric weighing matrices developed over abelian groups [2].

In this paper, we describe a search for group-developed weighing matrices with order 60 and weight 25. Such matrices are a generalization of perfect ternary arrays with order 60 and energy 25.

Perfect ternary arrays are of particular interest in communications theory. An r-dimensional ternary array is a $s_1 \times s_2 \times ... \times s_r$ array with entries chosen from $\{0, 1, -1\}$. Such a ternary array, A, is perfect if its out-of-phase periodic autocorrelation coefficients are zero. The *energy*, k, of A is the number of nonzero entries; its *order* is $n = \Pi s_i$. The correlation properties imply orthogonality of rows, so A is equivalent to a $W(n, k)$ developed over the group $\mathbb{Z}_{s_1} \times \mathbb{Z}_{s_2} \times \cdots \times \mathbb{Z}_{s_r}$ [4].

Group-developed weighing matrices are also closely related to difference sets. A subset Δ of a finite group G is called a (v, k, λ) difference set if: $|G| = v$; $\Delta = k$; and for each $g \in G \backslash \{e\}$, the equation $g = d_1 d_2^{-1}$ has exactly λ solutions (d_1, d_2) in $\Delta \times \Delta$. The connection between weighing matrices and difference sets is best described in the language of group rings.

2 Algebraic methods

Assume that G is a finite group and that R is a ring. The corresponding group ring is $RG = \left\{\sum_{g \in G}(\alpha_g g);\ \alpha_g \in R\right\}$. We denote the identity by I. If the polynomial $\delta = \sum \alpha_g g$ is in RG, then we write $\delta^{(-1)} = \sum \alpha_g g^{-1}$. We extend automorphisms of G linearly to form automorphisms of RG.

Polynomials α and β are translates if there exist $g, h \in G$ such that $g\alpha h = \beta$. They are equivalent if there is an automorphism ϕ, extended from G, such that $\phi(\alpha)$ is a translate of β.

We are particularly interested in the integral and rational group rings, $\mathbb{Z}G$ and $\mathbb{Q}G$. Elements of these rings are *multisets in* G if each coefficient is a nonnegative integer. Multisets with m coefficients equaling 1 and all other coefficients equaling 0 are called *m-sets in* G; we identify a set $B \subseteq G$ with the $|B|$-set $\widehat{B} = \sum_{b \in B} 1b$.

Weighing matrices can be represented as polynomial solutions to the equation $\widehat{D}\widehat{D}^{(-1)} = \alpha + \beta\widehat{G}$ in some group ring $\mathbb{Z}G$. We identify solutions by solving corresponding equations in a variety of quotient rings.

Theorem 1. *Suppose that W is a weighing matrix $W(n, k)$ developed over a group G. The matrix W is equivalent to a a disjoint pair of sets P, M in G such that $(\widehat{P} - \widehat{M})(\widehat{P} - \widehat{M})^{(-1)} = kI$.*

For a proof of this theorem, see Arasu and Dillon [4]. Informally, we note that a group-developed weighing matrix is fully determined by its first row; that row corresponds to coefficients for the polynomial $(\widehat{P} - \widehat{M}) \in \mathbb{Z}G$.

Reference [4] provides fundamental restrictions on such polynomials. If $(\widehat{P} - \widehat{M})$ represents a G-developed weighing matrix $W(|G|, k)$, then: $k = s^2$ for some integer s; and $\{|P|, |M|\} = \left\{\frac{s^2-s}{2}, \frac{s^2+s}{2}\right\}$.

We frequently consider the related multiset $\widehat{D} = \widehat{P} - \widehat{M} + \widehat{G}$. We assume $\widehat{D}$ satisfies these restrictions: $|P| = \frac{k+\sqrt{k}}{2}$; $|M| = \frac{k-\sqrt{k}}{2}$; and $\widehat{D}\widehat{D}^{(-1)} = kI + (2\sqrt{k} + n)\widehat{G}$.

Suppose that $\widehat{D}$ corresponds to a G-developed weighing matrix $W(60, 25)$. In this situation, $|P| = 15$, $|M| = 10$, $|D| = 65$, and $\widehat{D}\widehat{D}^{(-1)} = 25 + 70\widehat{G}$. Liebler [13] has identified such a polynomial in the group ring $\mathbb{Z}A_5$. We wish to determine if similar polynomials exist in $\mathbb{Z}G$, where G is solvable.

Difference sets are characterized by similar polynomials. A (v, k, λ) difference set $\Delta \subseteq G$ must satisfy $\widehat{\Delta}\widehat{\Delta}^{(-1)} = (k - \lambda)I + \lambda G$.

2.1 Quotient images

To identify interesting polynomials over solvable groups, we utilize projections into quotient groups. Assume that $N \triangleleft G$ and that $\pi : G \to G/N$ is the usual projection, $\pi(g) = gN$. If $\widehat{D} \in \mathbb{Z}G$ satisfies $\widehat{D}\widehat{D}^{(-1)} = \alpha I + \beta\widehat{G}$, then $\pi(\widehat{D})$ satisfies the analogous equation: $\pi(\widehat{D})\pi(\widehat{D})^{(-1)} = \alpha I + \beta|N|\widehat{(G/N)}$.

In particular, if $\widehat{D} = \widehat{P} - \widehat{M} + \widehat{G}$ is equivalent to a weighing matrix

$W(n,k)$, then $\pi(\widehat{D})$ satisfies

$$\pi(\widehat{D})\pi(\widehat{D}^{(-1)}) = kI + (2\sqrt{k}+n)|N|(\widehat{G/N}). \qquad (1)$$

We define a $W(n,k)$ quotient image to be any reasonable solution of this equation. Specifically, a multiset $\delta = \sum_{m \in G/N} (a_m m)$ satisfying equation (1) is a $W(n,k)$ G/N quotient image if $\sum a_m = n + \sqrt{k}$ and $a_m \leq 2|N|$ for each m. We refer to the coefficients a_m of a quotient image as G/N *intersection numbers.* Difference sets admit similar definitions.

Suppose that $N \lhd K \lhd G$ and that G admits a weighing matrix. A quotient image must exist in G/K. If $N \lhd G$, then quotient images exist in G/N and in $(G/N)/(K/N) \cong G/K$. Formally, each invariant series of G generates a corresponding sequence of quotient images.

