Two new ${}_3F_2$ summation formulas

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Abstract

Built on earlier works of Larcombe on a certain class of non-terminating expansions of the sine function, we set up two new ${}_3F_2$ summation formulas via integration.

In his previous paper [4], one of a series of theses on the expansion of $sin(m\alpha)$ in odd powers of $\sin(\alpha)$, i.e.,

$$\sin(m\alpha) = \sum_{n=0}^{\infty} S_n^{(m)} \sin^{2n+1}\alpha, \quad (|\alpha| < \pi/2) \tag{1}$$

Larcombe established a closed form of the coefficient $S_n^{(m)}$. The importance of $S_n^{(m)}$, as stated in [2, 3, 6], consists in its relationship with the famous Catalan number. Thanks to Luo's and Larcombe's works, we now know that the earliest awareness of this fact can be traced to a Chinese mathematician *Antu Ming* who lived more than two hundred years ago, not Euler or Catalan as it had been previously known in the literature. We refer the reader to [2] for this historical point and [4, Remark 6] for a short but good introduction on *Antu Ming*.

For convenience purpose, we state without proof the result of Larcome as follows.

Lemma 1. [4, Theorem 2] *For even integer $m \geq 2, |\alpha| < \pi/2$,*

$$\sin(m\alpha) = \sum_{n=0}^{\infty} (-1)^{m/2} 2^{1-2n} Q(n;m) m c_{n-1} \sin^{2n+1}\alpha \tag{2}$$

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where for $m \geq 4$*,*

$$Q(n;m) = \frac{(2n+3)(2n+5)(2n+7)\cdots(2n+m-1)}{(2n-3)(2n-5)(2n-7)\cdots(2n-m+1)}$$

and $Q(n;2) \equiv 1$*,* c_n *denotes the* $(n+1)$*th term*

$$c_n = \frac{1}{n+1}\binom{2n}{n}$$

of the Catalan sequence $\{c_0, c_1, c_2, c_3, \cdots\} = \{1, 1, 2, 5, \cdots\}$*, the binomial coefficient* $\binom{n}{k} = n!/(k!(n-k)!)$*. In particular, order that* $c_{-1} = -1/2$*.*

Very recently, applying the technique of integration to the expansion of sine function, Larcombe [5, Theorem 3] finally obtained that:

Lemma 2. *For integer* $m \geq 0$*,*

$$_3F_2\left(\frac{1-m}{2}, \frac{1+m}{2}, 1; \frac{3}{2}, \frac{3}{2}; 1\right) = \begin{cases} \frac{2}{m^2}, & m = 2 \pmod 4; \\ 0, & m = 0 \pmod 4. \end{cases} \tag{3}$$

Here $_3F_2(\cdots)$ denotes the hypergeometric series associated with special parameters, which is in general defined as

$$_rF_s(a_1, \ldots, a_r; b_1, \ldots, b_s; z) = \sum_{n=0}^{\infty} \frac{(a_1, \cdots, a_r)_n}{(b_1, \cdots, b_s)_n} \frac{z^n}{n!}$$

with the convergence condition(s), where the notation

$$(a)_n = \prod_{k=1}^{n}(a+k-1), (a_1, \cdots, a_r)_n = (a_1)_n (a_2)_n \cdots (a_r)_n.$$

It should be noted that the summation formula (3) is the special case $c = 1, a = (1-m)/2, b = (1+m)/2$ of the following summation formula. Actually, it can also be dated back to the Thomae's $_3F_2$ transform formula [1, pp.143,Corollary 3.3.6]. See [1] for more details.

Theorem 1. [1, pp.148,Theorem 3.5.5] *For any parameters* a, b, c, e, f *such that* $a + b = 1, e + f = 2c + 1$*,*

$$_3F_2(a, b, c; e, f; 1) = \frac{\pi\, \Gamma(e)\Gamma(f)}{2^{2c-1}\Gamma((a+e)/2)\Gamma((a+f)/2)\Gamma((b+e)/2)\Gamma((b+f)/2)}, \tag{4}$$

where $\Gamma(x)$ *denotes the usual Gamma function.*

Despite of this, motivated by Larcombe's result, we will set up two new ${}_3F_2$ summation formulas in this short paper. The word "new" means what we obtain are not only not covered by Theorem 1 but also extensions of Larcombe's result (3). For this, we need

Lemma 3. *Let $n \geq 1$ be an odd integer and $p \geq 0$ an integer, and define*

$$I(n,p) = \int_0^{\pi/2} \sin^n \alpha \cos^p \alpha d\alpha.$$

Then

$$I(n,p) = \frac{(p-1)!!\,(n-1)!!}{(n+p)!!}, \tag{5}$$

where $n!! = \prod_{i=0}^{\lfloor n/2 \rfloor}(n-2i)$, $\lfloor x \rfloor$ denotes the maximum integer less than x.

