Extremal results on arc-traceable tournaments

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Abstract

A tournament $T = (V, A)$ is *arc-traceable* if for each arc $xy \in A$, xy lies on a directed path containing all the vertices of V, i.e., a hamiltonian path. In this paper we give two extremal results related to arc-traceability in tournaments. First, we show that a non-arc-traceable tournament T which is m-arc-strong must have at least $2^{m+1} + 4m - 3$ vertices, and we construct an example that shows that this result is best possible. Next, we consider the maximum number of arcs in a strong tournament that are not part of any hamiltonian path. We use the structure of non-arc-traceable tournaments to prove that no strong tournament contains more than $\frac{n^2-4n+3}{8}$ arcs that are not part of a hamiltonian path, and we give the unique example that shows that this bound is best possible.

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1 Introduction

A directed graph (or *digraph*) $D = (V, A)$ consists a set V of vertices and a collection A of ordered pairs of vertices called *arcs*. In this paper, we will always assume that V is finite and that A includes at most one copy of each arc and that all arcs are ordered pairs of distinct vertices. When working with more than one digraph or when the context is not clear, the vertex set and arc set of a digraph will be referred to as $V(D)$ and $A(D)$ and to minimize notation an element of $A(D)$ simply as uv. A good reference for general results on digraphs is the book by Bang-Jensen and Gutin [1]. The in-neighborhood of a vertex v is the set $\{u \mid uv \in A\}$ and denoted $N^-(v)$. The cardinality of the set $N^-(v)$ is the in-degree of v and is denoted $d^-(v)$. Out-neighborhood and out-degree are defined analogously and we define δ_D^- and δ_D^+ as the minimum values of $d^-(v)$ and $d^+(v)$ over all vertices $v \in V(D)$. A strong digraph is one in which for any two distinct vertices x and y, there exists a directed path from x to y. A set $S \subset V$ is a *cut-set* if the sub-digraph induced on $V \setminus S$ is not strong. For a digraph D, D is *k-connected* if the size of the smallest cut-set is at least k, and the size of the smallest cut-set of D is denoted $\kappa(D)$. Analogously, a set $S \subset A$ is an *arc-cut-set* if $D' = (V, A \setminus S)$ is not strongly connected and a digraph is *m-arc-strong* if the smallest arc-cut-set S has $|S| \geq m$. We write $\kappa'(D)$ to represent the size of the smallest arc cut-set for the digraph D. When two vertices x and y are given, a set S of arcs or vertices such that there is no path from x to y in D avoiding S will be called an x, y-separating set.

A *tournament* is a digraph such that for each two distinct vertices u and v, exactly one of the arcs uv and vu is present. Alternately, a tournament on n vertices can be thought of as an arbitrary orientation of the complete undirected graph K_n. Using the latter perspective, it is clear that every induced subdigraph of a tournament is also a tournament. We will call such a subdigraph a subtournament and for $S \subset V$, we will denote the subtournament induced on S by $T[S]$. If $V \setminus S = \{v\}$, we will instead write T_v. The vertices of a tournament T that is not strong can be decomposed into strong components, and these strong subtournaments form the vertex set of a transitive tournment. In such a case, we will order the strong components from 1 to r such that every vertex in the i^{th} component dominates every vertex in the j^{th} component for all $j > i$. We will designate the i^{th} component by $T^{(i)}$ and refer to $T^{(1)}$ as the initial strong component of T and $T^{(r)}$ as the terminal strong component.

A digraph $D = (V, A)$ is *arc-traceable* if for each arc $xy \in A$, xy lies on a directed path containing all the vertices of V, i.e., a hamiltonian path. If D is non-arc-traceable there is some arc that is not part of any hamiltonian path, and any such arc will be called *non-traceable.* For tournaments, a relationship between connectivity and arc-traceability is clear; it is easy to show that every 2-connected tournament is arc-traceable. In [4], the structure of strong non-arc-traceable tournaments was investigated. The main result of that work was Theorem 1.1.

Theorem 1.1 (Busch, Jacobson and Reid [4]). *If T is a strong tournament, and $xy \in A(T)$ is not on a hamiltonian path, then:*

(i) There exists a vertex z such that T_z is not strong.

(ii) T_z has k strong components, $k \geq 4$.

(iii) x is in the initial strong component of T_z, and y is in the terminal strong component of T_z.

(iv) z is dominated by the 2^{nd} strong component of T_z and z dominates the $(k-1)^{st}$ strong component of T_z.