Twelve of the thirteen groups with order 60 are solvable. The solvable groups admit normal subgroups of order 5, implying quotient groups with order 12.

Proposition 1. *If a weighing matrix W(60,25) exists in a solvable group, then a W(60,25) quotient image exists in a group of order 12.*

2.2 Matrix representations of groups

A representation of degree p is a homomorphism from a group to the multiplicative group of $(p \times p)$ complex matrices. A representation of degree 1 is a character. We extend representations linearly, forming homomorphisms from group rings to rings of complex matrices.

Assume δ represents a $W(n,k)$ quotient image in a quotient group H; that is, it satisfies equation (1). Let ϕ be a representation of degree $d \geq 1$; equation (1) becomes:

$$\phi(\delta)\phi(\delta^{(-1)}) = kI_d + (2\sqrt{k}+n)\frac{n}{|H|}\phi(\widehat{H}). \qquad (2)$$

If ϕ is a nontrivial irreducible representation, then $\phi(\widehat{H}) = 0$. Consequently, any quotient image obeys the restrictions $\phi(\delta)\phi(\delta^{(-1)}) = k\phi(I)$ for every nontrivial irreducible representation, and $\chi_0(\delta) = n + \sqrt{k}$ for the trivial representation.

2.3 Decompositions of group rings

The rational group ring of a finite group, H, is semisimple. Consequently, $\mathbb{Q}H$ is a direct sum of minimal 2-sided ideals, called *simple components.* In order to utilize this decomposition, we treat quotient images in H as elements of $\mathbb{Q}H$ which happen to have integral coefficients.

If H is abelian, then each simple component is generated by a *central primitive idempotent,* E_i. Therefore, quotient images in H decompose as: $\delta = \sum \delta_i E_i$, where δ_i is a polynomial in the ideal $\langle E_i \rangle$. The idempotents are sums over equivalence classes of irreducible characters:

$$\{E_i\} = \left\{ \sum_{\chi_m \in <\chi_i>} \frac{1}{|G|} \sum_{g \in G} \chi_m(g^{(-1)})g \right\}.$$

Characters in the class $\langle \chi_i \rangle$ are called *components* of E_i.

If H is non-abelian, we split $\mathbb{Q}H$ into commutative and non-commutative components. Assume that H' is the commutator (derived) subgroup of H, and define the idempotent polynomial $e_{H'} = \frac{1}{|H'|}\widehat{H'}$.

Proposition 2 (Proposition 3.6.11 in [15]). *If RH is a semisimple group algebra, then $RH = RHe_{H'} \oplus RH(1 - e_{H'})$, where $R_{H'} \cong R(H/H')$ is the sum of all commutative simple components of RH, and $RH(1 - e_{H'})$ is the sum of all the others.*

A quotient image in H splits as $\delta = \delta_0 e_{H'} + \delta_1(1 - e_{H'})$. We determine δ by identifying abelian quotient images $\delta_0 \in \mathbb{Q}(H/H')$ and corresponding orthogonal polynomials δ_1 in $\mathbb{Q}H(1 - e_{H'})$. In section 2.4, we describe techniques for identifying δ_0 as a sum of idempotents. To identify δ_1, we use induced matrix representations.

Assume H' has a linear character T, and that the cosets of H' in H are $\{H'x_0, H'x_1, \ldots, H'x_p\}$. The representation of H induced by T is:

$$T^*(h) = \begin{bmatrix} T(x_0hx_0^{-1}) & \ldots & T(x_0hx_p^{-1}) \\ \vdots & \vdots & \vdots \\ T(x_phx_0^{-1}) & \ldots & T(x_phx_p^{-1}) \end{bmatrix},$$

where $T(x_ihx_j^{-1}) = 0$ if $(x_ihx_j^{-1}) \notin H'$. If T is a nonprincipal irreducible character, then T^* maps δ to $T^*(\delta_1)I$ (Lemma 3.1, [6]). We identify the non-abelian component δ_1 by applying equation (2) to induced representations.

2.4 Restrictions on abelian quotient images

Assume δ is a $W(n,k)$ quotient image in an abelian group H. Expand δ as a linear combination of idempotents, $\delta = \sum \delta_i E_i$. Suppose χ_j is an irreducible character. If χ_j is a component of E_i, then $\chi_j(E_i) = 1$; otherwise, $\chi_j(E_i) = 0$. Therefore, $\chi_j(\delta) = \sum_i \chi_j(\delta_i)\chi_j(E_i) = \chi_j(\delta_j)$. We conclude that $\chi_j(\delta)$ is an algebraic integer in $\mathbb{Z}[\zeta_j]$ for some j^{th} root of unity, ζ_j. As δ is a quotient image, equation (2) determines the magnitude $\chi_j(\delta)$. This approach is based on work by Iiams [12], Liebler [14], and others. For more details, see references [5], [6] and [8].

Proposition 3 (Magnitude of coefficients). *If $\delta = \sum \delta_i E_i$ is a $W(n,k)$ weighing matrix in an abelian group H, then $\chi_0(\delta) = \chi_0(\delta_0) = n + \sqrt{k}$, and, for each nontrivial irreducible character χ_m, $|\chi_m(\delta_m)| = |\chi_m(\delta)| = \sqrt{k}$.*

Thus, the search for an abelian quotient image is equivalent to a search for coefficients from $\mathbb{Z}[\zeta_m]$, m dividing $|H|$, with magnitude $\sqrt{k}$. Translation of multisets corresponds to translation of $\chi_m(\delta_m)$ by roots of unity. In other words, if $d = \chi_m(\delta_m)$, then $\chi_m(g\delta) = \chi_m(g)d$. As $d\bar{d} = k$, (d) divides (k); it is sufficient to factor the ideal (k) over $\mathbb{Q}[\zeta_m]$. Details of this approach, and the relevant ideal factorizations, are discussed in [6]. We proceed directly to the computation of small quotient images.

3 Small quotient images

Our procedure is to collect (up to equivalence) all $W(60,25)$ quotient images in small groups, then correlate them to determine polynomials representing weighing matrices. In this section, we compute several small quotient images. We also summarize the other relevant small quotient images; details can be found at website http://math.bd.psu.edu/faculty/becker or at website http://www.cosc.brocku.ca/staff/houghten .

