On Friendly Index Sets of Root-unions of Stars By Cycles

Yong-Song Ho
Nan Chiao High School
Singapore

Sin-Min Lee
Department of Computer Science
San Jose State University
San Jose, CA 95192, USA

Ho Kuen Ng
Department of Mathematics
San Jose State University
San Jose, CA 95192, USA

In memory of Professor S. C. Shee

__Abstract__ Let G be a graph with vertex set V(G) and edge set E(G). A labeling $f : V(G) \rightarrow Z_2$ induces an edge labeling $f^* : E(G) \rightarrow Z_2$ defined by $f^*(xy) = f(x) + f(y)$, for each edge $xy \in E(G)$. For $i \in Z_2$, let $v_f(i) = \text{card}\{v \in V(G) : f(v) = i\}$ and $e_f(i) = \text{card}\{e \in E(G) : f^*(e) = i\}$. A labeling f of a graph G is said to be friendly if $|v_f(0) - v_f(1)| \leq 1$. The friendly index set of the graph G, FI(G), is defined as $\{|e_f(0) - e_f(1)| :$ the vertex labeling f is friendly$\}$. This is a generalization of graph cordiality. We introduce a graph construction called the root-union, and investigate when gaps exist in the friendly index sets of root-unions of stars by cycles.

Key words: vertex labeling, friendly labeling, cordiality, friendly index set, cycle, star, arithmetic progression.

AMS 2000 MSC: 05C78, 05C25

1. Introduction

Let G be a graph with vertex set V(G) and edge set E(G). Let A be an abelian group. A labeling f: $V(G) \rightarrow A$ induces an edge labeling $f^* : E(G) \rightarrow A$ defined by $f^*(xy) = f(x) + f(y)$, for each edge $xy \in E(G)$. For each $i \in A$, let $v_f(i) = \text{card}\{v \in V(G) : f(v) = i\}$ and $e_f(i) = \text{card}\{e \in E(G) : f^*(e) = i\}$. Let $c(f) = (c_{ij})$ be the matrix indexed by $(i, j) \in A \times A$, such that $c_{ij} = |e_f(i) - e_f(j)|$. A labeling f of a graph G is said to be A-friendly if $|v_f(i) - v_f(j)| \leq 1$ for all $(i, j) \in A \times A$. If c(f) is a (0, 1)-matrix for an A-friendly labeling f, then f is said to be A-cordial.

The notion of A-cordial labelings was first introduced by Hovey [10], who generalized the concept of cordial graphs of Cahit [2]. Cahit considered $A = Z_2$ and he proved the following: every tree is cordial; K_n is cordial if and only if $n \leq 3$; $K_{m,n}$ is cordial for all m and n; the wheel W_n is cordial if and only if $n \neq 3 \pmod 4$; C_n is cordial if and only if $n \neq 2 \pmod 4$; and an Eulerian graph is not cordial if its size is congruent to 2 (mod 4). Benson and Lee [1] found a large class of cordial regular windmill graphs that include the friendship graphs as a subclass.

Lee and Liu [12] investigated cordial complete k-partite graphs. Kuo, Chang and Kwong [11] determined all m and n for which mK_n is cordial. In 1989, the second author, Ho and Shee [9] completely characterized cordial generalized Petersen graphs. Ho, Lee and Shee [8] investigated the construction of cordial graphs by Cartesian product and composition. Seoud and Abdel [18] proved certain cylinder graphs are cordial. Several constructions of cordial graphs were proposed in [17, 18, 19, 20, 21]. For more details of known results and open problems on cordial graphs, see [4, 7].

In this paper, we will exclusively focus on $A = Z_2$, and drop the reference to the group. When the context is clear, we will also drop the subscript f. In [6] the following concept was introduced.

<u>**Definition.**</u> The friendly index set FI(G) of a graph G is defined as $\{|e_f(0) - e_f(1)| :$ the vertex labeling f is friendly$\}$.

Note that if 0 or 1 is in FI(G), then G is cordial. Thus the concept of friendly index sets could be viewed as a generalization of cordiality. Cairnie and Edwards [5] have determined the computational complexity of cordial labeling and Z_k-cordial labeling. They proved that to decide whether a graph admits a cordial labeling is NP-complete. Even the restricted problem of deciding whether a connected graph of diameter 2 has a cordial labeling is NP-complete. Thus in general it is difficult to determine the friendly index sets of graphs.

In [13, 14, 15, 16] the friendly index sets of a few classes of graphs, including complete bipartite graphs and cycles, are determined. The following result was established.

<u>**Theorem 1.1.**</u> For any graph with q edges, the friendly index set $FI(G) \subseteq \{0, 2, 4, \ldots, q\}$ if q is even, and $FI(G) \subseteq \{1, 3, \ldots, q\}$ if q is odd.

<u>**Example 1.**</u> Figure 1 illustrates the friendly index set of the cycle C_8 with two parallel chords $FI(PC(8, 2)) = \{0, 2, 4, 6\}$.

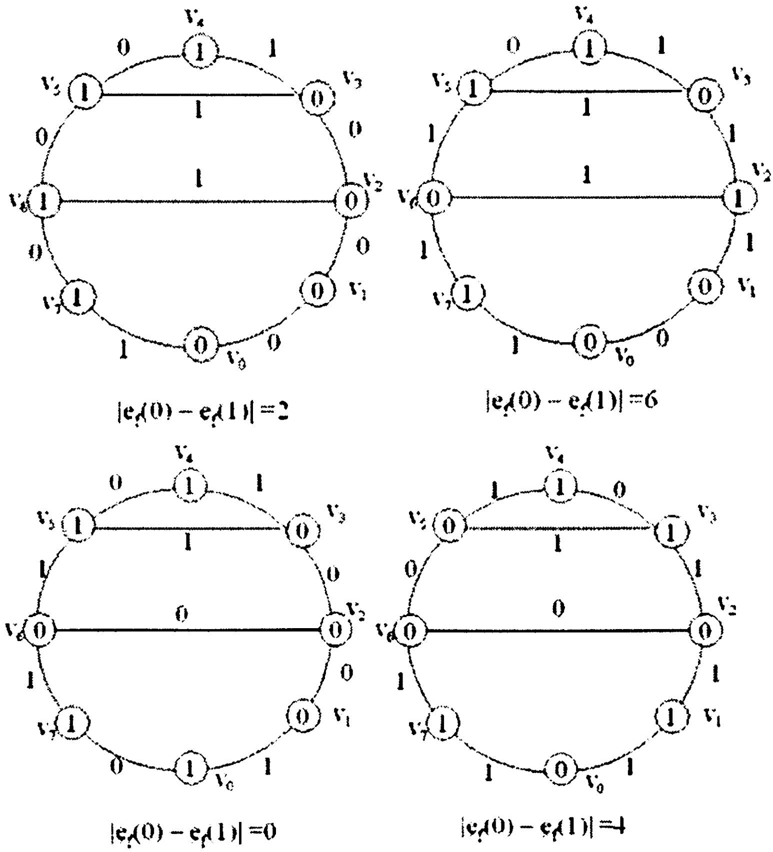


Figure 1.

Example 2. $FI(K_{3,3}) = \{1, 9\}$ and $FI(C_3 \times K_2) = \{1, 3, 5\}$.

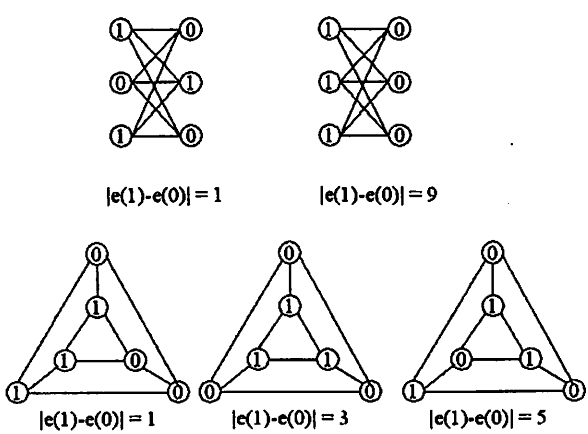


Figure 2.

In [13] it was shown that

Theorem 1.2. The friendly index set of a cycle is given as follows:

(i) $FI(C_{2n}) = \{0, 4, 8, \ldots, 2n\}$ if n is even.
$FI(C_{2n}) = \{2, 6, 10, \ldots, 2n\}$ if n is odd.

(ii) $FI(C_{2n+1}) = \{1, 3, 5, \ldots, 2n-1\}$.

In [13], the authors proposed the following

Conjecture. The numbers in FI(T) for any tree T form an arithmetic progression.

We observed the same phenomenon for cycles and cycles with parallel chords. In [16], we showed that for a cycle with an arbitrary non-empty set of parallel chords, the values in its friendly index set form an arithmetic progression with common difference 2. If the chords are not parallel, the numbers in the friendly index set might not form an arithmetic progression. See [15] on the friendly index sets of Möbius ladders.

2. Root-union of stars by a cycle, with center as root

For a (p, q)-graph G and a graph H with a root v, we introduce the following **root-union of (H, v) by G** construction as follows:

Take p copies of (H, v). For each copy, identify its root with a vertex of G. We denote this graph by G ® (H, v), and drop the reference to v if the context is clear.

In this section we consider the friendly index set of the root-union of stars by a cycle, where the center of the star is the root.

Lemma 2.1. For any friendly vertex labeling of C_n ® St(m), where n is even and m is odd, the induced edge labeling has an even e(1) and an even e(0).