In Section 2, we consider the relationship between arc-connectivity and arc-traceability. For any positive integer m, we construct an m-arc-strong tournament that is not arc-traceable, and we show that this tournament is minimal with respect to the number of vertices. In Section 3, we consider tournaments which contain many non-traceable arcs. We use the structure of non-arc-traceable tournaments to show that at most $\frac{n^2-4n+3}{8}$ of the arcs of a strong tournament are not on a hamiltonian path, and note that as a result of an example given in [3], this result is best possible.

2 Non-arc-traceable m-arc-strong tournaments

As a consequence of Theorem 1.1, for any non-traceable arc xy in a strong tournament T, there exists a y,x-separating set of vertices of size 1 (the vertex z in the Theorem). We begin this section by showing that a similar result for a y,x-separating set of arcs is impossible. In particular, we show that for any $m > 0$ there exists a strong tournament T that has a non-traceable arc xy with m arc disjoint paths from y to x. As an example, in the tournament in Figure 1, xy is not on any hamiltonian path, and there exist 2 arc disjoint paths from y to x.

For an arbitrary m, we construct a strong tournament by reversing the arcs of a set of m arc disjoint paths in a transitive tournament of order $2^{m+1}+1$. Let T be a transitive tournament or order $2^{m+1}+1$ with vertices

Figure 1: A tournament with non-traceable arc xy and two arc disjoint paths from y to x. All arcs not-shown are from left to right.

labeled $v_0, \dots, v_n$ (so $n = 2^{m+1}$) such that $d^-(v_i) = i$. Now, consider the paths

$$P_i = v_0 v_{(2^i)} v_{(2\cdot 2^i)} v_{(3\cdot 2^i)} \dots, v_n, \text{ for } 1 \le i \le m.$$

We reverse the arcs in each of these paths to obtain the tournament $T_{[m]}$, and will refer to the reversed path P_i as U_i. Note that $v_{\frac{n}{2}}$ is on each U_i, and thus for any $i < \frac{n}{2} < j$, every path from v_j to v_i must contain $v_{\frac{n}{2}}$. We also note that $v_0 v_1 v_3 \dots v_{n-5} v_{n-3} v_{n-1} U_1$ is a hamiltonian cycle, and hence $T_{[m]}$ is strong. Additionally, it is easy to see that $T_{[m]}$ is isomorphic to it's reversal under the isomorphism $\phi(v_i) = v_{n-i}$. Finally, observe that we can view the construction of $T_{[m+1]}$ recursively; take two copies of $T_{[m]}$ sharing a single vertex (v_n from one copy, and v_0 in the other copy) and reverse the 2-path P_{m+1}. This recursive perspective permits straightforward induction arguments, as we shall see throughout this section.

Lemma 2.1. *The arc $v_0 v_n$ with $n = 2^{m+1}$ is not on any hamiltonian path of the tournament $T_{[m]}$.*

Proof. For $m = 1$, $T_{[1]}$ is a strong tournament on 5 vertices, and it is easy to verify that $v_0 v_4$ is not on any hamiltonian path of $T_{[1]}$. Now, assume the result for m and consider the tournament $T_{[m+1]}$. Let P be a path of maximal length in $T_{[m+1]}$ containing the arc $v_0 v_{2n}$ with $2n = 2^{m+2}$. If this path does not include the vertex v_n, then no hamiltonian path includes the arc $v_0 v_{2n}$ and the result holds. So assume that P includes v_n, and as $T_{[m+1]}$ is isomorphic to its reversal, without loss of generality assume that v_n follows the arc $v_0 v_{2n}$ on P. Let v_t be the terminal vertex of P and v_i the initial vertex of P. Since v_n separates v_a from v_b for each $a < n < b$, every vertex between v_i and v_0 on P must have index $j < n$. Similarly, every vertex between v_{2n} and v_n on P must have index $j > n$ and every vertex following v_n must have index $j < n$. Thus, all the vertices of P with index $j \le n$ are contained in the subpaths $v_i \dots v_0$ and $v_n \dots v_t$. We now consider the combined sequence of vertices $Q = v_i, \dots, v_0, v_n, \dots, v_t$. Each of the vertices in this sequence has index $j \le n$, and every pair of consecutive vertices other than $v_0 v_n$ are joined by an arc of $T_{[m+1]}$. Since $v_0 v_n \in A(T_{[m]})$, we can use the recursive perspective described earlier and think of this sequence as a path of $T_{[m]}$. Since the path Q contains the arc $v_n v_0$, the induction hypothesis implies that this can not be a hamiltonian path of $T_{[m]}$, and so there is a vertex v_a, $a < n$, that is not in the sequence Q. But since the sequence Q contains every vertex of P with index $j \le n$, the vertex v_a is not included in the original path P. Consequently, the longest path containing the arc $v_0 v_{2n}$ is not hamiltonian, and $v_0 v_{2n}$ is non-traceable in $T_{[m+1]}$. □

