Clique algorithms for finding substructures in generalized quadrangles

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Abstract

The search for special substructures in combinatorial objects that have a lot of symmetry, such as searching for maximal partial ovoids or spreads in generalized quadrangles, can often be translated to a well-known algorithmic problem, such as a maximum clique problem in a graph. These problems are typically NP-hard. However, using standard backtracking strategies together with pruning techniques based on problem specific properties, it is possible to obtain non-trivial results which are mathematically interesting. In some cases also heuristic techniques can lead to interesting results. In this paper we describe some techniques as well as new results obtained for maximal partial ovoids and spreads in generalized quadrangles.

1 Preliminaries

We denote an undirected graph by $G = (V, E)$, where V is the set of vertices and E is the set of edges. Two vertices are said to be *adjacent* if they are connected by an edge. A *clique* is a set of pairwise adjacent vertices; an *independent set* is a set of pairwise non-adjacent vertices. A clique in a graph G is an independent set in its complement $\overline{G}$. A *maximal* clique is a clique that is not contained in a larger clique. A *maximum clique* is a clique of maximum cardinality in the graph. Maximal and

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maximum independent sets are defined similarly. The algorithmic problems of searching for maximum cliques or independent sets are well known to be NP-hard [8].

A (finite) *generalized quadrangle* (GQ) is an incidence structure $\mathcal{S} = (P, B, \mathrm{I})$, in which P and B are disjoint (non-empty) sets of objects, called *points* and *lines* respectively, and for which I is a point-line *incidence relation* satisfying the following axioms: (i) each point is incident with $t+1$ lines ($t \geq 1$) and two distinct points are incident with at most one line; (ii) each line is incident with $s+1$ points ($s \geq 1$) and two distinct lines are incident with at most one point; (iii) if x is a point and L is a line not incident with x, then there is a unique pair $(y, M) \in P \times B$ for which $x \mathrm{I} M \mathrm{I} y \mathrm{I} L$. For the theory of generalized quadrangles we refer to [18].

The integers s and t are the *parameters* of the generalized quadrangle $\mathcal{S}$, which is said to have *order* (s, t); if $s = t$, $\mathcal{S}$ is said to have order s. Generalized quadrangles with $s > 1$ and $t > 1$ are called *thick*. Interchanging points and lines in $\mathcal{S}$ yields a generalized quadrangle $\mathcal{S}^D$ of order (t, s), which is called the *dual* of $\mathcal{S}$.

Two points x and y of P are called *collinear* ($x \sim y$) if there is a line L in B incident with both. Dually, two lines L and M are called *concurrent* if there is a point x in P incident with both. For $x \in P$ denote $x^\perp = \{y \in P \mid\mid y \sim x\}$ and note that $x \in x^\perp$. The *trace* of a pair (x, y) of distinct points is defined as $x^\perp \cap y^\perp$ and is denoted by $\{x, y\}^\perp$. We get $|\{x, y\}^\perp| = s+1$ or $t+1$ according to $x \sim y$ or $x \not\sim y$. More generally, if A is an arbitrary subset of P, we define $A^\perp$ as $A^\perp = \bigcap\{x^\perp \mid\mid x \in A\}$. For $x \neq y$ the *span* of the pair (x, y) is defined as $\{x, y\}^{\perp\perp} = \{u \in P \mid\mid u \in z^\perp,\ \forall z \in x^\perp \cap y^\perp\}$. The pair of points (x, y) is called *regular* if $x \sim y$, $x \neq y$, or if $x \not\sim y$ and $|\{x, y\}^{\perp\perp}| = t+1$. The point x is *regular* provided (x, y) is regular for all $y \in P, y \neq x$.

An *ovoid* of $\mathcal{S}$ is a set $\mathcal{O}$ of points of P such that each line of B is incident with a unique point of $\mathcal{O}$. Equivalently, an ovoid is a set of $st+1$ pairwise non-collinear points. A *partial ovoid* (sometimes also called *cap*) is a set of points of P such that each line of B is incident with at most one point of $\mathcal{O}$, i.e. a set of pairwise non-collinear points. Dually, a *spread* of $\mathcal{S}$ is a set $\mathcal{R}$ of lines of B such that each point of P is incident with a unique line of $\mathcal{R}$ or, equivalently, a set of $st+1$ pairwise non-concurrent lines; note that a spread partitions the point set into classes of collinear points. A *partial spread* of $\mathcal{S}$ is a set $\mathcal{R}$ of lines of B such that each point of P is incident with at most one line of $\mathcal{R}$, i.e. a set of pairwise non-concurrent lines. A (partial) spread in $\mathcal{S}$ is a (partial) ovoid in $\mathcal{S}^D$. A partial ovoid (or spread) is called *maximal* or *complete* if it is not contained in a larger partial ovoid (or spread). Two (partial) ovoids (or spreads) are called *equivalent* if there is an automorphism of S that transforms one into the other.