Proof. Since the graph has (m + 1)n vertices, $v(0) = v(1) = (m+1)n/2$. Assume that k of the vertices on C_n are lableled 1, and thus the other (n – k) vertices are labeled 0. Now consider the vertices with degree 1. Note that km of them are adjacent to a 1-vertex. Assume that j of these km pendant vertices are labeled 1. Then the other (km – j) vertices are labeled 0. Now note that the other (n – k)m pendant vertices are adjacent to a 0-vertex. Since $v(1) = (m+1)n/2$, there are $(m+1)n/2 - k - j$ of them labeled 1, with the remaining $(n-k)m - (m+1)n/2 + k + j$ labeled 0. Thus, for these pendant edges, there are $(m+1)n/2 - k - j + km - j = (m+1)n/2 + (m-1)k - 2j$ labeled 1, and $j + (n-k)m - (m+1)n/2 + k + j = 2j + (m-1)n/2 - (m-1)k$ labeled 0. Since n is even and m is odd, both of these numbers are even.

From [13], the cycle C_n must have an even number of edges labeled 1. Since n is even, C_n must also have an even number of edges labeled 0.

Note. The first paragraph of the above proof only requires that (m + 1)n be even. Its arguments and symbols will be repeatedly used in this section.

Lemma 2.2. If n is even and m is odd, then $FI(C_n \circledR St(m))$ can only contain multiples of 4.

Proof. From the Lemma 2.1, let $e(1) = 2i$. Then $e(0) = (m + 1)n - 2i$, and $e(1) - e(0) = 2i - (m + 1)n + 2i = 4i - (m + 1)n$, where both terms are multiples of 4.

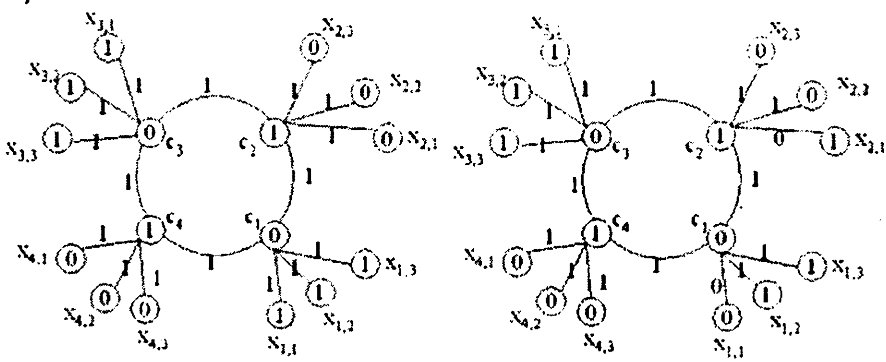
Theorem 2.1. If n is even and m is odd, then $FI(C_n \circledR St(m)) = \{0, 4, 8, \dots, (m + 1)n\}$.

Proof. By Theorem 1.1, $FI(C_n \circledR St(m)) \subseteq \{0, 2, 4, 6, 8, \dots, (m + 1)n\}$. By Lemma 2.2, only multiples of 4 can be in the friendly index set. Thus $FI(C_n \circledR St(m)) \subseteq \{0, 4, 8, \dots, (m + 1)n\}$. It suffices to show that all these values are attainable.

Label the vertices of C_n alternately by 0's and 1's. All pendant vertices adjacent to a 1-vertex are labeled 0, and all pendant vertices adjacent to a 0-vertex are labeled 1. This vertex labeling is obviously friendly with all edges labeled 1. Thus $e(1) - e(0) = (m + 1)n$.

In the above labeling, there are mn/2 pendant vertices labeled 0 and mn/2 pendant vertices labeled 1. Pair them into mn/2 pairs, and interchange the labels of each pair successively. After each interchange, there are two additional 0-edges, decreasing the value of $e(1) - e(0)$ by 4. Thus $e(1) - e(0) = (m + 1)n - 4i$, where $i = 0, 1, 2, \dots, mn/2$, showing that all the values in the set are attainable.

Example 3. Figure 3 shows that $C_4 \circledR St(3)$ has friendly index set $\{0, 4, 8, 12, 16\}$.



$e(1)=16$, $e(0)=0$ $|e(1)-e(0)| = 16$ $e(1)=14$, $e(0)=2$ $|e(1)-e(0)| = 12$

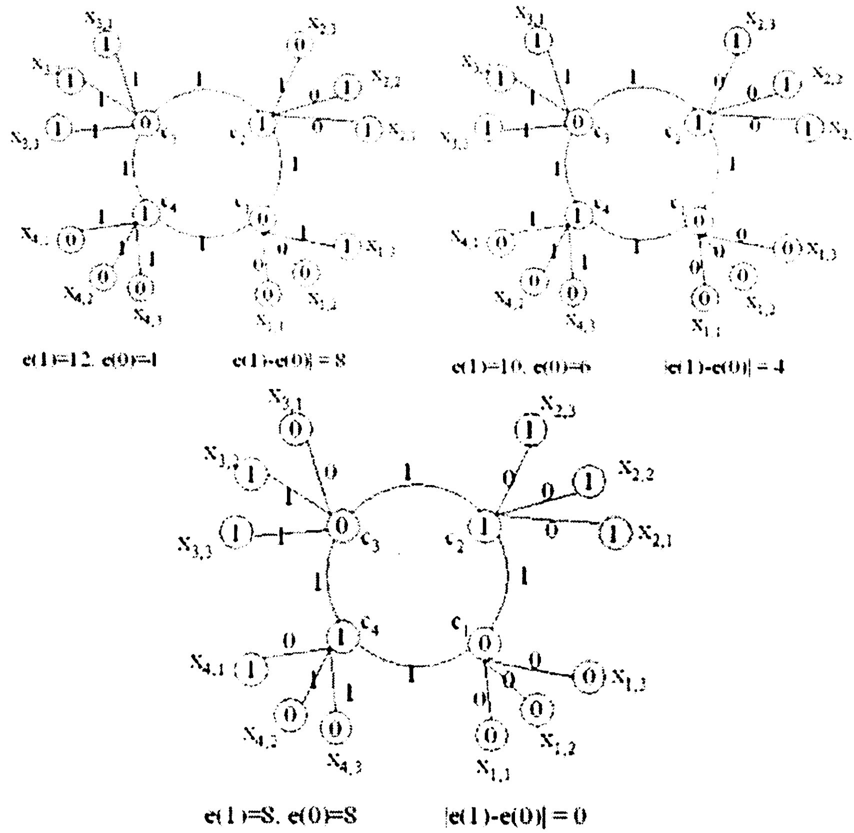


Figure 3.

__Lemma 2.3.__ For any friendly vertex labeling of C_n ® St(m), where n is odd and m is even, e(0) ≥ m/2 and e(1) ≥ m/2.

Proof. Note that the number of vertices = the number of edges = (m + 1)n is an odd number. Assume that k of the vertices on C_n are labeled 1, and thus the other (n – k) vertices are labeled 0. Since changing all vertex labels to their complements maintains friendliness and the values of both e(0) and e(1), we may assume that the cycle C_n has more 0-vertices than 1-vertices, i.e., k ≤ (n – 1)/2.

Now consider the vertices with degree 1. Note that km of them are adjacent to a 1-vertex. Assume that j of these km pendant vertices are labeled 1. Note that j ≤ km ≤ (n – 1)m/2. Then the other (km – j) vertices are labeled 0. Now note that the other (n – k)m pendant vertices are adjacent to a 0-vertex. Since v(1) = (m + 1)n/2 ± ½, there are (m + 1)n/2 ± ½ – k – j of them labeled 1, with the remaining (n – k)m – ((m + 1)n/2 ± ½ – k – j) labeled 0. Thus, e(1) ≥ (m + 1)n/2 ± ½ – k – j ≥ (m + 1)n/2 – ½ – k – j ≥ (m + 1)n/2 – ½ – (n – 1)/2 – (n – 1)m/2 = (m + 1)/2 – ½ = m/2, proving half of the Lemma.

Now the number of pendant edges with a 0-edge label is $j + (n - k)m - ((m + 1)n/2 \pm ½ - k - j) \geq 2j + (m - 1)n/2 - (m - 1)k - ½ \geq 2j + (m - 1)n/2 - (m - 1)(n - 1)/2 - ½ \geq 2j + (m - 1)/2 - ½ = 2j + m/2 - 1 \geq m/2 - 1$. Since n is odd, not all the edges of C_n can be labeled 1, i.e., at least one edge of C_n is labeled 0. Thus $e(0) \geq m/2$, finishing the proof.

Note. When m = 0, the graph is a cycle. Theorems 1.2 gives its friendly index set.

Theorem 2.2. If n is odd and $m \geq 2$ is even, then $FI(C_n ® St(m)) = \{1, 3, 5, \dots, (m + 1)n - m\}$.

Proof. By Theorem 1.1, $FI(C_n ® St(m)) \subseteq \{1, 3, \dots, (m + 1)n\}$. By Lemma 2.3, $|e(1) - e(0)| \leq (m + 1)n - m/2 - m/2 = (m + 1)n - m$. It suffices to show that all these values are attainable.

Label the vertices of C_n alternately by 0's and 1's, starting and ending with 0. Call the last vertex c_n. For all vertices of cycle but c_n, if its label is x, label all its adjacent pendant vertices $(1 - x)$. For the last vertex c_n of the cycle, label m/2 of its pendant vertices 0, and the other m/2 of its pendant vertices 1. This is a vertex-friendly labeling. Note that $e(1) = (n - 1) + (n - 1)m + m/2$ and $e(0) = 1 + m/2$, giving $e(1) - e(0) = (m + 1)n - m - 2$. In this labeling, there are $(n - 1)m/2$ pendant 0-vertices adjacent to a 1-vertex of C_n, and $(n - 1)m/2$ pendant 1-vertices adjacent to a 0-vertex of C_n other than c_n. Pair them into $(n - 1)m/2$ pairs, and interchange the labels of each pair successively. After each interchange, there are two additional 0-edges, decreasing the value of $e(1) - e(0)$ by 4. Thus $e(1) - e(0) = (m + 1)n - m - 2 - 4i$, where $i = 0, 1, 2, \dots, (n - 1)m/2$.