Thus, for any m, we can construct a tournament with $2^{m+1}+1$ vertices with a non-traceable arc xy such that there exist m arc disjoint paths from y to x. Next, we show that $2^{m+1}+1$ is the minimal number of vertices among strong tournaments with this property.

Lemma 2.2. *Let T be a strong tournament containing a non-traceable arc xy such that there exist m arc disjoint paths from y to x. Then T has order $k \geq 2^{m+1}+1$.*

Proof. Again, the proof is by induction. For $m = 1$, the result is obtained by observing that the unique strong tournaments on 3 and 4 vertices are arc-traceable. Next, assume the result for m and consider the smallest strong tournament T, with non-traceable arc xy and $m+1$ arc disjoint paths from y to x. Let k be the order of T and assume that $k < 2^{m+2}+1$.

As xy is non-traceable, T must have the structure given by Theorem 1.1. Furthermore, the minimality of T implies that T_z has exactly four strong components, and that the second and third components both consist of a single vertex. Let X be the set of vertices in the first and second strong components of T_z, and $Y = V(T_z) \setminus X$. Clearly, either $|X| < 2^{m+1}$ or $|Y| < 2^{m+1}$. Without loss of generality, assume $|X| < 2^{m+1}$ and consider the tournament $T[X \cup \{z\}]$. If $zx \in A(T[X \cup \{z\}])$, then reverse this arc to form the tournament T'. Otherwise, simply let $T' = T[X \cup \{z\}]$. Since z is on every path from y to x, and there are $m+1$ arc disjoint paths from y to x, there also exist $m+1$ arc disjoint paths from z to x in T and at most one of these contains the arc zx (if it is an arc of T). Thus, there are at least m arc disjoint paths from z to x in T'. Clearly, T' is strong and has fewer than $2^{m+1}+1$ vertices, and so by the induction hypothesis, xz is on some hamiltonian path of this tournament. Let P be such a hamiltonian path, and split P into two smaller paths P_1, consisting of all the vertices up to and including x and P_2, consisting of all the vertices of P that follow z. The structure of T' and the assumption that xy is non-traceable guarantees that both P_1 and P_2 are paths of order at least 1. The only vertex of T' on neither P_1 or P_2 is z, so each vertex of X is on either P_1 or P_2. Next, the structure of T requires that $T[Y \cup \{z\}]$ must be strong, so let C be a hamiltonian cycle of this tournament and let Q_1 be the subpath of C from y to z, inclusive, and Q_2 the subpath of C from the vertex immediately succeeding z to the vertex immediately preceding y. We allow for the possibility that Q_2 may have order 0. Thus, every vertex of $Y \cup \{z\}$ is on either Q_1 or Q_2. We now construct $H = P_1Q_1P_2Q_2$, and we note that the vertex preceding z on this path is a vertex of Q_1 and so H does not use the arc of T from x to z. We claim that H is a hamiltonian path of T. First, the terminal vertex of P_1 is x and the initial vertex of

Q_1 is y, and $xy \in A(T)$ by assumption. Next, the terminal vertex of Q_1 is z, and the initial vertex of P_2 is the vertex immediately following z on P. Lastly, the terminal vertex of P_2 is a vertex of X, while the initial vertex of Q_2 (if any) is a vertex of Y, and X dominates Y. So, H is indeed a path of T. Finally, H includes all of X and Y as well as the vertex z, so H is a hamiltonian path including the arc xy. But xy is non-traceable, contradicting our assumption that $k < 2^{m+2} + 1$. □

The previous results apply to a particular arc xy and as a result we make no claim about the number of arc disjoint paths between every two distinct vertices. In fact, for each m, $T_{[m]}$ contains a vertex of in-degree 1 (v_1) as well as a vertex of out-degree 1 (v_{n-1} where $n = 2^{m+1}$), so we have yet to produce even a 2-arc-strong tournament that is not arc-traceable. We now seek to construct such an m-arc-strong non-arc-traceable tournament. Doing so requires only a minor variation on the construction of $T_{[m]}$.