With a generalized quadrangle $\mathcal{S}$ a so-called *collinearity graph* or *point* *graph* G_S can be associated as follows: the points of P correspond to the vertices of G_S and two vertices are adjacent if and only if the corresponding points are collinear. The graph G_S is a strongly regular graph with parameters $v = (s+1)(st+1)$, $k = s(t+1)$, $\lambda = s-1$, $\mu = t+1$. Considering the point graph, an ovoid of $\mathcal{S}$ is a maximum independent set of size $st+1$ in G_S, or equivalently, a maximum clique in its complement $\overline{G_S}$. A spread of $\mathcal{S}$ is a maximum independent set of size $st+1$ in the collinearity graph G_{S^D} of $\mathcal{S}^D$. Maximal partial ovoids and spreads are maximal independent sets in G_S or G_{S^D}.

In this paper we present algorithms for finding a largest maximal partial ovoid or spread in a generalized quadrangle, for exploring the spectrum of sizes for which maximal partial ovoids or spreads exist, and for classifying up to equivalence all maximal partial ovoids or spreads of a certain size in a generalized quadrangle. Results will be given for some of the classical generalized quadrangles (with q a power of a prime):

The quadrics $Q(4,q)$ and $Q^-(5,q)$: Let $Q(4,q)$ resp. $Q^-(5,q)$ be a nonsingular quadric of projective index 1 in the projective space PG$(4,q)$ resp. PG$(5,q)$. Then the points of the quadric together with the lines of the quadric form a generalized quadrangle with parameters $(s,t) = (q,q)$ resp. (q,q^2). Every $Q(4,q)$ has ovoids; for q even it has spreads and for q odd it does not. $Q^-(5,q)$ always has spreads, but it never has ovoids.

The symplectic generalized quadrangle $W(q)$: The points of PG$(3,q)$, together with the totally isotropic lines with respect to a symplectic polarity, form a generalized quadrangle with parameters $(s,t) = (q,q)$. Note that $W(q)$ is isomorphic to the dual of $Q(4,q)$. Moreover $Q(4,q)$, or $W(q)$, is selfdual iff q is even. $W(q)$ has ovoids for q even and has no ovoids for q odd; it always has spreads.

The Hermitian varieties $H(3,q^2)$ and $H(4,q^2)$: Let H be a nonsingular Hermitian variety of the projective space PG$(3,q^2)$ resp. PG$(4,q^2)$. Then the points of H together with the lines on H form a generalized quadrangle with parameters $(s,t) = (q^2,q)$ resp. (q^2,q^3). Note that $H(3,q^2)$ is isomorphic to the dual of $Q^-(5,q)$. $H(3,q^2)$ has ovoids and has no spreads. $H(4,q^2)$ has no ovoids and $H(4,4)$ has no spreads [3]. Whether $H(4,q^2)$ has spreads for $q > 2$ is still an open problem.

The paper is organized as follows. Section 2 describes exhaustive search algorithms, where we use standard clique searching algorithms and add pruning strategies based on specific properties of the generalized quadrangle. This approach leads to exact answers concerning e.g. the size of the largest maximal partial ovoid or spread, or the classification of all maximal partial ovoids and spreads of a given size. Another approach, described in Section 3, is based on heuristic techniques and turns out to be very effective for exploring the spectrum of sizes for which maximal partial ovoids or spreads exist. In Section 4 we illustrate the effect of the proposed techniques and present some results obtained by our computer searches. These include new exact values improving on earlier theoretical bounds for the size of the largest maximal partial ovoids and spreads, as well as their complete classification up to equivalence, and new values for the spectra of sizes for which maximal partial ovoids and spreads exist.

2 Exhaustive search algorithms

2.1 Standard backtracking and pruning methods

The basic form of most published algorithms (e.g. [4]) for the maximal or maximum clique problem is a backtracking search which tries in every recursion step to extend a partial clique by adding the vertices of a set A of allowed remaining vertices in a systematic way. When reaching a point where the set A is empty, a new maximal clique has been found.