We consider a quotient image, δ, in the cyclic group $\mathbb{Z}_4 = \langle z : z^4 = e \rangle$. The ring $\mathbb{Q}\mathbb{Z}_4$ splits as a direct sum of simple components $(E_0) \oplus (E_2) \oplus (E_4)$, generated by the idempotents: $E_0 = \frac{1}{4}[1 + z + z^2 + z^3] = \frac{1}{4}[1,1,1,1]$; $E_2 = \frac{1}{4}[1,-1,1,-1]$; and $E_4 = \frac{1}{4}[2,0,-2,0]$. Thus, $\delta = \delta_0 E_0 + \delta_2 E_2 + \delta_4 E_4$.

From Proposition 3, we know that: $\delta_0 = n + \sqrt{k} = 65$; $\chi_2(\delta_2)$ equals 5 or -5; and $\chi_4(\delta_4) \in \mathbb{Z}[i]$ with magnitude 5. The ideal (5) factors over $\mathbb{Z}[i]$ as $(5) = (3+4i)(3-4i)$, so we assume $\chi_4(\delta_4)$ is a simple multiple of $\chi_4^{-1}(3+4i)$ or $\chi_4^{-1}(3+4i^3)$. Consequently, $\delta = 65E_0 \pm 5E_2 + \delta_4 E_4$, where $z^m \delta_4 \in \{5, 3+4z, 3+4z^3\}$.

Table 1: $W(60,25)$ quotient images in small groups

Group	Quotient image polynomial
$\mathbb{Z}_2 = \langle z \rangle$	$35 + 30z$
$\mathbb{Z}_3 = \langle z \rangle$	$25 + 20z + 20z^2$
$\mathbb{Z}_4 = \langle z \rangle$	$20 + 15z + 15z^2 + 15z^3$ $19 + 17z + 16z^2 + 13z^3$
$\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle x, y \rangle$	$20 + 15x + 15y + 15xy$
$\mathbb{Z}_5 = \langle z \rangle$	$17 + 12z + 12z^2 + 12z^3 + 12z^4$ $9 + 14z + 14z^2 + 14z^3 + 14z^4$
$\mathbb{Z}_6 = \langle z \rangle$	$15 + 10(z + z^2 + z^3 + z^4 + z^5)$
$\mathbb{Z}_{10} = \langle z \rangle$	$5 + 6\widehat{\mathbb{Z}_{10}}$ $8\langle z^2 \rangle + 6z\langle z^2 \rangle - 5$ $7\widehat{\mathbb{Z}_{10}} - 5z$ $5z + 5z < z^2 > +7 < z^2 >$

As there is a group automorphism exchanging z and z^3, we assume δ is equivalent to a polynomial in which $z^m \delta_4 \in \{5, 3 + 4z\}$. Evaluating δ with the possible values of δ_2 and δ_4, we find only two inequivalent integral polynomials.

Proposition 4. *Up to equivalence, the only W(60,25) quotient images in* $\mathbb{Z}_4 =< z >$ *are* $20 + 15z + 15z^2 + 15z^3$ *and* $19 + 17z + 16z^2 + 13z^3$.

Similar calculations determine quotient images in other small abelian groups. A theorem by Dillon [9] shows that every quotient image in a dihedral group can be derived from a quotient image in the cyclic group of the same order. There is, for example, a unique D_4 quotient image; it is analogous to one of the two $\mathbb{Z}_4$ quotient images.

Table 1 contains a listing of all inequivalent $W(n, k)$ in a selection of small groups. Note that each quotient image has the form $\pi(\widehat{P} - \widehat{M}) + r\widehat{G}$, where r represents the order of some normal subgroup. The inequivalent quotient images in the cyclic group of order 20 appear in Table 2. An exposition of the corresponding calculations, in the context of difference sets, appears in reference [1].

In section 2.3, we saw that quotient images in a nonabelian group G split as $\delta = \delta_0 e_{G'} + \delta_1(1 - e_{G'})$. Reference [6] describes computation of difference set quotient images with this decomposition. Specifically, it describes computation of the nonabelian component δ_1 by applying equation (2) to induced representations. That paper also describes the correlation of δ_0 and δ_1. Here, we simply summarize the results for groups of order 12.

Table 2: $W(60,25)$ quotient images in $\mathbb{Z}_{20} = \langle x, y : x^4 = y^5 = 1\rangle$

Solution	Quotient image polynomial
1.	$(2+3\langle x\rangle)\langle y\rangle - 5$
2.	$(3x+4)(1+x^2)(\langle y\rangle - 1) + (3+5x+x^3)$
3.	$[3+4x+4x^2+3x^3](y^2+y^3) + [4+x+3x^2+1x^3] + \dots$ $\dots + [5+3x+2x^2+4x^3](y+y^4)$
4.	$[4+5x+3x^2+5x^3] + [3+3x+4x^2+2x^3](y^2+y^3) + \dots$ $\dots + [5+2x+2x^2+3x^3](y+y^4)$
5.	$[2+3x+x^2+3x^3] + [4+2x+4x^2+4x^3](y^2+y^3) + \dots$ $\dots + [5+4x+3x^2+2x^3](y+y^4)$
6.	$6+3x+5x^2+3x^3] + [4+4x+2x^2+2x^3](y+y^4) + \dots$ $\dots + [3+2x+3x^2+4x^3](y^2+y^3)$

There are only two inequivalent $W(60,25)$ A_4 quotient images: $5+5A_4$; and $9+5(12)(34)+5(13)(24)+6(14)(23)+[6+4(12)(34)+6(13)(24)+4(14)(23)](123)+[4+4(12)(34)+6(13)(24)+6(14)(23)](123)^2$. The other groups of order 12 admit presentations of the form $\langle x, y\rangle$, where x has order 6. We group the corresponding quotient images in Proposition 5.

Proposition 5. *Present the groups of order 12 (except A_4) as follows: $Q_{12} =< x, y : x^6 = y^4 = y^2x^3 = x^yx = e >$; $\mathbb{Z}_6 \times \mathbb{Z}_2 =< x, y : x^6 = y^2 = e >$; $D_{12} =< x, y : x^6 = y^2 = x^yx = e >$; and $\mathbb{Z}_{12} =< x, y : x^6 = y^2x^{-1} = x^yx^{-1} = e >$. Up to equivalence, the only quotient images of weighing matrices W(60,25) in these groups are the following:*

Table 3, Case 1 : Q_{12}, $\mathbb{Z}_{12}$, $\mathbb{D}_{12}$, and $\mathbb{Z}_6 \times \mathbb{Z}_2$;

Table 3, Case 2 : Q_{12} and $\mathbb{Z}_{12}$;

Table 3, Case 3 : Q_{12}.