Specifically, let $V' = V(T_{[m]})\backslash\{v_1, v_{n-1}\}$ and let Z_1 and Z_{n-1} be disjoint sets of size $2m-1$ with $V' \cap Z_i = \emptyset$ for $i = 1, n-1$. We define the tournament $T'_{[m]}$ of order $2^{m+1} + 4m - 3$ with vertex set $V' \cup Z_1 \cup Z_{n-1}$ by letting $T'_{[m]}[Z_i]$ be a regular tournament for $i = 1, n-1$ (so within this subtournament every vertex has in-degree and out-degree $m-1$), and $T'_{[m]}[V' \cup \{z_1, z_{n-1}\}] \equiv T_{[m]}$ for every $z_1 \in Z_1$ and $z_{n-1} \in Z_{n-1}$. Essentially, we are replacing the vertices v_1 and v_{n-1} of $T_{[m]}$ with regular tournaments on the sets Z_1 and Z_{n-1}.

Lemma 2.3. *$T'_{[m]}$ is m-arc-strong.*

Proof. It suffices to show that v_0 both reaches and is reached by every other vertex of $T'_{[m]}$ using m arc disjoint walks. The result then follows by Menger's Theorem [5] since any arc-cut-set separating v_0 and a vertex v must include at least one arc from each walk.

First consider any vertex $u \in Z_{n-1}$ and let $N^-(u) \cap Z_{n-1} = \{u_1, \ldots, u_{m-1}\}$. Then, $W_i = v_0 u_i u$ for $1 \le i \le m-1$ and $W_m = v_0 u$ are m arc disjoint walks from v_0 to u. Additionally, for any vertex $v \notin Z_{n-1}$, we can choose m distinct vertices $w_1, \ldots, w_m$ from Z_1 and form m arc-disjoint walks $W_i = v_0 w_i v$ for $1 \le i \le m$ from v_0 to v.

To show that v_0 is reached by every other vertex by m arc disjoint walks, we use the m arc-disjoint paths $U_1, \ldots U_m$ of $T_{[m]}$ as defined at the beginning of this section. Clearly, $U_1, \ldots, U_m$ are arc disjoint paths from v_n to v_0. To show that v_0 is reached by every vertex of $w \in Z_1$ by m arc disjoint walks, let $N^+(w) \cap Z_1 = \{w_1, \ldots w_{m-1}\}$ and let $W_i = w w_i U_i$ for

$1 \leq i \leq m-1$ and $W_m = wU_m$. For $v \notin Z_1 \cup \{v_n\}$, choose m distinct vertices $u_1, \ldots u_m$ from Z_{n-1} and then $W_i = vu_iU_i$ for $1 \leq i \leq m$ are arc-disjoint walks from v to v_0. □

Lemma 2.4. *The arc v_0v_n with $n = 2^{m+1}$ of $T'_{[m]}$ is non-traceable.*

Proof. Assume the result is false, so there is some hamiltonian path H of $T'_{[m]}$ that contains the arc v_0v_n. Since $N^-(u) \setminus Z_1 = \{v_0\}$ for any $u \in Z_1$, every path ending at a vertex of Z_1 is either a path of $T[Z_1]$, or includes an arc v_0w for some $w \in Z_1$. Since H cannot be a path of the latter type, every subpath of H ending at a vertex of Z_1 is a path of $T[Z_1]$ and hence the first $2m-1$ vertices of H are precisely the vertices in the set Z_1. Similarly, every subpath of H beginning at a vertex of Z_{n-1} is a path of $T[Z_{n-1}]$ and hence the last $2m-1$ vertices of H are precisely the vertices in Z_{n-1}. Let z_1 be the last vertex of Z_1 on H, and let z_{n-1} be the first vertex of Z_{n-1} on H. Then the subpath of H from z_1 to $z_n - 1$ is a hamiltonian path of $T'_{[m]}[V' \cup \{z_1, z_{n-1}\}] \equiv T_{[m]}$ containing the arc v_0v_n, contradicting Lemma 2.1. □

Finally, we conclude this section with a proof that $T'_{[m]}$ has the fewest vertices among all non-arc-traceable m-arc-strong tournaments.