Pruning strategies are used to avoid going through every single clique of the graph. Typically this consists in a bounding function which gives an upper bound on the number of vertices that can still be added to the current partial clique. E.g. when searching for maximum cliques, a straightforward idea is to backtrack when the set A becomes so small that even if all its vertices could be added to form a clique, the size of that clique would not exceed the size of the largest clique found so far; in that case the bound is simply $|A|$. Other pruning strategies involve vertex colorings. In a vertex coloring adjacent vertices must be assigned different colors, so if a graph or an induced subgraph can be colored with c colors, then the graph or subgraph cannot contain a clique of size $c+1$; in this case the bound is the number of colors used to color the vertices of A. In practice a fixed coloring of the original graph is used, since determining a coloring for the induced subgraph $\langle A \rangle$ each time usually is too expensive.

Recently Östergård [17] presented a new maximum clique algorithm that allows to introduce a new pruning strategy. Let $v_1, v_2, \ldots, v_n$ be an ordering of the vertices of the graph, let $S_i = \{v_i, \ldots, v_n\}$ and let $c(i)$ denote the size of the largest clique in S_i. For any $1 \leq i \leq n-1$, either $c(i) = c(i+1)$ or $c(i) = c(i+1)+1$. Moreover $c(i) = c(i+1)+1$ if and only if there is a clique of size $c(i+1)+1$ in S_i that contains v_i. The algorithm starts with $c(n) = 1$ and computes $c(i)$, $i = n-1, \ldots, 1$ by searching for such a clique. Finally the size of a maximum clique is given by $c(1)$. The values of $c(i)$ can be used for pruning the search as follows. Searching for a clique of size larger than s, the search can be pruned if $j + c(i) \leq s$, where j denotes the size of the current partial clique and i is the index of the next vertex v_i to be added to the current partial clique.

Next to these standard clique finding algorithms we can use additional pruning strategies which take into account the special structure of the collinearity graph or the incidence structure of a generalized quadrangle. The rest of this section will discuss such specific techniques.

2.2 Isomorph rejection

Since the classical generalized quadrangles have automorphism groups that act transitively on the pairs of non-collinear points, every (partial) ovoid is equivalent to a (partial) ovoid containing a given pair of non-collinear points. Hence in the clique finding algorithm we can restrict the search to cliques containing a certain fixed edge. This reduces the search space with a factor of $O(vk')$, where $v = (s+1)(st+1)$ is the number of points of $\mathcal{S}$ and $k' = s^2t$ is the number of points not collinear with an arbitrary point. In some cases it is possible to fix even more pairwise adjacent vertices, e.g. for the generalized quadrangle $Q^-(5,q)$ 3 vertices can be fixed. This straightforward approach of fixing a certain number of vertices is already a very effective way to reduce the search space.

More advanced isomorph-rejection techniques, such as the techniques described in [22], allow to reduce the search space even further. Having determined in a step of the search process the set stabilizer of the current partial clique in the automorphism group of the quadrangle, it suffices to try only one point of each orbit of the stabilizer for extending the current partial clique in the next recursive steps instead of trying to add all vertices of the allowed set. An existing software package, such as *nauty* [12], can be used to compute the set stabilizer and its orbits.

2.3 Spread coloring

When using a coloring bound in a maximum clique algorithm, one is faced with the problem that determining the chromatic number of a graph is also an NP-hard problem. However, an upper bound for the chromatic number can also be used as a coloring bound. Hence an approximation algorithm, often a simple greedy algorithm, is used to obtain a reasonable upper bound for the chromatic number. But in the case of generalized quadrangles theoretical arguments lead to good – even optimal – colorings for some types of generalized quadrangles.

For instance, classical constructions for ovoids in $Q(4,q)$ are known [18]. Since $Q(4,q)$ is isomorphic to the dual of $W(q)$, the points of an ovoid in $Q(4,q)$ correspond to the lines of a spread in $W(q)$, hence to a partitioning of the vertices of $\overline{G_{W(q)}}$ into classes of pairwise non-adjacent vertices. In other words, this is a partitioning of $\overline{G_{W(q)}}$ into color classes which can be used for pruning in a maximum clique algorithm. It is obvious that this construction uses $st+1$ colors to color the graph. In the same way a coloring with $st+1$ colors for $\overline{G_{Q^-(5,q)}}$ can be obtained from classical constructions of ovoids in its dual $H(3,q^2)$ [18].

It is easy to see that the obtained vertex coloring is optimal, i.e. uses a minimum number of colors. This can be proven as follows. From the geometry of the generalized quadrangles it follows that the points of a line in the generalized quadrangle form a clique of maximum size in the collinearity graph G_S. Indeed, the points of a line are pairwise collinear and hence the corresponding vertices form a clique of size $s+1$, while a set of more than $s+1$ pairwise collinear points would require a triangle in the generalized quadrangle, which is not allowed. Hence a color class in the graph $\overline{G_S}$ has size at most $s+1$. Since there are $(s+1)(st+1)$ vertices, at least $st+1$ colors are needed to color $\overline{G_S}$. The coloring obtained as described above uses exactly this number of colors and thus is an optimal coloring.