Table 3: Order 12 quotients

	Quotient image in $H = \langle x, y\rangle$
1	$5+5\widehat{H}$
2a	$7+5x+5x^2+3x^3+5x^4+5x^5+y(4+7x+4x^2+6x^3+8x^4+6x^5)$
2b	$7+5x+5x^2+3x^3+5x^4+5x^5+y(5+9x+5x^2+5x^3+6x^4+5x^5)$
3a	$6+9x+4x^2+4x^3+6x^4+6x^5+y(5+6x+4x^2+5x^3+4x^4+6x^3)$
3b	$7+5x+8x^2+3x^3+5x^4+7x^5+y(6+6x+5x^2+4x^3+4x^4+5x^5)$

4 Computation of large quotient images

In this section, we describe computer-based methods for correlating small quotient images to identify weighing matrices. We describe a detailed search for $W(60, 25)$ weighing matrices in two groups, and describe plans to continue the search in the other solvable groups of order 60. Complementary quotient images provide the algebraic structure for our correlations.

4.1 Complementary quotient images

If G is an internal direct product of N by H, then $G/N \cong H$ and $G/H \cong N$. Informally, we view G as an external product of its quotient groups ($G \cong G/H \times G/N$). A related statement is true for semidirect products. The *core* of H is the largest $G-$normal subgroup of H. If $G \cong N \rtimes H$, then G is also isomorphic to some subgroup of $H \times G/core(H)$. A proof of this claim appears in reference [7]. We view H and $G/core(H)$ as the most important homomorphic images of G. They contain *complementary* quotient images of combinatorial structures in G.

Corollary 1. *If a finite semidirect product, $G = N \rtimes H$, contains a group-developed weighing matrix, then correlation of complementary quotient images in H and $G/core(H)$ partially identifies the matrix.*

4.2 Direct products

For a direct product, $G = N \times H$, the cross-referencing process is relatively easy. We distribute the coefficients of $\delta \in \mathbb{Z}G$ across a $|N| \times |H|$ matrix, where rows are indexed by elements of N and columns are indexed by elements of H.

Suppose $N = \mathbb{Z}_5 = \langle z \rangle$ and H is a group of order 12, $H =< x, y : x^6 = 1 >$. Label entries in a 5×12 matrix to correspond with the 60 elements.

e	x	$\dots$	x^5	y	yx	$\dots$	yx^5
z	zx	$\dots$	zx^5	zy	zyx	$\dots$	zyx^5
z^2	z^2x	$\dots$	z^2x^5	z^2y	z^2yx	$\dots$	z^2yx^5
z^3	z^3x	$\dots$	z^3x^5	z^3y	z^3yx	$\dots$	z^3yx^5
z^4	z^4x	$\dots$	z^4x^5	z^4y	z^4yx	$\dots$	z^4yx^5

The first column is indexed by elements of N. The remaining columns are indexed by elements in cosets of N. The first row is indexed by elements of H, while the remaining rows reflect cosets of H.

Now distribute the coefficients of $\delta \in \mathbb{Z}G$ to corresponding positions in the matrix. Let π be the projection $\pi : \mathbb{Z}G \to \mathbb{Z}G/N$. We view $\pi(\delta)$ as a polynomial in the variables $xN, x^2N, \ldots, yx^5N$. The coefficients of $\pi(\delta)$ equal the corresponding column-sums of our matrix. Similarly, a projection onto $\mathbb{Z}G/H$ produces a polynomial in $\mathbb{Z}G/H \cong \mathbb{Z}N$; its coefficients are row-sums in the matrix.

If δ represents a weighing matrix $W(60, 25)$, then the row- and column-sums of our 5×12 matrix are coefficients from complementary quotient images in N and H, respectively. We present a matrix for $N \times H = \mathbb{Z}_5 \times Q_{12} = \langle z \rangle \times \langle x, y \rangle$ as Table 4. From Table 1 we know the possible $\mathbb{Z}_5$ quotient images, and from Proposition 5 we know the possible Q_{12} images.

For convenience, we express quotient images as vectors with reduced coefficients. We order the $\mathbb{Z}_5$ coefficients as $[e, z, z^2, z^3, z^4]$. We then compute vectors of possible row-sums by subtracting $12\widehat{\mathbb{Z}_5}$ from $\mathbb{Z}_5$ quotient images. For example, the quotient image $9 + 14z + 14z^2 + 14z^3 + 14z^4$ from Table 1 appears as $R2 = [-3, 2, 2, 2, 2]$ in Table 4.

Similarly, the Q_{12} quotient image $7 + 5x + 5x^2 + 3x^3 + 5x^4 + 5x^5 + y(4 + 7x + 4x^2 + 6x^3 + 8x^4 + 6x^5)$ from Proposition 5, reduced by $5\widehat{Q_{12}}$, produces the vector of column sums $C2a = [2, 0, 0, -2, 0, 0, -1, 2, -1, 1, 3, 1]$. The other possible column sums are displayed in Table 5.

Table 4: Row sums for $\mathbb{Z}_5 \times Q_{12}$.

R1	**R2**	1	2	...	6	7	8	...	12
5	−3	e	x	...	x^5	y	yx	...	yx^5
0	2	z	zx	...	zx^5	zy	zyx	...	yx^5
0	2	z^2	z^2x	...	z^2x^5	z^2y	z^2yx	...	z^2yx^5
0	2	z^3	z^3x	...	z^3x^5	z^3y	z^3yx	...	z^3yx^5
0	2	z^4	z^4x	...	z^4x^5	z^4y	z^4yx	...	z^4yx^5

Table 5: Column sums for $\mathbb{Z}_5 \times Q_{12}$.

column	1	2	3	4	5	6	7	8	9	10	11	12
C1	5	0	0	0	0	0	0	0	0	0	0	0
C2a	2	0	0	−2	0	0	−1	2	−1	1	3	1
C2b	2	0	0	−2	0	0	0	4	0	0	1	0
C3a	1	4	−1	−1	1	1	0	1	−1	0	−1	1
C3b	2	0	3	−2	0	2	1	1	0	−1	−1	0

A number of assumptions, reflecting equivalence of weighing matrices, are incorporated in these tables. As the order of our factor subgroups are relatively prime, we consider equivalence with respect to N and H separately. We translate $N = \mathbb{Z}_5$ quotient images (row sums) by a group element of order 5. Effectively, we choose our favorite number for the first row sum. Repeated application of the $\mathbb{Z}_5$ automorphism $\alpha : z \longrightarrow z^2$ brings any other desired row into the second position.