Theorem 2.1. *If T is a non-arc-traceable m-arc-strong tournament, then T has order $k \geq 2^{m+1} + 4m - 3$.*

Proof. For $m = 1$, the result is immediate by observing that all strong tournaments are 1-arc-strong and that $2^{1+1} - 4(1) - 3 = 5$ is the size of the smallest non-arc-traceable strong tournament. So we can assume that $m \geq 2$. Let xy be a non-traceable arc of T. Define S_x as the initial strong component of T_x and S_y as the terminal strong component of T_y. Note that $x \notin S_y$ and $y \notin S_x$. As T is m-arc-strong, every vertex of T has in- and out-degree at least m, and thus each vertex of T_v has in- and out-degree at least $m-1$ for any $v \in V(T)$. Thus, $\delta^-_{T_x} \geq m-1$ and $\delta^+_{T_y} \geq m-1$ and consequently both $|S_x| \geq 2m-1$ and $|S_y| \geq 2m-1$.

Next, we claim that $S_x \cap S_y = \emptyset$. This follows from the fact that $(T_x)_y$ is not strong (if it is, then we can find a hamiltonian path beginning or ending with the arc xy), and the observation that S_x is contained in the initial strong component of $(T_x)_y$, and S_y is contained in the terminal strong component of $(T_x)_y$.

Lastly, observe that no path from y to x can use any vertex of $S_x \cup S_y$, as every path from y to S_x must contain the vertex x, and dually every path from S_y to x must contain y. Thus we can form a tournament T' by replacing the entire set S_x with a single vertex u_x and replacing the entire set S_y with a single vertex u_y without disturbing any path from y to x. Thus, there remain m arc disjoint paths from y to x in T'. Furthermore, if we let x dominate u_x and let u_y dominate y, then T' is strong. By a similar argument to the one used in Lemma 2.4, any hamiltonian path of T' containing the arc xy can be extended to a hamiltonian path of T containing this arc. As xy is not on any hamiltonian path of T by assumption, it is therefore not on any hamiltonian path of T'. By Lemma 2.2, T' has at least $2^{m+1}+1$ vertices and so T has at least $(2^{m+1}+1)-2+2(2m-1) = 2^{m+1}+4m-3$ vertices. □

3 The maximal number of non-traceable arcs in a strong tournament

We now turn our attention to tournaments with many non-traceable arcs. Let T be the transitive tournament on the set $V = \{v_0, \dots v_{n-1}\}$ where $d^-(v_i) = i$. It then follows that $v_iv_j \in A$ if and only if $i < j$. It is well known that this tournament has a unique hamiltonian path, and hence it follows immediately that there are $\binom{n}{2} - (n-1) = \frac{n^2-3n+2}{2}$ non-traceable arcs in the transitive tournament of order n. Since every tournament contains a hamiltonian path, this is also clearly maximal. Strong tournaments, however, contain many distinct hamiltonian paths. Recently, Busch [2] showed that a strong tournament of order n has at least $5^{\frac{n-1}{3}}$ distinct hamiltonian paths, improving a result of Moon [6]. This seems to suggest that strong tournaments have few non-traceable arcs. This is not the case, however. For any odd n, let $T_{\max}$ be the tournament obtained from a transitive n-tournament by reversing the arcs $v_{i-2}v_i$ for each even i, $2 \le i \le n-1$. This tournament is an example of an *upset tournament*, a tournament which can be obtained from the transitive tournament by reversing the arcs in a single path from the vertex with in-degree zero to the vertex with out-degree zero. In [3], arc-traceable upset tournaments were characterized, and $T_{\max}$ was shown to have $\frac{n^2-4n+3}{8}$ non-traceable arcs. Further, it was shown that this tournament was maximal among all upset tournaments with respect to the number of non-traceable arcs. We now extend this result and prove that all strong n-tournaments have at most this number of non-traceable arcs. Once again, we use the structure of non-arc-traceable tournaments given by Theorem 1.1. However, we note that the structure described in Theorem 1.1 is necessary, but not sufficient for a tournament to be non-arc-traceable. We begin by developing additional structure that is required of non-arc-traceable tournaments, and using that structure to identify other non-traceable arcs in a tournament T with at least one non-traceable arc xy. Recall that T_z is the subtournament $T - z$ and $T_z^{(i)}$ is the i^{th} strong component of this tournament.