Now do the same procedure as in the previous paragraph, except that for the last vertex c_n of the cycle, label $(m/2 + 1)$ of its pendant vertices 1, and the other $(m/2 - 1)$ of its pendant vertices 0. This is still a vertex-friendly labeling. Note that $e(1) = (n - 1) + (n - 1)m + m/2 + 1$ and $e(0) = 1 + m/2 - 1$, giving $e(1) - e(0) = (m + 1)n - m$. Again, the same interchanges will give $e(1) - e(0) = (m + 1)n - m - 4i$, where $i = 0, 1, 2, \dots, (n - 1)m/2$.

Example 4. Figure 4 shows that $FI(C_3 ® St(2)) = \{1, 3, 5, 7\}$.

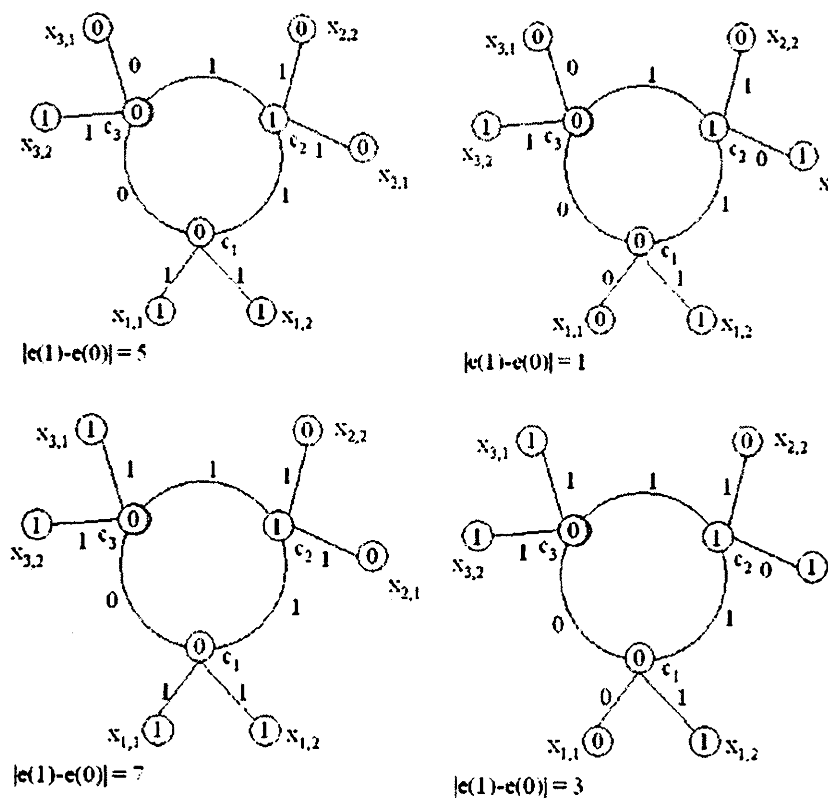


Figure 4.

__Lemma 2.4.__ Consider any friendly vertex labeling of C_n ® St(m), where n is even. If half of the vertices of C_n are labeled 0, then e(0) and e(1) are both even.
Proof. We use the notation in the proof of Lemma 2.1. Note that k = n/2 under our assumption here. The cycle C_n has an even number of 1-edges and hence an even number of 0-edges. The number of pendant 1-edges is $(m + 1)n/2 + (m - 1)k - 2j = (m + 1)n/2 + (m - 1)n/2 - 2j = mn - 2j$, an even number. The number of pendant 0-edges is 2j, also an even number.

__Lemma 2.5.__ Consider any friendly vertex labeling of C_n ® St(m), where n is even. If e(1) is odd, then $e(1) \geq m + 1$ and $e(0) \geq m + 1$.
Proof. Again we use the notation in the proof of Lemma 2.1. By Lemma 2.4, we know that $k \neq n/2$. Since changing all vertex labels to their complements maintains friendliness and the values of both e(0) and e(1), we may assume that the cycle C_n has more 0-vertices than 1-vertices, i.e., $k \leq n/2 - 1$. Then $j \leq km \leq (n/2 - 1)m$. From $k \neq n/2$, we also note that the edges of C_n cannot all be labeled 1, because that would require the vertex labels to alternate between 0's and 1's. Since n is even, there must be at least two edges in C_n labeled 0.

From the proof of Lemma 2.1, the number of pendant 1-edges $\geq (m + 1)n/2 - k - j \geq (m + 1)n/2 - (n/2 - 1) - (n/2 - 1)m = m + 1$. The number of pendant 0-edges is $2j + (m - 1)n/2 - (m - 1)k = 2j + (m - 1)(n/2 - k) \geq 2j + (m - 1) \geq m - 1$. Then e(0) = the number of pendant 0-edges + the number of 0-edges in $C_n \geq (m - 1) + 2 = m + 1$.

__Note.__ When m = 0, the graph is a cycle. Theorems 1.2 gives its friendly index set.

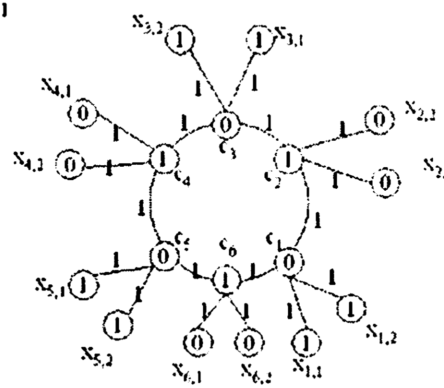
__Theorem 2.3.__ If n and m are both even, $m \geq 2$, then $FI(C_n ® St(m)) = \{(m + 1)n, (m + 1)n - 4, (m + 1)n - 8, \ldots\} \cup \{(m + 1)n - 2(m + 1), (m + 1)n - 2(m + 1) - 4, (m + 1)n - 2(m + 1) - 8, \ldots\}$, i.e., $\{0, 2, 4, \ldots, (m + 1)n - 2m\} \cup \{(m + 1)n - 2m + 4, (m + 1)n - 2m + 8, \ldots, (m + 1)n\}$.

Proof. The first set in the statement of the Theorem has both e(0) and e(1) even, while the second set has both e(0) and e(1) odd. By Theorem 1.1, $FI(C_n ® St(m)) \subseteq \{0, 2, \ldots, (m + 1)n\}$. By Lemma 2.5, if e(1) is odd, then $|e(1) - e(0)| \leq (m + 1)n - (m + 1) - (m + 1) = (m + 1)n - 2(m + 1)$. It suffices to show that all these values are attainable.

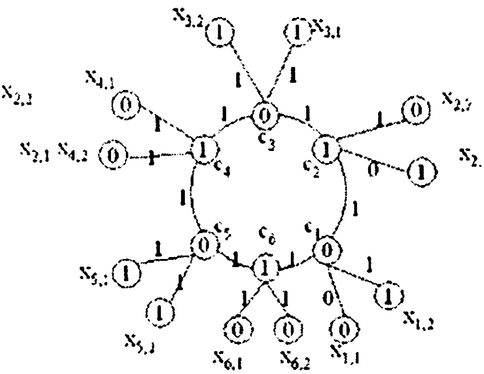
Label the vertices of C_n alternately by 0's and 1's. All pendant vertices adjacent to a 1-vertex are labeled 0, and all pendant vertices adjacent to a 0-vertex are labeled 1. This vertex labeling is obviously friendly with all edges labeled 1. Thus $e(1) - e(0) = (m + 1)n$. In this labeling, there are mn/2 pendant vertices labeled 0 and mn/2 pendant vertices labeled 1. Pair them into mn/2 pairs, and interchange the labels of each pair successively. After each interchange, there are two additional 0-edges, decreasing the value of $e(1) - e(0)$ by 4. Thus $e(1) - e(0) = (m + 1)n - 4i$, where $i = 0, 1, 2, \ldots, mn/2$, showing that all the values in the first set are attainable.

Start with the same labeling as that in the previous paragraph, except that the last vertex of the cycle, say c_n, is labeled 0, i.e., the first, last and second last vertices of the cycle are labeled 0. For each vertex of the cycle other than c_n, if its label is x, label all its pendant vertices $1 - x$. For c_n, label one of its pendant vertices 1, and the other $(m - 1)$ pendant vertices 0. This vertex labeling is friendly, with $e(1) = (n - 2) + (n - 1)m + 1 = (m + 1)n - m - 1$, and $e(0) = 2 + (m - 1) = m + 1$. Thus $e(1) - e(0) = (m + 1)n - 2(m + 1)$. Similar interchanges of pendant vertex labels give $e(1) - e(0) = (m + 1)n - 2(m + 1) - 4i$, where $i = 0, 1, 2, \ldots, m(n - 2)/2$, showing that all the values in the second set are attainable.

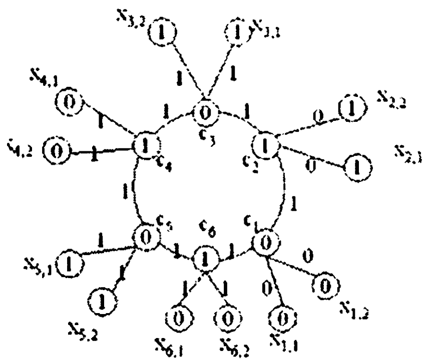
__Example 5.__ Figure 5 shows that $FI(C_6 ® St(2)) = \{0, 2, 4, 6, 8, 10, 12, 14, 18\}$. We note that 16 is missing in the friendly index set.