Proof. At first, using the integration by part, we can set up a recurrence for $I(n,p)$ as below:

$$I(n,p) = \frac{p-1}{n+p} I(n,p-2).$$

On the other hand, it is readily found by direct calculation that for any integer n,

$$I(2n+1,1) = \frac{1}{2n+2};$$

$$I(2n+1,0) = \frac{4^n}{(n+1)(2n+1)c_n},$$

where c_n is the $(n+1)th$ Catalan number. Hence, the result in question follows at once by induction on n together with these initial conditions.

Lemma 4. *Let m, p be two nonnegative integers and define*

$$J(m,p) = \int_0^{\pi/2} \sin(m\alpha) \cos^p \alpha d\alpha.$$

Then

$$J(m,p) = \frac{1}{2^p} \sum_{i=0}^{p} \binom{p}{i} J(m+2i-p,0), \tag{6}$$

where

$$J(m+2i-p,0) = \begin{cases} \dfrac{1}{m+2i-p}, & for p\ odd; \\[2ex] \left[1-(-1)^{\frac{m+2i-p}{2}}\right] \dfrac{1}{m+2i-p}, & for\ p\ even. \end{cases}$$

Proof. The result can be shown by induction on m and p together with pure integration, which is thus left to the reader.

These two lemmas lead us to our main results, described as follows.

Theorem 2. *Let $p \geq 1$ be odd and $m \geq 2$ even integers. Then*

$$ {}_3F_2\left(\frac{1-m}{2}, \frac{1+m}{2}, 1; \frac{3}{2}, \frac{3+p}{2}; 1\right) = \frac{p+1}{m2^p} \sum_{i=0}^{p} \binom{p}{i} \frac{1}{m+2i-p}. \tag{7} $$

Proof. First, multiply both sides of (1) by $\cos^p \alpha$ to get

$$ \sin(m\alpha)\cos^p \alpha = \sum_{n=0}^{\infty} S_n^{(m)} \sin^{2n+1} \alpha \cos^p \alpha. $$

Then integrate both sides of this identity on $[0, \pi/2]$. The result is

$$ J(m,p) = \sum_{n=0}^{\infty} S_n^{(m)} I(2n+1, p), \tag{8} $$

where $I(2n+1,p)$ and $J(m,p)$ are given by Lemmas 3 and 4, which reduces further (8) to

$$ \frac{1}{2^p} \sum_{i=0}^{p} \binom{p}{i} \frac{1}{m+2i-p} = \sum_{n=0}^{\infty} \frac{m\,(p-1)!!(-1)^{\frac{m}{2}} 2^{1-2n} Q(n;m) c_{n-1}}{(2n+p+1)(2n+p-1)\cdots(2n+4)(2n+2)}. $$

Now, set

$$ a_n = \frac{2^{1-2n} Q(n;m) c_{n-1}}{(2n+p+1)(2n+p-1)\cdots(2n+4)(2n+2)}. $$

Note that $a_0 = \frac{(-1)^{\frac{m}{2}}}{(p+1)!!}$. Then it is easily seen that

$$ \frac{a_{n+1}}{a_n} = \frac{(n+\frac{1-m}{2})(n+\frac{1+m}{2})}{(n+\frac{3}{2})(n+\frac{3+p}{2})}. $$

This proves the result in question if represented in terms of hypergeometric notations.

Theorem 3. *Let $m \geq 2, p \geq 0$ be two even integers. Then*

$$ {}_3F_2\left(\frac{1-m}{2}, \frac{1+m}{2}, 1; \frac{3}{2}, \frac{3+p}{2}; 1\right) $$
$$ = \frac{p+1}{m2^p} \sum_{i=0}^{p} \left[1 - (-1)^{\frac{m+2i-p}{2}}\right] \frac{\binom{p}{i}}{m+2i-p}. \tag{9} $$

Proof. Apply the same technique as in Theorem 2 to get

$$J(m,p)=\sum_{n=0}^{\infty} S_n^{(m)} I(2n+1,p).$$

By Lemmas 3 and 4, we get the summation

$$\frac{1}{2^p}\sum_{i=0}^{p}\left[1-(-1)^{\frac{m+2i-p}{2}}\right]\frac{\binom{p}{i}}{m+2i-p}$$
$$=m(p-1)!!(-1)^{\frac{m}{2}}\sum_{n=0}^{\infty}\frac{2^{1-2n}\,Q(n;m)c_{n-1}(2n)!!}{(2n+p+1)!!}.$$

Proceeding as before, write

$$a_n=\frac{2^{1-2n}Q(n;m)c_{n-1}(2n)!!}{(2n+p+1)!!}.$$

It is easy to verify that

$$\frac{a_{n+1}}{a_n}=\frac{(n+\frac{1-m}{2})(n+\frac{1+m}{2})}{(n+\frac{3}{2})(n+\frac{3+p}{2})},\ a_0=\frac{(-1)^{\frac{m}{2}}}{(p+1)!!}.$$

The desired result emerges on being written in terms of hypergeometric notation.

We remark finally that, the special case $p=0$ of (9) is just Larcombe's Theorem 3 in [5]. In the meantime, the related parameters in Theorems 2 and 3 do not satisfy the requirement of Theorem 1 that $e+f=2c+1$, so they can not be deduced by Theorem 1.

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