Lemma 3.1. *Let T be a strong tournament with some vertex $z \in V(T)$ such that T_z has $k \geq 4$ strong components. Further, let $V(T_z^{(2)})$ dominate z and let z dominate $V(T_z^{(k-1)})$ in T. For $x \in V(T_z^{(1)})$ and $y \in V(T_z^{(k)})$, xy is part of some hamiltonian path if and only if (i) the vertices of $T_z^{(1)}$ can be partitioned by paths P_1, Q_1 where P_1 begins at a vertex dominated by z and Q_1 ends at x or (ii) the vertices of $T_z^{(k)}$ can be partitioned by paths P_k, Q_k where Q_k begins at y and P_k ends at a vertex that dominates z.*

Proof. We prove the sufficiency of condition (i), as condition (ii) is equivalent to condition (i) in the reversal of T. Assume that condition (i) holds. Let P_i be a hamiltonian path of $T_z^{(i)}$ for $2 \leq i \leq k-1$, and let Q_k be any path in $T_z^{(k)}$ from y to a vertex that dominates z. Finally, let P_k be a hamiltonian path of $T_z^{(k)} \setminus V(Q_k)$. Then $H = Q_1 Q_k z P_1 P_2 P_2 \ldots P_k$ is a hamiltonian path of T containing the arc xy.

For the converse, choose $x \in V(T_z^{(1)})$ and $y \in V(T_z^{(k)})$ and assume the arc xy is traceable in T. Let H be a hamiltonian path of T containing the arc xy. First, observe that H contains at most one other arc uv with $u \in V(T_z^{(1)})$ and $v \notin V(T_z^{(1)})$, as z must lie between any two such arcs on H. If H does not contain another arc with this property, then the initial vertex of H must be a vertex of $T_z^{(2)}$. In this case, the portion of the path H that lies in $T_z^{(1)}$ is a hamiltonian path of the subtournament $T_z^{(1)}$ that begins at a vertex dominated by z and ends at x. Removing any arc of this subpath yields two paths that satisfy condition (i). So, we may assume that H contains an arc $uv \neq xy$ with $u \in T_z^{(1)}$ and $v \notin T_z^{(1)}$. In this case, the portion of H that lies in $T_z^{(1)}$ consists of two vertex disjoint paths, P_1 and Q_1 (assume that P_1 precedes Q_1 on H). If xy precedes uv on H, then P_1 ends at x and the vertex immediately preceding Q_1 on H must be z, and so condition (i) is satisfied. If uv precedes xy on H, then there must be an arc $u'v'$ with $u' \notin V(T_z^{(k)})$ and $v' \in V(T_z^{(k)})$ such that $u'v'$ precedes xy on H. In the reversal of T, we find that yx precedes $v'u'$ on the reversal of H, and condition (i) is satisfied in the reversal of T. Thus, condition (ii) is satisfied in T. □

We note that a corollary of this result gives a sufficient condition for non-arc-traceability in strong tournaments. The results from Section 2 clearly show that the converse of the following corollary is false.

Corollary 3.1. *Let T be a strong tournament having the structure given by Theorem 1.1. If $|N^+(z) \cap V(T_z^{(1)})| = 1$ and $|N^-(z) \cap V(T_z^{(k)})| = 1$, then T is not arc-traceable.*

Proof. Define vertices x and y as follows: $N^+(z) \cap V(T_z^{(1)}) = \{x\}$ and $N^-(z) \cap V(T_z^{(k)}) = \{y\}$. Then by Lemma 3.1, xy is non-traceable. □

Lemma 3.2. *Let T be a strong tournament having a cut-vertex z. If $X = \{x \in V(T_z^{(1)})$: the vertices of $T_z^{(1)}$ cannot be partitioned into two paths P and Q such that P begins at a vertex dominated by z and Q ends at $x\}$, then $|X| \leq \frac{a+1}{2}$ where $a = |V(T_z^{(1)})|$. Similarly, $|Y| \leq \frac{b+1}{2}$ for the analogous set Y, where $b = |V(T_z^{(k)})|$.*

Proof. The result is clear if $X = \emptyset$, so assume that $X = \{x_0, x_1, \ldots, x_m\}$. If $|N^+(z) \cap X| \geq 1$, then assume that z dominates x_0. Let P_i be a longest path not containing x_i that begins at a vertex dominated by z. As z dominates x_0 or some vertex $z' \notin X$, P_i is a path containing at least one vertex for each i, $1 \leq i \leq m$. Let $S_i = V(T_z^{(1)}) \setminus V(P_i)$ for $1 \leq i \leq m$. We claim first that $S_i \setminus \{x_i\}$ dominates $V(P_i)$. Assume otherwise, and let v be the last vertex along P_i such that v dominates w for some $w \in S_i \setminus \{x_i\}$. If v is the terminal vertex of P_i, then $P_i w$ is a longer path than P_i beginning at a vertex dominated by z. Otherwise, let v^+ be the vertex immediately following v on P_i. By the maximality of v, we can replace the arc vv^+ of P_i with the 2-path vwv^+ and again obtain a path that begins at a vertex dominated by z that is longer than P_i. Note, since $T_z^{(1)}$ is strong, S_i must be reachable from $V(P_i)$, and thus some vertex of $V(P_i)$ must dominate x_i. Choose such a vertex and call it v_i.