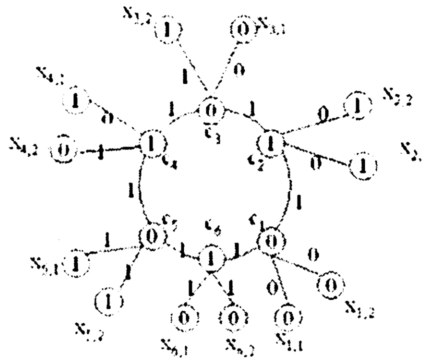
e(1)=18, e(0)=0 |e(1)-e(0)| = 18



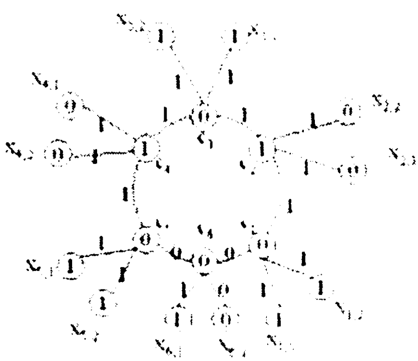
e(1)=16, e(0)=2 |e(1)-e(0)| = 14



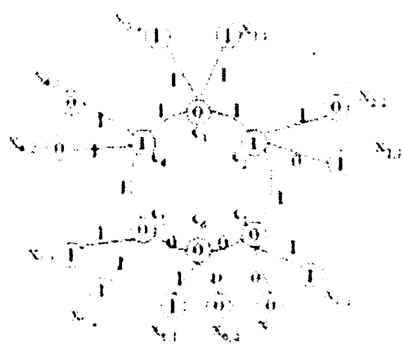
e(1)=14, e(0)=4 |e(1)-e(0)| = 10



e(1)=12, e(0)=6 |e(1)-e(0)| = 6



e(1)=15, e(0)=3 |e(1)-e(0)| = 12



e(1)=13, e(0)=5 |e(1)-e(0)| = 8

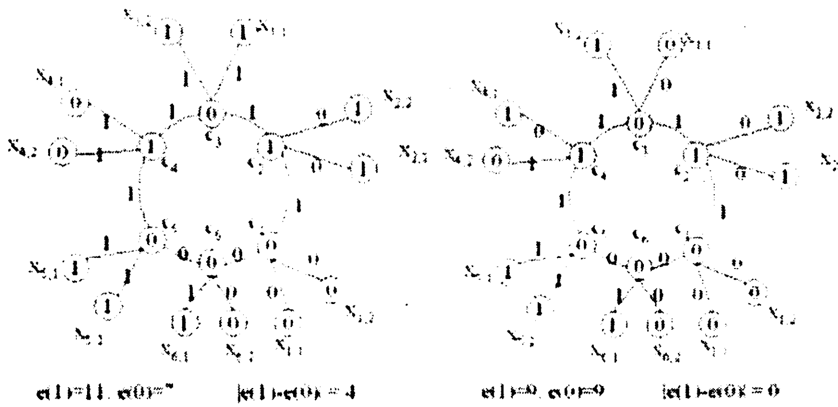


Figure 5.

<u>**Lemma 2.6.**</u> Consider any friendly vertex labeling of $C_n ® St(m)$, where m and n are both odd. Then $e(0) \geq (m + 1)/2$ and $e(1) \geq (m + 1)/2$.

Proof. Again we use the notation in the proof of Lemma 2.1, noting that $(m + 1)n$ is even. Since changing all vertex labels to their complements maintains friendliness and the values of both e(0) and e(1), we may assume that the cycle C_n has more 0-vertices than 1-vertices, i.e., $k \leq (n - 1)/2$. Then $j \leq km \leq (n - 1)m/2$.

From the proof of Lemma 2.1, the number of pendant 1-edges $\geq (m + 1)n/2 - k - j \geq (m + 1)n/2 - (n - 1)/2 - (n - 1)m/2 = (m + 1)/2$, proving half of this Lemma.

The number of pendant 0-edges is $2j + (m - 1)n/2 - (m - 1)k \geq 2j + (m - 1)n/2 - (m - 1)(n - 1)/2 = 2j + (m - 1)/2 \geq (m - 1)/2$. Since n is odd, not all the edges of C_n can be labeled 1, i.e., at least one edge of C_n is labeled 0. Then $e(0) \geq (m - 1)/2 + 1 = (m + 1)/2$.

<u>**Lemma 2.7.**</u> Take any two values in $FI(C_n ® St(m))$, where m and n are both odd. Their difference is a multiple of 4.

Proof. In C_n, there must be an even number of edges labeled 1, and thus an odd number of edges labeled 0. Again use the notation in the proof of Lemma 2.1. The number of pendant edges labeled 0 is $2j + (m - 1)n/2 - (m - 1)k \equiv ((m - 1)/2)(n - 2k) \equiv ((m - 1)/2)n \pmod 2$. Thus whether e(0) in $C_n ® St(m)$ is odd or even is completely determined by m and n, and whatever it is, e(1) must have the same parity. Then $e(1) - e(0) = e(1) + e(0) - 2e(0) = (m + 1)n - 2e(0)$. The sum or difference of any two such values must be $\equiv 0 \pmod 4$.

<u>**Theorem 2.4.**</u> If n and m are both odd, then $FI(C_n ® St(m)) = \{0, 4, 8, \ldots, (m + 1)(n - 1)\}$.

Proof. By Theorem 1.1, $FI(C_n ® St(m)) \subseteq \{0, 2, \ldots, (m + 1)n\}$. By Lemma 2.6, $|e(1) - e(0)| \leq (m + 1)n - (m + 1)/2 - (m + 1)/2 = (m + 1)n - (m + 1)$. Combining this with Lemma 2.7, we see that $FI(C_n ® St(m)) \subseteq \{(m + 1)n - (m$

+ 1), (m + 1)n – (m + 1) – 4, (m + 1)n – 8, … }, or $FI(C_n ® St(m)) \subseteq$ {(m + 1)n – (m + 1) – 2 , (m + 1)n – (m + 1) – 6, (m + 1)n – 10, … }. Thus it suffices to show that all the values in the first set are attainable.

Label the vertices of the cycle alternately by 0's and 1's, starting and ending with 0. Since n is odd, the last vertex of the cycle, say c_n, is labeled 0. For all vertices of the cycle but c_n, if its label is x, label all its adjacent pendant vertices (1 – x). For the last vertex c_n of the cycle, label (m + 1)/2 of its pendant vertices 1, and the other (m – 1)/2 of its pendant vertices 0. This is a vertex-friendly labeling, with e(1) = (n – 1) + (n – 1)m + (m + 1)/2 = (n – 1)(m + 1) + (m + 1)/2, and e(0) = 1 + (m – 1)/2. Thus e(1) – e(0) = (n – 1)(m + 1) = (m + 1)n – (m + 1). At this point, we know that the second set in the previous paragraph is not possible.

In this labeling, there are m(n – 1)/2 pendant vertices labeled 0 and m(n – 1)/2 pendant vertices labeled 1 that are adjacent to vertices of the cycle other than c_n. Pair them into m(n – 1)/2 pairs, and interchange the labels of each pair successively. After each interchange, there are two additional 0-edges, decreasing the value of e(1) – e(0) by 4. Thus e(1) – e(0) = (m + 1)n – (m + 1) – 4i, where i = 0, 1, 2, … , m(n – 1)/2. If m = 1, the least possible value of e(1) – e(0) is (1 + 1)n – (1 + 1) – 4(1(n – 1)/2) = 0. If m ≥ 3, the least possible value of e(1) – e(0) is (m + 1)n – (m + 1) – 2m(n – 1) = m + n – mn – 1 < 0. Thus $FI(C_n$ ® St(m)) = {(m + 1)n – (m + 1), (m + 1)n – (m + 1) – 4, (m + 1)n – 8, … , 0}.

<u>**Example 6.**</u> Figure 6 shows that $FI(C_3$ ® St(1)) = {0, 4}.

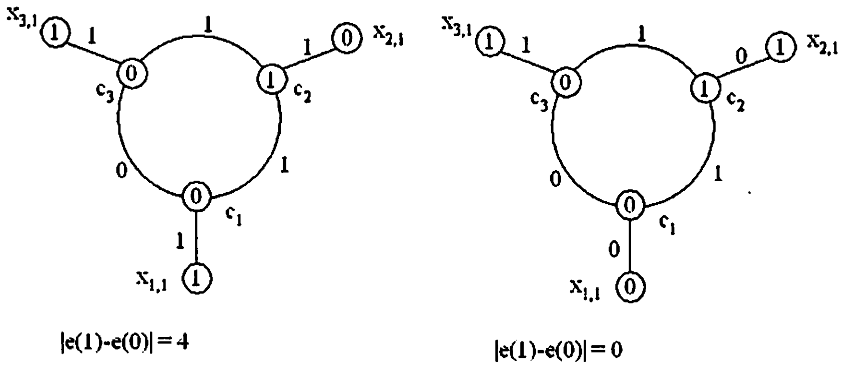


Figure 6.

<u>**Example 7.**</u> Figure 7 shows that $FI(C_5$ ® St(3)) = {0, 4, 8, 12, 16}.