Therefore, we assume that the row sums (in order) of the $\mathbb{Z}_5 \times Q_{12}$ coefficient matrix are $R1$ or $R2$ in Table 4. We define the weight of a vector to be the number of 1's which it contains. The automorphism α allows us to assume row 2 contains at least as many 1's as each of the following rows; that is, we can assume $wt(row2) \geq wt(row3, \ldots, row5)$.

The order of column sums is fixed, as we made a number of simplifying assumptions while determining the Q_{12} quotient images. If the column sums are all distinct, we can make no further assumptions. If, however, they are repeated, we utilize automorphisms which fix the column sums, but not the actual columns.

Recall that the column sums are ordered $[e, x, \ldots, x^5, y, yx, \ldots, yx^6]$. There are six distinct automorphisms of Q_{12} which fix columns 1 through 6, while moving each of columns 7...12 to column 7. There is also an automorphism which exchanges columns 2 and 6, while fixing columns 7...12. The vector $C1$, which contains many zeroes, is fixed by each of these automorphisms. Therefore, when considering case 1, we make assumptions about column weights while fixing the column sums. Specifically, we assume that $wt(col\ 2) \geq wt(col\ 6)$ and that $wt(col\ 7) \geq wt(cols\ 8, \ldots, 12)$.

Vector $C2a$ is fixed by fewer automorphisms, so we make fewer assumptions about column weights when considering case 2a. Conclusions about cases 1, 2a, 2b, 3a, and 3b of Proposition 5 are summarized in Table 6.

Similar arguments apply to matrices developed over direct products $\mathbb{Z}_5 \times H$, with $H \in \{\mathbb{Z}_{12}, Q_{12}, D_{12}, \mathbb{Z}_2 \times \mathbb{Z}_6\}$. Table 6 summarizes the allowable column weight assumptions in 5×12 coefficient matrices for these groups.

4.3 Stripings of quotient images

We wish to determine all entries of a 5×12 matrix M, which represents a possible weighing matrix in $G = N \times H = \langle z \rangle \times \langle x, y \rangle$. In principle, complementary quotient images in N and H should be sufficient to initiate a computer search. Many groups, however, admit additional quotient groups which further restrict the search space.

Such restrictions take the form of partial row-sums in M. We refer to partial rows as *rowlets*; a collection of restrictions on rowlets is a *striping*

Table 6: Column assumptions in $\mathbb{Z}_5 \times H$ coefficient matrix

H	Case	Automorphims	Implied simplifications
$\mathbb{Z}_{12}$	1	All	$wt(C\ 2) \geq wt(C\ 6)$ $wt(C\ 7) \geq wt(C\ 10)$.
	2a, 2b		No valid assumptions.
Q_{12}	1	All	$wt(C\ 2) \geq wt(C\ 6)$ $wt(C\ 7) \geq wt(C\ 8, \ldots, 12)$.
	2a, 2b	$x \to x^{-1}, y \to yx^2$	$wt(C\ 2) \geq wt(C\ 6)$
	3a, 3b		No valid assumptions.
D_{12}	1	All	$wt(C\ 2) \geq wt(C\ 6)$ $wt(C\ 7) \geq wt(C\ 8, \ldots, 12)$.
	2a, 2b	$x \to x^{-1}, y \to yx^2$	This case is impossible.
$\mathbb{Z}_2 \times \mathbb{Z}_6$	1	All	$wt(C\ 2) \geq wt(C\ 6)$ $wt(C\ 7) \geq wt(C\ 10)$.

of M. We illustrate this approach for specific groups. The details reflect group structure calculations performed with GAP [10]. We occasionally refer to groups by their identifier in the GAP library "Small Groups."

4.3.1 Stripings of $\mathbb{Z}_5 \times Q_{12}$

Suppose M contains coefficients of a $W(60, 25)$ weighing matrix in $G = \mathbb{Z}_5 \times Q_{12} = \langle z \rangle \times \langle x, y \rangle$. Possible row- and column-sums of M appeared in Tables 4 and 5; these were based on the presentation $Q_{12} =< x, y : x^6 = y^4 = 1, y^2 = x^3, x^y = x^{-1} >$. Suppose we project G onto the quotient group $T = G/\langle x \rangle \cong \mathbb{Z}_{10}$. Informally, we map x to the identity, producing $T \cong \langle z \rangle \times \langle y : y^2 = 1 \rangle$. Applied to a weighing matrix, the projection produces a $\mathbb{Z}_{10}$ quotient image. Applied to M, the projection combines entries which were previously distinguished by differing powers of x. The following rowlet sums $\{m_i\}$ must be the coefficients of a $\mathbb{Z}_{10}$ quotient image:

$$\begin{aligned} m_1 &= M_{1,1} + M_{1,2} + M_{1,3} + M_{1,4} + M_{1,5} + M_{1,6} && (3)\\ m_2 &= M_{1,7} + M_{1,8} + M_{1,9} + M_{1,10} + M_{1,11} + M_{1,12} && (4)\\ m_3 &= M_{2,1} + M_{2,2} + M_{2,3} + M_{2,4} + M_{2,5} + M_{2,6} && (5)\\ &\vdots && (6)\\ m_{10} &= M_{5,7} + M_{5,8} + M_{5,9} + M_{5,10} + M_{5,11} + M_{5,12} && (7) \end{aligned}$$

The projection of G onto $G/\langle x^2 \rangle \cong \mathbb{Z}_{20}$ produces a $\mathbb{Z}_{20}$ quotient image.