Now, let Q_i be the longest path of $T[S_i]$ that ends at x_i, and let $U_i = S_i \setminus V(Q_i)$. Using P_i and Q_i and the definition of X, we see that $U_i \neq \emptyset$ for $1 \leq i \leq m$ and $U_i \subset S_i \setminus \{x_i\}$, so U_i dominates $V(P_i)$. By a similar argument used above, we also note that U_i is dominated by each vertex of $V(Q_i)$. Thus, we conclude that $V(Q_i) \setminus \{x_i\}$ dominates both U_i and $V(P_i)$.

Additionally, we observe that the terminal strong component of $T[U_i]$ contains no vertex of X and hence $U_i \setminus X \neq \emptyset$. To see this, let w be a vertex in the terminal strong component of $T[U_i]$ and let H be any hamiltonian path of $T[U_i]$ ending at w. Construct the paths $Q_i H$ and P_i, which partition the vertices of $T_z^{(1)}$. The initial vertex of P_i is a vertex dominated by z, and w is the terminal vertex of $Q_i H$. So, w is not in X.

Finally, we claim that $U_i \cap U_j = \emptyset$ for all $i \neq j$. Assume otherwise, and choose $i \neq j$ with $u \in U_i \cap U_j$. Without loss of generality assume that x_i dominates x_j. Since $u \in U_i \cap U_j$, and $U_i \cap U_j$ dominates $V(P_i) \cup V(P_j)$, u dominates both v_i and v_j. Similarly, as $V(Q_i) \cup V(Q_j)$ dominate $U_i \cap U_j$, x_i and x_j both dominate u. Now, since $V(Q_j)$ dominates u, $v_i \notin V(Q_j)$, and hence $v_i \in V(P_j) \cup U_j$. Also, if $x_i \in U_j$, then $V(Q_j)$ dominates x_i. But x_j is the terminal vertex of Q_j and x_i dominates x_j by assumption. As a result, we conclude that $x_i \notin U_j$. Further, x_i dominates u and U_j dominates $V(P_j)$ so $x_i \notin V(P_j)$. Thus, $x_i \in V(Q_j) \setminus \{x_j\}$ and so x_i dominates both U_j and $V(P_j)$. Since $v_i \in V(P_j) \cup U_j$, this requires that x_i dominates v_i, contradicting the choice of v_i as a vertex of P_i that dominates x_i.

The above arguments show that $U_i \backslash X \neq \emptyset$ for $1 \leq i \leq m$ and $U_i \cap U_j = \emptyset$ for $i \neq j$, which establishes that

$$\left| \left(\bigcup_{i=1}^{m} U_i \right) \setminus X \right| = \sum_{i=1}^{m} |U_i \setminus X| \geq m = |X| - 1.$$

Thus, we have $a \geq |X| + m = 2m + 1 = 2|X| - 1$ and so $|X| \leq \frac{a+1}{2}$.

The bound for the set Y is obtained using an identical argument in the reversal of T. □

Corollary 3.2. *Let T be a strong tournament having the structure given by Theorem 1.1. The number of non-traceable arcs from $T_z^{(1)}$ to $T_z^{(k)}$ is at most $\frac{(ab+a+b+1)}{4}$ where $a = |V(T_z^{(1)})|$ and $b = |V(T_z^{(k)})|$.*

Proof. Let B be the set of non-traceable arcs from $T_z^{(1)}$ to $T_z^{(k)}$. Let X (respectively, Y) be the set of vertices such that the vertices of $T_z^{(1)}$ (respectively, $T_z^{(k)}$) can not be split into paths P and Q such that P begins at a vertex dominated by z (respectively, P ends at a vertex dominating z) and Q ends at a vertex of X (respectively, Q begins at a vertex of Y). Clearly, by Lemma 3.1, $|B| = |X||Y|$ and by Lemma 3.2, $|X||Y| \leq \frac{(a+1)(b+1)}{4}$. □

Theorem 3.1. *If T is a strong n-tournament, then T has at most $\frac{n^2-4n+3}{8}$ non-traceable arcs, with equality if and only if T is isomorphic to $T_{\max}$.*

Proof. For $n = 3$ or $n = 4$, there is a unique strong n-tournament, and in either case this tournament is arc-traceable and so has at most $\frac{3^2-4(3)+3}{8} = 0$ or $\frac{4^2-4(4)+3}{8} = \frac{3}{8}$ non-traceable arcs, respectively. When $n = 3$, equality is achieved and this tournament is isomorphic to $T_{\max}$.