2.4 Forcing vertices using look-ahead

In some situations the pruning in a clique finding algorithm in a collinearity graph can be improved by using the information about the incidence structure of the generalized quadrangle as well as the graph. For instance, when classifying the ovoids in a generalized quadrangle or when checking whether a generalized quadrangle has an ovoid, the following idea proves to be useful.

Consider a step in the recursive process where the current partial ovoid gives rise to a line for which only one point still belongs to the allowed set A. If that point is not added to the current partial ovoid, then the resulting partial ovoid can never be extended to an ovoid, so we can prune these possibilities and force the point to be added to the current partial ovoid.

2.5 Pruning based on span and trace properties

The following theorem describes a relationship between an ovoid and the trace and span of a regular pair of points in a generalized quadrangle of order s.

Theorem 2.1 ([18]) *Let $\mathcal{S}$ be a generalized quadrangle of order s, having a regular pair (x,y) of non-collinear points. If $\mathcal{O}$ is an ovoid of $\mathcal{S}$, then*

$|\mathcal{O} \cap \{x,y\}^{\perp\perp}|$, $|\mathcal{O} \cap \{x,y\}^{\perp}| \in \{0,2\}$, *and* $|\mathcal{O} \cap (\{x,y\}^{\perp} \cup \{x,y\}^{\perp\perp})| = 2$. *If the GQ $\mathcal{S}$ of order s, $s \neq 1$, contains an ovoid $\mathcal{O}$ and a regular point z not on $\mathcal{O}$, then s is even.*

Note that the classical generalized quadrangle $W(q)$, with q even, satisfies these conditions. Moreover, all points of $W(q)$, q even, are regular.

For a generalized quadrangle satisfying the conditions of Theorem 2.1, the following observations can be used to prune a recursive process classifying all ovoids. Consider a step in the recursive process. Let A denote the set of allowed points in this step. Let y denote the point which is added to the current partial ovoid in this step. Let $\mathcal{O}'$ denote the current partial ovoid in this step (after adding y), which will be completed to an ovoid $\mathcal{O}$. Then:

1.ST If there is a point $x \in \mathcal{O}'$, such that (x,y) is a regular pair, then we already have $|\mathcal{O}' \cap \{x,y\}^{\perp\perp}| = 2$ (since $x, y \in \mathcal{O}'$). Hence, no other points from $\{x,y\}^{\perp\perp}$ can be used to extend $\mathcal{O}'$ to an ovoid and we can prune these possibilities from A.

2.ST If there is a point $x \in A$, such that (x,y) is a regular pair, we proceed as follows. Since $y \in \mathcal{O}$, we have $|\mathcal{O} \cap \{x,y\}^{\perp}| = 0$, implicating by Theorem 2.1 that $|\mathcal{O} \cap \{x,y\}^{\perp\perp}| = 2$. Since $y \in \{x,y\}^{\perp\perp}$, two possibilities remain.

   (a) If $|\mathcal{O}' \cap \{x,y\}^{\perp\perp}| = 2$, then it can never be extended to an ovoid by points from $\{x,y\}^{\perp\perp}$. We can prune these possibilities from A.

   (b) Suppose now that $\mathcal{O}' \cup \{x,y\}^{\perp\perp} = \{y\}$. Let $A' = A \cap \{x,y\}^{\perp\perp}$. Assume $|A'| = 1$. If this single point in A' is not added to $\mathcal{O}'$, then the resulting partial ovoid can never be extended to an ovoid, so we can force the point to be added to $\mathcal{O}'$.

3.ST If there are two points $x, z \in A$, such that (x,z) is a regular pair of non-collinear points, then $|\mathcal{O} \cap \{x,z\}^{\perp\perp}| \in \{0,2\}$. We count now the points of $\mathcal{O}'$ contained in $\{x,z\}^{\perp\perp}$. There are three possible cases.

   (a) If $|\mathcal{O}' \cap \{x,z\}^{\perp\perp}| = 2$, then we conclude, as above, that $\mathcal{O}'$ can never be extended by points from $\{x,z\}^{\perp\perp}$, so we can prune these possibilities from A.

   (b) If $|\mathcal{O}' \cap \{x,z\}^{\perp\perp}| = 1$, then $|\mathcal{O} \cap \{x,z\}^{\perp\perp}| = 2$. Let $A' = A \cap \{x,z\}^{\perp\perp}$. As above, if $|A'| = 1$, then its single point must be forced to be added to $\mathcal{O}'$.