For each row i in M, these partial row sums equal $\mathbb{Z}_{20}$ intersection numbers:

$$\begin{aligned} r_{i,1} &= M_{i,1} + M_{i,3} + M_{i,5} & (8) \\ r_{i,2} &= M_{i,2} + M_{i,4} + M_{i,6} & (9) \\ r_{i,3} &= M_{i,7} + M_{i,9} + M_{i,11} & (10) \\ r_{i,4} &= M_{i,8} + M_{i,10} + M_{i,12} & (11) \end{aligned}$$

We call this a striping of M. It restricts the rowlet-sums arising from $\mathbb{Z}_{10}$ quotient images by introducing information from $\mathbb{Z}_{20}$ quotient images.

4.3.2 Stripings of $\mathbb{Z}_5 \times D_{12}$

Next, we assume that M contains coefficients of a weighing matrix developed over $G = \mathbb{Z}_5 \times D_{12} =< z > \times < x, y : x^6 = y^2 = 1; x^y = x^{-1} >$. As our presentation of G parallels the presentation of $\mathbb{Z}_5 \times Q_{12}$ in section 4.2, the structure of M is identical to Table 4. The possible row sums are reduced vectors representing $\mathbb{Z}_5$ quotient images; they are the vectors $R1$ and $R2$ in Table 4.

There is only one possible vector of column sums; the vector $C1 = [5, 0, \ldots, 0]$ corresponds to the unique D_{12} quotient image in Proposition 5.

Frequently, a group admits several isomorphic quotients via different projections. These yield inequivalent stripings of the coefficient matrix M. For the present group, the stripings of M reflect three distinct normal subgroups of order 6.

As with $\mathbb{Z}_5 \times Q_{12}$, the projection $G/ < x > \cong \mathbb{Z}_{10}$ produces rowlets satisfying (in some order) equations (3) through (7).

The normal subgroup generated by yx and yx^3 provides another quotient group isomorphic to $\mathbb{Z}_{10}$. Applying the corresponding projection to M, we find that positions e, x^2, x^4, yx, yx^3 and yx^5 must be combined in each row. We have the following partial row sums for each row i:

$$\begin{aligned} a_{i,1} &= M_{i,1} + M_{i,3} + M_{i,5} + M_{i,8} + M_{i,10} + M_{i,12} & (12) \\ a_{i,2} &= M_{i,2} + M_{i,4} + M_{i,6} + M_{i,7} + M_{i,9} + M_{i,11} & (13) \end{aligned}$$

This is a striping of M; it imposes additional restrictions on rowlet sums.

Finally, the normal subgroup $\langle x^2, y \rangle$ generates a striping with these partial row sums:

$$\begin{aligned} b_{i,1} &= M_{i,1} + M_{i,3} + M_{i,5} + M_{i,7} + M_{i,9} + M_{i,11} & (14) \\ b_{i,2} &= M_{i,2} + M_{i,4} + M_{i,6} + M_{i,8} + M_{i,10} + M_{i,12} & (15) \end{aligned}$$

4.4 Semidirect products

Suppose G is a semidirect product, $G \cong N \rtimes H$, and contains a weighing matrix $W(60, 25)$. Consequently, N is a normal subgroup with order 5, while H is not normal in G. In this case, we cannot assume that row sums of M reflect quotient images in a group G/H.

Instead, we consider complementary quotient images in $G/N \cong H$ and $G/core(H)$. We illustrate this approach with a specific example: $G = \mathbb{Z}_5 \rtimes Q_{12} \cong < z, x, y : z^5 = x^6 = 1, y^2 = x^3, x^y = x^{-1}, z^x = z^{-1}, z^y = z^2 >$.

Assume that $N = \langle z \rangle$ and $H = \langle x, y \rangle$; the core of H is then $\langle x^2 \rangle \cong \mathbb{Z}_3$. The quotient groups G/N and $G/core(H)$ are isomorphic to Q_{12} and GAP small group [20,3], respectively. We view Q_{12} intersection numbers as column-sums in M. We view $[20, 3]$ intersection numbers as twenty partial row sum in that matrix. In fact, we re-use the matrix striping developed for $\mathbb{Z}_5 \times Q_{12}$ in section 4.3.1. The partial sums simply represent a different order-20 quotient image.

There is one critical difference between searches for weighing matrices in $N \times H$ and in $N \rtimes H$. When H is not normal, we must be very careful that assumptions about orderings of rowlets do not contradict assumptions about orderings of column sums. In direct product searches, the structure of $G/H \cong N$ quotient images and row-sums provide more assumptions about row-sums and rowlet orders.

5 Implementation of computer searches

We have completed computer searches for weighing matrices $W(60, 25)$ developed over several groups. The organization of those searches followed the theoretical development above. Brief summaries of these searches and their results are given below.

In each group $G = N \rtimes H$, we search for a polynomial δ in $\mathbb{Z}G$ which can be developed as a weighing matrix. We distribute polynomial coefficients across a 5×12 matrix according to appropriate rowlet sums and stripings. Once we determine a matrix M satisfying those restrictions, we view it as the first row of a potential 60×60 weighing matrix W. Specifically, we label the rows and columns of W in the order $[e, x, \ldots, x^5, y, \ldots, yx^5, z \ldots, zyx^5, \ldots, z^4yx^5]$ and set the first entry in column g equal to the corresponding element in M. We then set each entry $w_{g,f}$ equal to $w_{e,fg^{-1}}$, completing the group-developed matrix. We test the resulting matrix by direct multiplication; a valid weighing matrix must satisfy $WW^T = 25I$.

5.1 Search for weighing matrices in $\mathbb{Z}_5 \times Q_{12}$

Suppose a weighing matrix $W = W(60, 25)$ exists in the group $\mathbb{Z}_5 \times Q_{12}$. The first row of W is generated by a 5×12 matrix M; all other rows of W are (known) permutations of this first row. The matrix M must satisfy:

- each entry is 0, +1, or −1;
- the total number of 1's is 15 and the total number of −1's is 10;
- the row sums are (in order) either $R1$ or $R2$ as specified in Table 4;
- the column sums are (in order) either $C1$, $C2a$, $C2b$, $C3a$ or $C3b$ as specified in Table 5;
- row 2 contains at least as many 1's as each of rows 3, 4, and 5;
- for cases $C1$, $C2a$ and $C2b$, column 2 contains at least as many 1's as column 6;
- for case $C1$, column 7 contains at least as many 1's as each of columns 8 through 12; and
- restrictions imposed by our striping of M in section 4.3.1.

We consider the last point in detail.