For $n > 4$, assume that T is non-arc-traceable. Let xy be a non-traceable arc of T, and let T have the structure given by Theorem 1.1. Let $A = V(T_z^{(1)})$ and $B = V(T_z^{(k)})$, with $a = |A|$ and $b = |B|$, and choose $u \in T_z^{(2)}$ and $w \in T_z^{(k-1)}$. If $x'y'$ is a non-traceable arc of T, neither $T - x'$ nor $T - y'$ are strong, and hence x' and y' must both be in the set $A \cup B \cup \{z\}$. Thus, $x'y'$ is either an arc of $T[A \cup \{u, z\}]$, an arc of $T[B \cup \{w, z\}]$, or an arc from A to B.

We claim that every arc of $T[A \cup \{u, z\}]$ or $T[B \cup \{w, z\}]$ that is non-traceable in T is also non-traceable in this subtournament. Assume that some arc α is traceable in $T[A \cup \{u, z\}]$, and let P be a hamiltonian path of this tournament containing the arc α. If P ends at a vertex other than z, then for any hamiltonian path Q of the subtournament induced on the vertices not on P, PQ is a hamiltonian path of T containing α. On the other hand, u has out-degree one in this subtournament, so if P ends at z it must end with the arc uz. If α is not the arc uz, then we can remove this arc from the end of P to obtain the path P' ending at u. Then we can choose a hamiltonian path Q of the subtournament induced on the remaining vertices of T_z that ends in $T_z^{(k)}$ at a vertex dominating z. We then combine these paths to form $P'Qz$, a hamiltonian path of T containing α. Finally, if $\alpha = uz$, then we can choose a hamiltonian path P' of $T_z^{(1)}$ that begins at a vertex dominated by z, and a path Q that includes all remaining vertices of T_z except u. Since the initial vertex of Q is not in $T_z^{(1)}$, $uzP'Q$ is a hamiltonian path of T containing the arc α. An identical argument in the reversal of T establishes the corresponding result for $T[B \cup \{w, z\}]$.

As both of the subtournaments $T[A \cup \{u, z\}]$ and $T[B \cup \{w, z\}]$ are strong, we can apply the induction hypothesis, and hence $T[A \cup \{u, z\}]$ and $T[B \cup \{w, z\}]$ have at most $\frac{(a+2)^2-4(a+2)+3}{8}$ and $\frac{(b+2)^2-4(b+2)+3}{8}$ non-traceable arcs, with equality if and only if each of these subtournaments are isomorphic to $T_{\max}$. Summing these two values, we obtain $\frac{a^2+b^2-2}{8}$.

Finally, by Corollary 3.2, there are at most $\frac{(a+1)(b+1)}{4}$ non-traceable arcs from A to B. Combining and observing that $a + b \leq n - 3$, the number of non-traceable arcs in T is at most

$$\begin{aligned}
\frac{a^2+b^2-2}{8} + \frac{ab+a+b+1}{4} &= \frac{(a^2+2ab+b^2)+2(a+b)}{8} \\
&= \frac{(a+b)^2+2(a+b)}{8} \\
&\leq \frac{(n-3)^2+2(n-3)}{8}
\end{aligned}$$

$$\leq \frac{n^2 - 6n + 9 + 2n - 6}{8}$$
$$\leq \frac{n^2 - 4n + 3}{8}$$

The proof is complete by noting that in the above equation, equality is established if and only if $n = a+b+3$ and the subtournaments $T[A \cup \{u, z\}]$ and $T[B \cup \{w, z\}]$ are both upset tournaments isomorphic to the appropriate size $T_{\max}$. From this it follows directly that a and b are both odd, and hence T also has odd order. Finally, the structure of $T[A \cup \{u, z\}]$ and $T[B \cup \{w, z\}]$ guarantee that T is also an upset tournament and isomorphic to $T_{\max}$. $\square$

Since a tournament of order n has $\frac{n^2-n}{2}$ arcs, this result shows that as many as $(\frac{n-3}{4n})|A(T)| = (\frac{1}{4} - \epsilon)|A(T)|$ arcs in a strong tournament can be non-traceable.

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