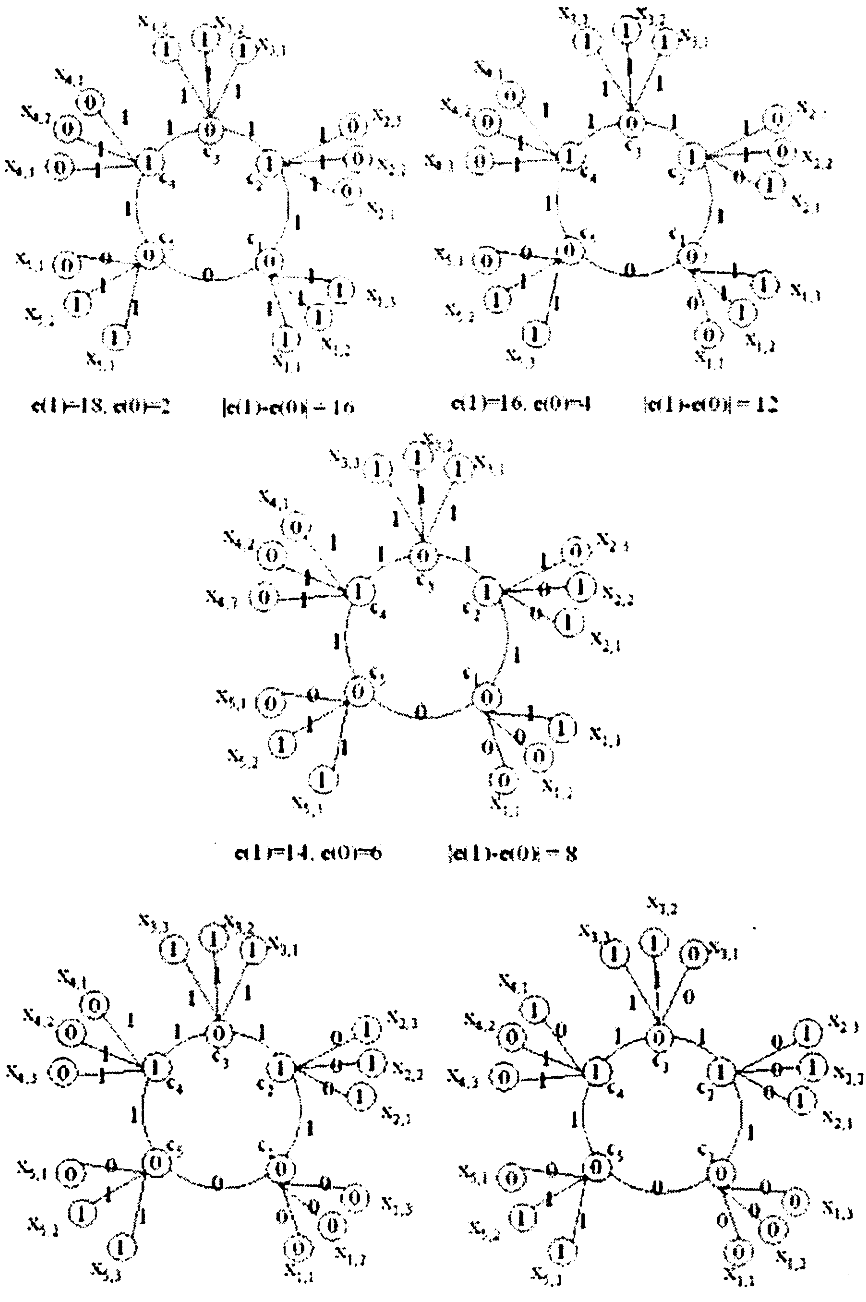

c(1)=18, c(0)=2 |c(1)-c(0)| = 16
c(1)=16, c(0)=4 |c(1)-c(0)| = 12
c(1)=14, c(0)=6 |c(1)-c(0)| = 8
c(1)=12, c(0)=8 |c(1)-c(0)| = 4
c(1)=10, c(0)=10 |c(1)-c(0)| = 0

Figure 7.

3. Root-union of stars by a cycle, with a leaf as root

In this section, we consider the friendly index set of the root-union of stars by a cycle, where a leaf of the star is the root. To distinguish these graphs from those in Section 2, we use the notation C_n ® (St(m), x_1). We will call an edge joining a vertex on the cycle to the center of a star a bridge, since it bridges the cycle C_n and a star St(m).

The arguments to be used are similar to, though a little more complicated than, those in Section 2. We number our results corresponding to those in Section 2. We will see that, except in the case when m and n are both even, the general results are essentially the same as those in the previous section.

Note that since there is a leaf, we must have $m \geq 1$. If $m = 1$, we can use the results in Section 2. So we will assume that $m \geq 2$.

<u>Lemma 3.1.</u> For any friendly vertex labeling of C_n ® (St(m), x_1), where n is even and m is odd, the induced edge labeling has an even e(1) and an even e(0).
Proof. Since the graph has $(m + 1)n$ vertices, $v(0) = v(1) = (m + 1)n/2$. Assume that h of the vertices on C_n are lableled 1, and thus the other $(n - h)$ vertices are labeled 0. Consider the 1-vertices on the cycle C_n. Assume that i of them are adjacent to a 1-vertex (the center) of a star, and $(h - i)$ of them are adjacent to a 0-vertex (the center) of a star. Consider the 0-vertices on the cycle C_n. Assume that j of them are adjacent to a 1-vertex (the center) of a star, and $(n - h - j)$ of them are adjacent to a 0-vertex (the center) of a star. Now consider the vertices with degree 1. Note that $(i + j)(m - 1)$ of them are adjacent to a 1-vertex. Assume that k of these $(i + j)(m - 1)$ pendant vertices are labeled 1. Then the other $((i + j)(m - 1) - k)$ vertices are labeled 0. Now note that the other $(n - i - j)(m - 1)$ pendant vertices are adjacent to a 0-vertex. Since $v(1) = (m + 1)n/2$, there are $((m + 1)n/2 - h - i - j - k)$ of them labeled 1, with the remaining $(n - i - j)(m - 1) - ((m + 1)n/2 - h - i - j - k) = (m - 1)n/2 - n - (i + j)(m - 2) + h + k$ labeled 0. Thus, besides the edges of the cycle C_n, there are $(m + 1)n/2 + (i + j)(m - 1) - 2i - 2k$ labeled 1, and $(m - 1)n/2 - (i + j)(m - 1) + 2i + 2k$ labeled 0. Since n is even and m is odd, both of these numbers are even.

From [13], the cycle C_n must have an even number of edges labeled 1. Since n is even, C_n must also have an even number of edges labeled 0.

<u>Note.</u> The first paragraph of the above proof only requires that $(m + 1)n$ be even. Its arguments and symbols will be repeatedly used in this section.

<u>Lemma 3.2.</u> If n is even and m is odd, then FI(C_n ® (St(m), x_1)) can only contain multiples of 4.
Proof. This proof is similar to that of Lemma 2.2.

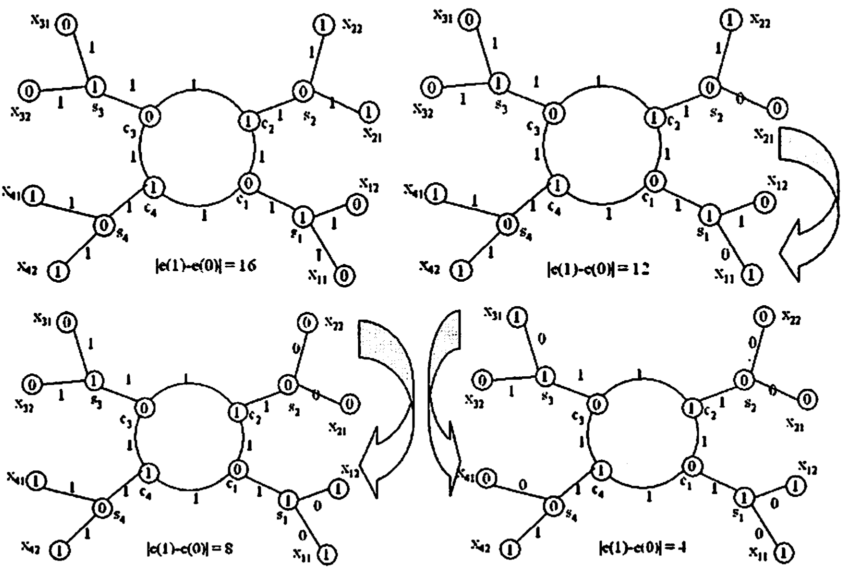
<u>Theorem 3.1.</u> If n is even and $m \geq 3$ is odd, then FI(C_n ® (St(m), x_1)) = $\{0, 4, 8, \ldots, (m + 1)n\}$.

Proof. By Theorem 1.1, $FI(C_n \circledR (St(m), x_1)) \subseteq \{0, 2, 4, 6, 8, \ldots, (m+1)n\}$. By Lemma 3.2, only multiples of 4 can be in the friendly index set. Thus $FI(C_n \circledR (St(m), x_1)) \subseteq \{0, 4, 8, \ldots, (m+1)n\}$. It suffices to show that all these values are attainable.

Label the vertices of C_n alternately by 0's and 1's. Label the centers of the stars so that all the bridges have induced label 1. All pendant vertices adjacent to a 1-vertex are labeled 0, and all pendant vertices adjacent to a 0-vertex are labeled 1. This vertex labeling is obviously friendly with all edges labeled 1. Thus $e(1) - e(0) = (m+1)n$.

In the above labeling, there are $(m-1)n/2$ pendant vertices labeled 0 and $(m-1)n/2$ pendant vertices labeled 1. Pair them into $(m-1)n/2$ pairs, and interchange the labels of each pair successively. After each interchange, there are two additional 0-edges, decreasing the value of $e(1) - e(0)$ by 4. Thus $e(1) - e(0) = (m+1)n - 4i$, where $i = 0, 1, 2, \ldots, (m-1)n/2$, showing that all the values in the set are attainable.

<u>**Example 8.**</u> Figure 8 shows that $FI(C_4 \circledR (St(3), x_1)) = \{0, 4, 8, 12, 16\}$.



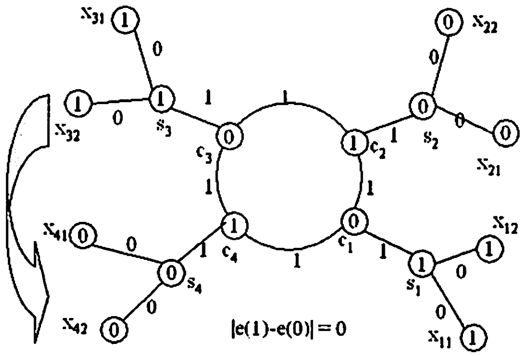


Figure 8.

Lemma 3.3. For any friendly vertex labeling of C_n ® (St(m), x_1), where n is odd and m is even, $e(0) \geq m/2$ and $e(1) \geq m/2$.