The rowlet sums defined by equations (3) through (7) must be some permutation a $\mathbb{Z}_{10}$ quotient image (reduced by $\widehat{6\mathbb{Z}_{10}}$). The closely related full row-sums represent $\mathbb{Z}_5$ quotient images, and have a fixed order. If the vector of row-sums is $R1 = [5, 0, 0, 0, 0]$, then the rowlet sums $[m_1, m_2, \ldots, m_{10}]$ are some permutation of either $[5, 0, 0, 0, 0, 0, 0, 0, 0, 0]$ or $[4, 1, -1, 1, -1, 1, -1, 1, -1, 1]$. Only a few of these permutations are acceptable. For example, if the row sum is $R1 = [5, 0, 0, 0, 0]$, then the only possible pairs (m_1, m_2) are $(5, 0)$, $(0, 5)$, $(4, 1)$ or $(1, 4)$. Similar restrictions apply to pairs from the remaining rows. If the $\mathbb{Z}_5$ image is $R2 = [-3, 2, 2, 2, 2]$, then the rowlet sums are some permutation of either $[-3, 0, 2, 0, 2, 0, 2, 0, 2, 0]$ or $[-4, 1, 1, 1, 1, 1, 1, 1, 1, 1]$.

The striping imposed by $\mathbb{Z}_{20}$ intersection numbers takes the form of four partial sums for each row, defined by equations (8) through (11). We consider the collection of these partial sums: $([r_{1,1}, r_{2,1}, r_{3,1}, r_{4,1}, r_{5,1}]$, $[r_{1,2}, \ldots, r_{5,2}]$, $[r_{1,3}, \ldots, r_{5,3}]$, $[r_{1,4}, \ldots, r_{5,4}])$. The possible values are implied by Table 2. For example, case 1 in Table 2 contains the quotient image $[(2 + 3\langle x\rangle)\langle y\rangle - 5]$. Subtracting three copies of the complete group, we obtain the polynomial $-3 + 2y + 2y^2 + 2y^3 + 2y^4$. Therefore, the collection of partial sums is some permutation of ([-3,2,2,2,2], [0,0,0,0,0], [0,0,0,0,0], [0,0,0,0,0]), giving $5 \cdot 4 = 20$ subcases.

Notice that each rowlet sum m_i is equal to $r_{i,j} + r_{i,k}$, for some $j \neq k$. This reflects the algebraic information that each $\mathbb{Z}_{20}$ quotient image projects onto a unique $\mathbb{Z}_{10}$ quotient image. Consequently, we can correlate intersection numbers, rowlet sums, and row sums. The striping of case 1 in Table 2 is a restriction on the rowlet sums $[-3, 0, 2, 0, 2, 0, 2, 0, 2, 0]$. Those rowlet sums describe possible refinements of the row sums $R2 = [-3, 2, 2, 2, 2]$. Note that in this case, the ordering of the numbers $r_{i,j}$ is unique.

Similarly, subtracting $3\widehat{\mathbb{Z}_{20}}$ from case 2 in Table 2 yields the polynomial $(y + y^2 + y^3 + y^4) + 2x + x^2(-3 + y + y^2 + y^3 + y^4) - 2x^3$. In this case, the collection of partial sums is some permutation of ([0,1,1,1,1], [2,0,0,0,0], [-3,1,1,1,1], [-2,0,0,0,0]), giving $5^4 \cdot 4! = 15000$ subcases. Entries from [0,1,1,1,1] and [-3,1,1,1,1] must belong to the same rowlet, and entries from [2,0,0,0,0] and [-2,0,0,0,0] to the same rowlet. Furthermore, no permutations within these sets are possible.

The remaining cases from Table 2 yield the following collections of partial sums. Case 3 yields ([1,0,0,1,-2],[-1,1,1,-1,0],[0,1,1,0,-2],[2,0,0,2,1]). This is compatible only with rowlet sum $[-4, 1, 1, 1, 1, 1, 1, 1, 1, 1]$. For each partial sum $[r_{1,i}, \ldots r_{5,i}]$, there are $5 \cdot 3 \cdot 2$ possible permutations. Notice that the set of permutations of $[r_{1,1}, \ldots r_{5,1}]$ is identical to the set of permutations of $[r_{1,3}, \ldots r_{5,3}]$. Thus case 3 gives a total of $4 \cdot 3 \cdot 30^4 = 9720000$ subcases. Case 4 yields ([1,0,0,2,2],[2,0,0,-1,-1],[0,1,1,-1,-1],[2,-1,-1,0,0]). This is compatible only with rowlet sum $[4, 1, -1, 1, -1, 1, -1, 1, -1, 1]$. Similarly, this also gives $12 \cdot 30^4 = 9720000$ subcases. Case 5 yields ([-1,1,1,2,2], [0,-1,-1,1,1],[-2,1,1,0,0],[0,1,1,-1,-1]). This is compatible only with rowlet sum $[-3, 0, 2, 0, 2, 0, 2, 0, 2, 0]$, similarly giving $12 \cdot 30^4 = 9720000$ subcases. Finally, case 6 yields ([3,1,1,0,0],[0,1,1,-1,-1],[2,-1,-1,0,0],[0,-1,-1,1,1]). This is compatible only with rowlet sum $[5, 0, 0, 0, 0, 0, 0, 0, 0, 0]$, similarly giving $12 \cdot 30^4$ subcases. Each striping describes possible restrictions on a unique set of rowlet sums and thus on a unique set of row sums.

Our backtrack search strategy is as follows. We calculate in advance each possible sequence of rowlet sums. For each possible rowlet sum, we generate in advance all possibilities for a rowlet with that sum. We also generate in advance all acceptable permutations of the partial row sums $r_{i,j}$ for each possible rowlet sum.

For each possible sequence of rowlet sums and acceptable permutations of the values $\{r_{i,j}\}$, we generate one rowlet at a time from the right-hand side of M. After generating each rowlet, we verify that the total accumulations of 1's and -1's in the matrix do not exceed our maximums. After the entire right-hand side has been generated, we verify each column sum. If applicable, we also verify the relative numbers of 1's in these columns. We then repeat this process for the left-hand side. As each row is completed, we verify that the weight of rows 3 through 5 do not exceed the weight of row 2.