Proof. Note that the number of vertices = the number of edges = $(m + 1)n$ is an odd number. We used essentially the same arguments and symbols as in the proof of Lemma 3.1. Note that changing all vertex labels to their complements maintains friendliness and the values of both e(0) and e(1), we may assume that there are more star centers labeled 0 than 1, i.e., $i + j \leq (n - 1)/2$, and $k \leq (i + j)(m - 1) \leq (m - 1)(n - 1)/2$. By friendliness, $v(1) = (m + 1)n/2 \pm ½$. Thus, besides the edges of the cycle C_n, the number of 1-edge labels $= (m + 1)n/2 \pm ½ + (i + j)(m - 1) - 2i - 2k \geq (m + 1)n/2 \pm ½ + (i + j)(m - 1) - 2i - 2(i + j)(m - 1) = (m + 1)n/2 \pm ½ - 2i - (i + j)(m - 1) \geq (m + 1)n/2 \pm ½ - 2i - (m - 1)(n - 1)/2 \geq (m + 1)n/2 \pm ½ - (n - 1) - (m - 1)(n - 1)/2 \geq m/2 \pm ½ + ½ \geq m/2$, proving half of the Lemma. Also, besides the edges of the cycle C_n, the number of 0-edge labels $\geq (m - 1)n/2 - ½ - (i + j)(m - 1) + 2i + 2k \geq (m - 1)n/2 - ½ - (i + j)(m - 1) \geq (m - 1)n/2 - ½ - (m - 1)(n - 1)/2 = (m - 1)/2 - ½ = m/2 - 1$. Since n is odd, not all the edges of C_n can have induced label 1, i.e., at least one edge of C_n is labeled 0. Thus $e(0) \geq m/2$, finishing the proof.

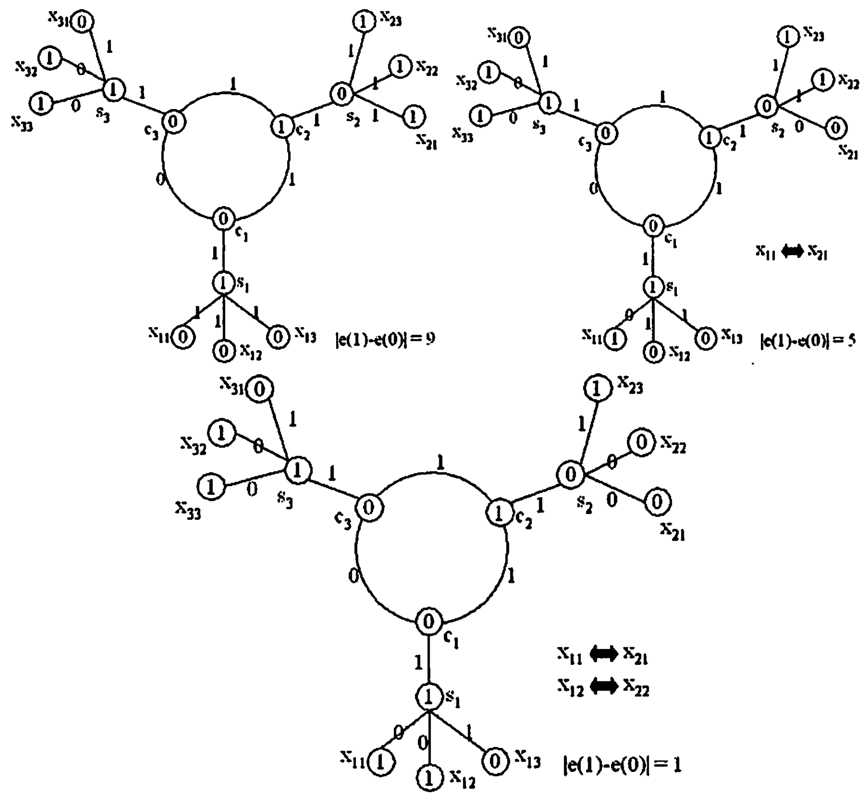
Theorem 3.2. If n is odd and $m \geq 4$ is even, then FI(C_n ® (St(m), x_1)) = {1, 3, 5, … , (m + 1)n − m}.

Proof. By Theorem 1.1, FI(C_n ® (St(m), x_1)) ⊆ {1, 3, … , (m + 1)n}. By Lemma 3.3, $|e(1) - e(0)| \leq (m + 1)n - m/2 - m/2 = (m + 1)n - m$. It suffices to show that all these values are attainable.

Label the vertices of C_n alternately by 0's and 1's, starting and ending with 0. Call the last vertex c_n. Label the centers of the stars so that all the bridges have induced label 1. Call the center of the star adjacent to the vertex c_n of the cycle s_n. For all star centers but s_n, if its label is x, label all its adjacent pendant vertices $(1 - x)$. For the last star center s_n, label $(m/2 - 1)$ of its pendant vertices 0, and the other $m/2$ of its pendant vertices 1. This is a vertex-friendly labeling. Note that $e(1) = (n - 1) + n + (m - 1)(n - 1) + (m/2 - 1) = m(n - 1) + n + (m/2 - 1)$, and $e(0) = 1 + m/2$, giving $e(1) - e(0) = (m + 1)n - m - 2$. In this labeling, there are $(m-1)(n-1)/2$ pendant 0-vertices adjacent to a star center labeled 1 other than s_n, and $(m-1)(n-1)/2$ pendant 1-vertices adjacent to a star center labeled 0. Pair them into $(m-1)(n-1)/2$ pairs, and interchange the labels of each pair successively. After each interchange, there are two additional 0-edges, decreasing the value of $e(1)-e(0)$ by 4. Thus $e(1)-e(0) = (m+1)n - m - 2 - 4i$, where $i = 0, 1, 2, \ldots, (m-1)(n-1)/2$.

Now do the same procedure as in the previous paragraph, except that for the last star center s_n, label $m/2$ of its pendant vertices 0, and the other $(m/2-1)$ of its pendant vertices 1. This is still a vertex-friendly labeling. Note that $e(1) = (n-1) + n + (m-1)(n-1) + m/2 = m(n-1) + n + m/2$, and $e(0) = 1 + m/2 - 1 = m/2$, giving $e(1)-e(0) = (m+1)n - m$. Again, the same interchanges will give $e(1)-e(0) = (m+1)n - m - 4i$, where $i = 0, 1, 2, \ldots, (m-1)(n-1)/2$. The smallest value of $e(1)-e(0)$ is $(m+1)n - m - 2(m-1)(n-1) = m + 3n - mn - 2$. If $m = 4$, this value is $2 - n < 0$. If $m \geq 6$, then $m \leq mn/3$ and $3n \leq mn/2$, showing that $m + 3n - mn - 2 < 0$. Thus the value 1 is attainable.

Example 9. Figure 9 shows that $FI(C_3 \circledR (St(4), x_1)) = \{1, 3, 5, 7, 9, 11\}$.



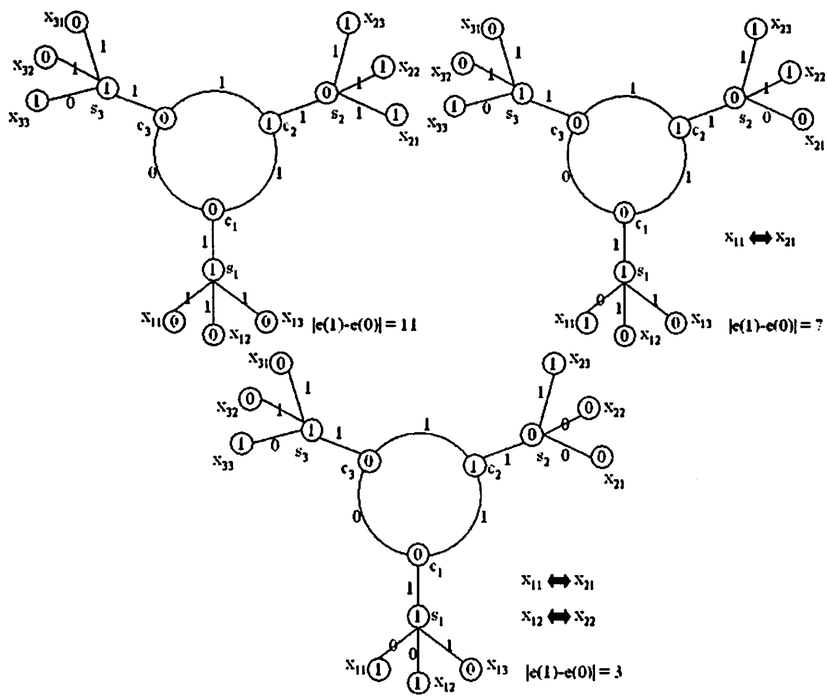


Figure 9.

<u>**Lemma 3.4.**</u> Consider any friendly vertex labeling of C_n ® (St(m), x_1)), where n is even. If half of the star centers are labeled 0, then e(0) and e(1) are both even.
Proof. We use the notation in the proof of Lemma 3.1. Note that $i + j = n/2$ under our assumption here. The cycle C_n has an even number of 1-edges and hence an even number of 0-edges. Besides the edges of C_n, the number of 1-edges is $(m + 1)n/2 + (i + j)(m - 1) - 2i - 2k = (m + 1)n/2 + (m - 1)n/2 - 2i - 2k = mn - 2i - 2k$, and the number of 0-edges is $(m - 1)n/2 - (i + j)(m - 1) + 2i + 2k = (m - 1)n/2 - (m - 1)n/2 + 2i + 2k = 2i + 2k$, both even numbers.

<u>**Lemma 3.5.**</u> Consider any friendly vertex labeling of C_n ® (St(m), x_1)), where n is even. If e(1) is odd, then $e(1) \geq m + 1$ and $e(0) \geq m - 1$.
Proof. Again we use the notation in the proof of Lemma 3.1. By Lemma 3.4, we know that $i + j \neq n/2$. Since changing all vertex labels to their complements maintains friendliness and the values of both e(0) and e(1), we may assume that there are more star centers labeled 0 than 1, i.e., $i + j \leq n/2 - 1$. Then $k \leq (i + j)(m - 1) \leq (m - 1)(n/2 - 1)$. Besides the edges of the cycle C_n, the number of 1-edge labels $= (m + 1)n/2 + (i + j)(m - 1) - 2i - 2k \geq (m + 1)n/2 + (i + j)(m - 1) - 2i - 2(i + j)(m - 1) = (m + 1)n/2 - 2i - (i + j)(m - 1) \geq (m + 1)n/2 - 2i - (m - 1)(n/2 - 1) \geq (m + 1)n/2 - (n - 2) - (m - 1)(n/2 - 1) = m + 1$, proving half of the Lemma. Also, besides the edges of the cycle C_n, the number of 0-edge labels $\geq (m - 1)n/2 - (i + j)(m - 1) + 2i + 2k \geq (m - 1)n/2 - (i + j)(m - 1) \geq (m - 1)n/2 - (m - 1)(n/2 - 1) = m - 1$, finishing the proof.

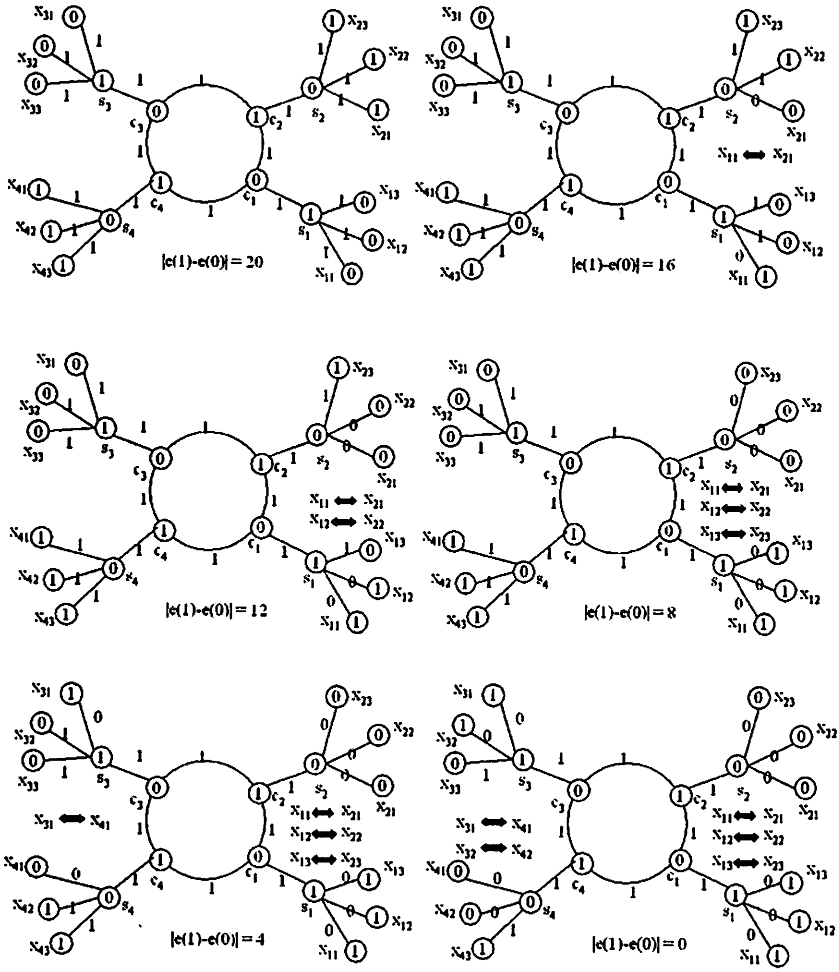
Theorem 3.3. If n and m are both even, $m \geq 4$, then $FI(C_n \circledR (St(m), x_1)) = \{(m + 1)n, (m + 1)n - 4, (m + 1)n - 8, \dots\} \cup \{(m + 1)n - 2(m - 1), (m + 1)n - 2(m - 1) - 4, (m + 1)n - 2(m - 1) - 8, \dots\}$, i.e., $\{0, 2, 4, \dots, (m + 1)n - 2(m - 2)\} \cup \{(m + 1)n - 2(m - 2) + 4, (m + 1)n - 2(m - 2) + 8, \dots, (m + 1)n\}$.

Proof. The first set in the statement of the Theorem has both e(0) and e(1) even, while the second set has both e(0) and e(1) odd. By Theorem 1.1, $FI(C_n \circledR (St(m), x_1)) \subseteq \{0, 2, \dots, (m + 1)n\}$. By Lemma 3.5, if e(1) is odd, then $|e(1) - e(0)| \leq (m + 1)n - (m - 1) - (m - 1) = (m + 1)n - 2(m - 1)$. It suffices to show that all these values are attainable.

Label the vertices of C_n alternately by 0's and 1's. Label the centers of the stars so that all the bridges have induced label 1. All pendant vertices adjacent to a 1-vertex are labeled 0, and all pendant vertices adjacent to a 0-vertex are labeled 1. This vertex labeling is obviously friendly with all edges labeled 1. Thus $e(1) - e(0) = (m + 1)n$. In this labeling, there are $(m - 1)n/2$ pendant vertices labeled 0 and $(m - 1)n/2$ pendant vertices labeled 1. Pair them into $(m - 1)n/2$ pairs, and interchange the labels of each pair successively. After each interchange, there are two additional 0-edges, decreasing the value of $e(1) - e(0)$ by 4. Thus $e(1) - e(0) = (m + 1)n - 4i$, where $i = 0, 1, 2, \dots, (m - 1)n/2$. When $i = (m - 1)n/2$, $e(1) - e(0) = (m + 1)n - 2(m - 1)n = (3 - m)n$. This shows that all the values in the first set are attainable.

Start with the same labeling as that in the previous paragraph. Label the vertices of C_n alternately by 0's and 1's. Call the last vertex c_n, which is labeled 1. Call the center of the star adjacent to the vertex c_n of the cycle s_n. Label the centers of the stars so that all the bridges have induced label 1, except that s_n is labeled 1, so that its bridge has induced label 0. For all star centers except s_n, if its label is x, label all its adjacent pendant vertices $(1 - x)$. For the pendant vertices adjacent to s_n, label one of them 0 and the other $(m - 2)$ of them 1. This is a vertex-friendly labeling. Note that $e(1) = n + (n - 1) + (m - 1)(n - 1) + 1 = (m + 1)n - m + 1$, and $e(0) = 1 + (m - 2) = m - 1$, giving $e(1) - e(0) = (m + 1)n - 2(m - 1)$. Besides the pendant vertices adjacent to s_n and a star center next to s_n, there are $(m - 1)(n - 2)/2$ pendant 0-vertices adjacent to a star center labeled 1, and $(m - 1)(n - 2)/2$ pendant 1-vertices adjacent to a star center labeled 0. Pair them into $(m - 1)(n - 2)/2$ pairs, and interchange the labels of each pair successively. After each interchange, there are two additional 0-edges, decreasing the value of $e(1) - e(0)$ by 4. Thus $e(1) - e(0) = (m + 1)n - 2(m - 1) - 4i$, where $i = 0, 1, 2, \dots, (m - 1)(n - 2)/2$. The smallest value of $e(1) - e(0)$ is $(m + 1)n - 2(m - 1) - 2(m - 1)(n - 2) = 2m + 3n - mn - 2$. If $m = 4$ and $n = 4$, this value is 2. If $m = 4$ and $n \geq 6$, this value is $-n + 6 \leq 0$. If $m \geq 6$ and $n \geq 4$, then $2m \leq mn/2$ and $3n \leq mn/2$, showing that $2m + 3n - mn - 2 < 0$. This shows that all the values in the second set are attainable.

Example 10. Figure 10 shows that $FI(C_4 \circledR (St(4), x_1)) = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 20\}$. We note here that 18 is missing in the friendly index set.


|e(1)-e(0)| = 20
x11 ⟺ x21
|e(1)-e(0)| = 16
x11 ⟺ x21
x12 ⟺ x22
|e(1)-e(0)| = 12
x11 ⟺ x21
x12 ⟺ x22
x13 ⟺ x23
|e(1)-e(0)| = 8
x31 ⟺ x41
x11 ⟺ x21
x12 ⟺ x22
x13 ⟺ x23
|e(1)-e(0)| = 4
x31 ⟺ x41
x32 ⟺ x42
x11 ⟺ x21
x12 ⟺ x22
x13 ⟺ x23
|e(1)-e(0)| = 0

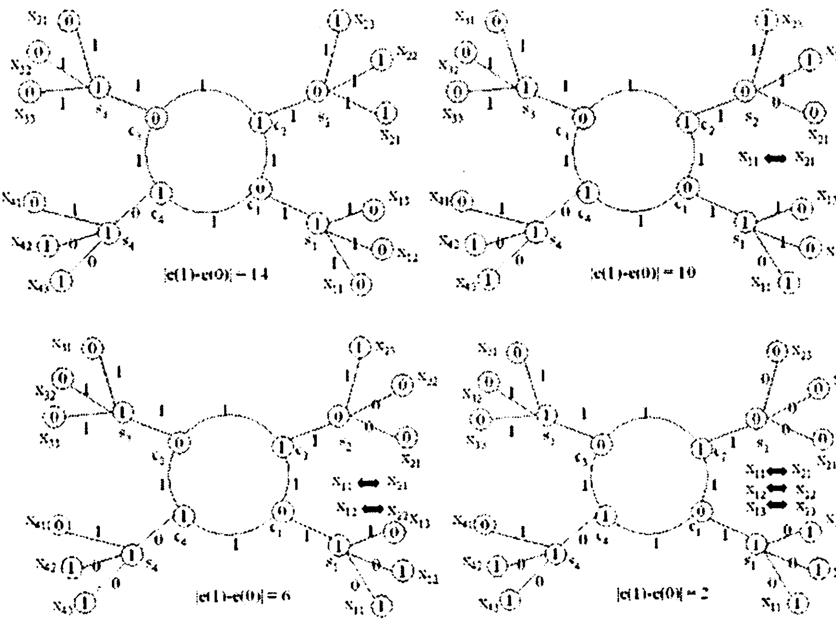


Figure 10.

<u>**Lemma 3.6.**</u> Consider any friendly vertex labeling of C_n ® (St(m), x_1), where m and n are both odd. Then $e(0) \geq (m+1)/2$ and $e(1) \geq (m+1)/2$.