Using this strategy we generated 57 375 080 matrices satisfying the restrictions above. This process took just over 90 days of computing time. We next unpacked each 5×12 matrix into the corresponding 60×60 matrix W and tested the condition $WW^T = 25I$. After less than 2 days computing time, we found that none of the matrices satisfied this condition. We conclude that there are no weighing matrices $W(60, 25)$ developed over $\mathbb{Z}_5 \times Q_{12}$.

5.2 Search for weighing matrices in $\mathbb{Z}_5 \times D_{12}$

Next, we describe our search for a weighing matrix $W = W(60, 25)$ developed over the group $\mathbb{Z}_5 \times D_{12}$. There are two notable differences from the situation described in section 5.1. First, the search for weighing matrices in $\mathbb{Z}_5 \times Q_{12}$ considered several vectors of column sums; in this new search, only one vector is possible. Second, stripings of the coefficient matrix M differ from the previous search; this reflects the differences in group structure.

In our present search, the vector $C1 = [5, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$ is the only possible list of column sums. Recall that Table 6 allows us limited assumptions about column weights. We assume that column 2 contains at least as many 1's as column 6 and that column 7 contains at least as many 1's as each of columns 8, ..., 12.

In this search, rowlet sums and stripings are determined by $\mathbb{Z}_{10}$ quotients alone, rather than by both $\mathbb{Z}_{10}$ and $\mathbb{Z}_{20}$ quotients. As with $\mathbb{Z}_5 \times Q_{12}$, one $\mathbb{Z}_{10}$ quotient determines 10 rowlets. These correspond to the left side and right side of each row in M. The rowlet sums are given by equations (3) through (7). In section 4.3.2, we described two distinct stripings of $\mathbb{Z}_5 \times D_{12}$.

The two stripings use $\mathbb{Z}_{10}$ intersection numbers to determine partial row sums $a_{i,1}, a_{i,2}, b_{i,1}$ and $b_{i,2}$, as defined in equations (12) through (15). The sequence of values for $a_{i,j}$ must be some permutation of the allowed $\mathbb{Z}_{10}$ intersection numbers. Similarly, the values for $b_{i,j}$ must be a permutation of some vector of $\mathbb{Z}_{10}$ intersection numbers.

As in the previous search, the row sums are either $R1 = [5, 0, 0, 0, 0]$ or $R2 = [-3, 2, 2, 2, 2]$. The sequence of rowlet sums for $R1$ is some permutation of $[5, 0, 0, 0, 0, 0, 0, 0, 0, 0]$ or $[4, 1, -1, 1, -1, 1, -1, 1, -1, 1]$; furthermore, since each row is composed of 2 rowlets, only 2 permutations of the first of these cases, and $2^5 = 32$ permutations of the second of these cases, can add up to the correct row sum. Similarly, if the row sum is $R2$ then the sequence of rowlet sums is some permutation of either $[-3,0,2,0,2,0,2,0,2,0]$ or $[-4,1,1,1,1,1,1,1,1,1]$; again, only $2+32$ permutations will add up to the correct row sum.

A similar restriction applies to the stripes. Notice that the sum of the entries in row i of the matrix is not only $a_{i,1} + a_{i,2}$ but also $b_{i,1} + b_{i,2}$. Thus, we are restricted to the same $2+32$ allowed permutations for $a_{i,1}, a_{i,2}$ (respectively, $b_{i,1}, b_{i,2}$) as for the rowlet sums. Note that sums of the entries in the rowlets, sums of the entries in the first striping, and sums of the entries in the second striping, may admit very different permutations.

We now describe our backtrack search strategy. Combinatorially, it appears that the problem is simplified. However, from the point of view of implementation, several aspects are now more difficult. In the previous search, the "stripes" resulting from the restriction imposed by $\mathbb{Z}_{20}$ were restricted to a single rowlet. Furthermore, these stripes were very restrictive, as they affected only 3 columns. Now our stripes affect 6 columns, and these 6 columns spread into 2 rowlets. Not only are the stripes less restrictive, but we must generate an entire row before verifying that entries in the stripe have the desired sum. In the previous search, we generated the entire right side of the matrix first, followed by the entire left side. In this case, that strategy causes the search space to grow too quickly. Our search strategy is modified as follows.

We calculate in advance each possible sequence of rowlet sums; recall that this same set of possibilities to be used for two distinct stripings. For each possible rowlet sum, we generate in advance all possibilities for a rowlet. For each possible sequence of rowlet sums, each possible sequence of $a_{i,j}$, and each possible sequence of $b_{i,j}$, we generate first the left rowlet and then the right rowlet of each row in turn. We verify that we have not exceeded the allowed total numbers of 1's and -1's in the matrix after each rowlet is generated. After row i is complete, we verify the sums of the entries corresponding to $a_{i,1}, a_{i,2}, b_{i,1}$ and $b_{i,2}$. We compare the weight of row 2 to each successive row, as it is generated. As the final row is completed, we verify column sums and relative column weights.

Using this strategy, we found 0 matrices that satisfied the given restrictions. This required just over 20 days of computing time on a Pentium-4 2GHz computer. We conclude that there are no weighing matrices $W(60, 25)$ developed over the group $\mathbb{Z}_5 \times D_{12}$.

6 Conclusion and future plans

This paper outlines a theoretical basis for a computer search; specifically, a computer search for weighing matrices developed over solvable groups.

We are particularly interested in weighing matrices of order 60 and weight 25. There are twelve solvable groups of order 60; each of these groups admits a quotient group of order 12. Therefore, we view weighing matrices $W(60, 25)$ as polynomials with coefficients distributed over 5×12 matrices. We correlate the group structure with these coefficient matrices, creating systems of restrictions which we call stripings. These stripings allow us to narrow our search space with information from quotient images in smaller groups.

The twelve groups of interest fall into natural clusters, based on similarity of stripings. We demonstrated that the groups $\mathbb{Z}_5 \times Q_{12}$ and $\mathbb{Z}_5 \times D_{12}$ admit very similar search strategies, and that these searches require the same basic data. We are currently continuing the search in several groups with related stripings.

We intend to complete the search in all solvable groups of order 60, following the theoretical outline in this paper. We believe this is practical, though development of stripings and search strategies will require considerable work. Some of the computer searches will require extensive computational time.

In this paper, we have shown that two solvable groups do not admit weighing matrices of order 60 and weight 25. We have shown that the complete search is practical. We have also outlined a search strategy which is applicable to related combinatorial structures, including perfect ternary arrays and difference sets.

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