Proof. Again we use the notation in the proof of Lemma 3.1, noting that $(m+1)n$ is even. Since changing all vertex labels to their complements maintains friendliness and the values of both e(0) and e(1), we may assume that there are more star centers labeled 0 than 1, i.e., $i + j \leq (n-1)/2$, and $k \leq (i+j)(m-1) \leq (m-1)(n-1)/2$. Then, besides the edges of the cycle C_n, the number of 1-edge labels $= (m+1)n/2 + (i+j)(m-1) - 2i - 2k \geq (m+1)n/2 + (i+j)(m-1) - 2i - 2(i+j)(m-1) = (m+1)n/2 - 2i - (i+j)(m-1) \geq (m+1)n/2 - 2i - (m-1)(n-1)/2 \geq (m+1)n/2 - (n-1) - (m-1)(n-1)/2 \geq (m+1)/2$, proving half of the Lemma. Also, besides the edges of the cycle C_n, the number of 0-edge labels $\geq (m-1)n/2 - (i+j)(m-1) + 2i + 2k \geq (m-1)n/2 - (i+j)(m-1) \geq (m-1)n/2 - (m-1)(n-1)/2 = (m-1)/2$. Since n is odd, not all the edges of C_n can have induced label 1, i.e., at least one edge of C_n is labeled 0. Thus $e(0) \geq (m+1)/2$, finishing the proof.

<u>**Lemma 3.7.**</u> Take any two values in FI(C_n ® (St(m), x_1)), where m and n are both odd. Their difference is a multiple of 4.

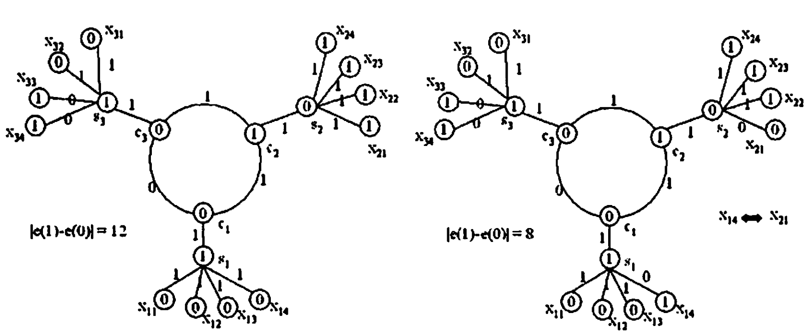
Proof. In C_n, there must be an even number of edges labeled 1, and thus an odd number of edges labeled 0. Again use the notation in the proof of Lemma 3.1. Besides the edges of C_n, the number of 0-edges $= (m-1)n/2 - (i+j)(m-1) + 2i + 2k \equiv (m-1)n/2 \pmod 2$. Thus whether e(0) in C_n ® (St(m), x_1) is odd or even is completely determined by m and n, and whatever it is, e(1) must have the same parity. Then e(1) – e(0) = e(1) + e(0) – 2e(0) = (m + 1)n – 2e(0). The sum or difference of any two such values must be ≡ 0 (mod 4).

Theorem 3.4. If n and m are both odd, then $FI(C_n ® (St(m), x_1)) = \{0, 4, 8, \dots, (m+1)(n-1)\}$.

Proof. By Theorem 1.1, $FI(C_n ® (St(m), x_1)) \subseteq \{0, 2, \dots, (m + 1)n\}$. By Lemma 3.6, $|e(1) - e(0)| \le (m + 1)n - (m + 1)/2 - (m + 1)/2 = (m + 1)n - (m + 1)$. Combining this with Lemma 3.7, we see that $FI(C_n ® (St(m), x_1)) \subseteq \{(m + 1)n - (m + 1), (m+1)n - (m+1) - 4, (m+1)n - 8, \dots\}$, or $FI(C_n ® (St(m), x_1)) \subseteq \{(m+1)n - (m+1) - 2, (m+1)n - (m+1) - 6, (m+1)n - 10, \dots\}$. Thus it suffices to show that all the values in the first set are attainable.

Label the vertices of C_n alternately by 0's and 1's, starting and ending with 0. Call the last vertex c_n. Label the centers of the stars so that all the bridges have induced label 1. Call the center of the star adjacent to the vertex c_n of the cycle s_n. For all star centers but s_n, if its label is x, label all its adjacent pendant vertices (1 – x). For the last star center s_n, label (m – 1)/2 of its pendant vertices 0, and the other (m – 1)/2 of its pendant vertices 1. This is a vertex-friendly labeling. Note that e(1) = (n – 1) + n + (m – 1)(n – 1) + (m – 1)/2 = (m + 1)n – m/2 – ½, and e(0) = 1 + (m – 1)/2 = m/2 + ½, giving e(1) – e(0) = (m + 1)n – (m + 1). In this labeling, there are (m – 1)(n – 1)/2 pendant 0-vertices adjacent to a star center labeled 1 other than s_n, and (m – 1)(n – 1)/2 pendant 1-vertices adjacent to a star center labeled 0. Pair them into (m – 1)(n – 1)/2 pairs, and interchange the labels of each pair successively. After each interchange, there are two additional 0-edges, decreasing the value of e(1) – e(0) by 4. Thus e(1) – e(0) = (m + 1)n – (m + 1) – 4i, where i = 0, 1, 2, ..., (m – 1)(n – 1)/2. The smallest value of e(1) – e(0) is (m + 1)n – (m + 1) – 2(m – 1)(n – 1) = m + 3n – mn – 3. If m = 3, this value is 0. If m ≥ 5 and n ≥ 3, then m ≤ mn/3 and 3n ≤ 3mn/5, showing that m + 3n – mn – 3 < 0. This shows that all the values in the first set are attainable.

Example 11. Figure 11 shows that $FI(C_3 ® (St(5), x_1)) = \{0, 4, 8, 12\}$.



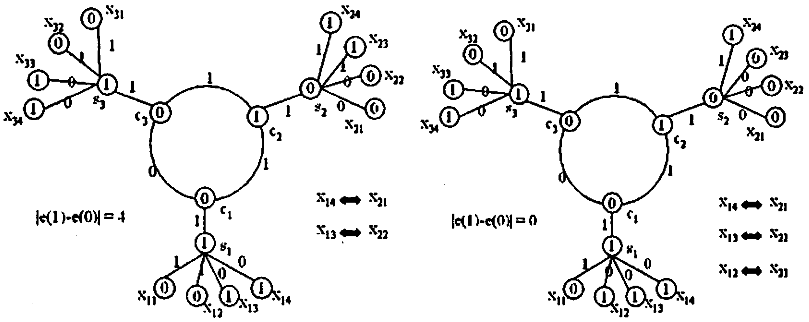


Figure 11.

4. Conclusion

In this paper, we introduced the root-union construction, and investigated the friendly index set of C_n ® St(m). Roughly speaking, about three quarters of these sets have values in arithmetic progressions while the others have gaps. The project on finding when gaps exist is still ongoing.

References

1. M. Benson and S-M. Lee, On cordialness of regular windmill graphs, *Congr. Numer.*, **68** (1989) 49-58.
2. I. Cahit, Cordial graphs: a weaker version of graceful and harmonious graphs, *Ars Combin.*, **23** (1987) 201-207.
3. I. Cahit, On cordial and 3-equitable graphs, *Utilitas Mathematica*, **37** (1990) 189-197.
4. I. Cahit, Recent results and open problems on cordial graphs, *Contemporary Methods in Graph Theory*, 209-230, Bibliographisches Inst., Mannheim, 1990.
5. N. Cairnie and K. Edwards, The computational complexity of cordial and equitable labelling, *Discrete Math.*, **216** (2000) 29-34.
6. G. Chartrand, S-M. Lee and P. Zhang, Uniformly cordial graphs, *Discrete Math.*, **306** (2006) 726-737.
7. A. Elumalai , On graceful, cordial and elegant labelings of cycle related and other graphs, Ph.D. dissertation of Anna University, 2004, Chennai, India.
8. Y.S. Ho, S-M. Lee and S.C. Shee, Cordial labellings of the Cartesian product and composition of graphs, *Ars Combin.*, **29** (1990) 169-180.
9. Y.S. Ho, S-M. Lee and S.C. Shee, Cordial labellings of unicyclic graphs and generalized Petersen graphs. *Congr. Numer.*, **68** (1989) 109-122.
10. M. Hovey, A-cordial graphs, *Discrete Math.*, **93** (1991) 183-194.
11. S. Kuo, G.J. Chang and Y.H.H. Kwong, Cordial labeling of mK_n, *Discrete Math.*, **169** (1997) 121-131.
12. S-M. Lee and A. Liu, A construction of cordial graphs from smaller cordial graphs, *Ars Combin.*, **32** (1991) 209-214.
13. S-M. Lee and H.K. Ng, On friendly index sets of bipartite graphs, to appear in *Ars Combin.*

14. S-M. Lee and H.K. Ng, On friendly index sets of spiders, manuscript.
15. S-M. Lee and H.K. Ng, On friendly index sets of prisms and Möbius ladders, manuscript.
16. S-M. Lee and H.K. Ng, On friendly index sets of cycles with parallel chords, to appear in *Ars Combin.*
17. H.Y. Lee, H.M. Lee and G.J. Chang, Cordial labelings of graphs, *Chinese J. Math.*, **20** (1992) 3, 263-273.
18. E. Seah, On the construction of cordial graphs, *Ars Combin.*, **31** (1991) 249-254.
19. M.A. Seoud and A.E.I. Abdel Maqsoud, On cordial and balanced labelings of graphs, *J. Egyptian Math. Soc.*, 7 (1999) 127-135.
20. S.C. Shee and Y.S. Ho, The cordiality of the path-union of n copies of a graph, *Discrete Math.*, **151** (1996) 1-3, 221-229.
21. S.C. Shee and Y.S. Ho, The cordiality of one-point union of n copies of a graph, *Discrete Math.*, **117** (1993) 225-243.