   (c) Suppose $|\mathcal{O}' \cap \{x,z\}^{\perp\perp}| = 0$. Denote $A_{\{x,z\}^{\perp}} = A \cap \{x,z\}^{\perp}$ and $A_{\{x,z\}^{\perp\perp}} = A \cap \{x,z\}^{\perp\perp}$. If $|A_{\{x,z\}^{\perp}}| = 1$, then $\mathcal{O}'$ can never be extended by the single point of $A_{\{x,z\}^{\perp}}$ and we can prune this possibility. If $|A_{\{x,z\}^{\perp}}| = 0$ and $|A_{\{x,z\}^{\perp\perp}}| = 2$, then the two points of $A_{\{x,z\}^{\perp\perp}}$ are forced to be added to the current partial ovoid.

3 Non-exhaustive search algorithms

3.1 Heuristic completion strategies

A simple greedy algorithm builds a maximal clique step by step by adding vertices from a set of allowed vertices until this set is empty.

Several strategies are possible for choosing a vertex to be added in each step. For instance, adding the vertex that leaves the largest number of vertices in the allowed set will tend to build large maximal cliques. A similar strategy, which is inspired by the pruning strategies using colorings, consists of adding the vertex that leaves the largest number of colors in the allowed set for the next step; this also results in large maximal cliques. On the other hand, choosing the vertex that leaves the least number of vertices or the least number of colors in the set of allowed vertices, is expected to result in small maximal cliques.

Starting from a maximal clique obtained by one of the above approaches, a simple restart strategy removes some of the vertices of the clique and again adds vertices until the clique is maximal. Both the removing and the adding can be done either randomly or following one of the above heuristics.

3.2 Limited discrepancy search

For cases where exhaustive search is computationally unfeasible, a technique called *limited discrepancy search* (LDS), introduced in [10], can be used to perform a partial exploration of the search space based on a heuristic. The main idea behind LDS is that only a few of the decisions made by the heuristic are "wrong" in the search for a largest (resp. smallest) clique. For a solution tree of height d there are only d ways in which the heuristic could make one wrong decision and $\binom{d}{k}$ ways it could make k. If k is small, a large (resp. small) clique can be found by systematically searching all paths in the solution tree that differ from the heuristic path in at most k decision points or *discrepancies*. LDS is a backtracking algorithm that searches the nodes of the solution tree in increasing order of such discrepancies.

4 Results

In this section we present results obtained by computer searches implementing the techniques described in Sections 2 and 3. All our programs are written in Java and call *nauty* [12] using the Java Native Interface (JNI) for the isomorph pruning. The timing results are obtained on a 1.6Ghz Pentium processor running Linux.

4.1 Effect of forcing vertices using look-ahead

In Tables 1 and 2 we illustrate the effect of forcing vertices when classifying all non-equivalent ovoids or spreads in a generalized quadrangle or when proving that no ovoid or spread exists. We give timings for versions of the program with and without isomorph pruning, and also versions of the program with and without a final filtering of the generated ovoids in order to obtain only the non-equivalent ovoids. For each quadrangle we also give the order $|G|$ of its collinearity graph, the running times of the different versions and the number $\#\mathcal{O}$ of ovoids obtained.

We present results for ovoids in the smallest cases of $Q(4,q)$ and $H(3,q^2)$, which are known to have ovoids, and for spreads in the case $H(4,4)$ (i.e. ovoids in $H(4,4)^D$), which are known not to exist [3].

Complete classifications of the ovoids of $Q(4,q)$ and $H(3,q^2)$ are not yet known in general. Recently Ball *et al* [1] proved that, for q prime, $Q(4,q)$ has a unique (up to equivalence) ovoid. For $Q(4,8)$ Penttila and Praeger [20] proved that there are 2 non-equivalent ovoids; an earlier computer classification was done by Fellegara [7] in 1962. Penttila and Royle [21] proved by a computer classification that $Q(4,9)$ has 2 non-equivalent ovoids. Ovoids in $Q(4,16)$ are elliptic quadrics, which was proved by O'Keefe and Penttila by means of a computer classification [14] and two years later without a computer [15]. In $Q(4,32)$ ovoids are either elliptic quadrics or Tits ovoids, a result obtained by O'Keefe, Penttila and Royle [16] with the aid of a computer. Recently Penttila [19] classified the ovoids in $H(3,9)$ by a computer search and found that there are 26 non-equivalent ovoids. The results presented in Tables 1 and 2 confirm these earlier results.

From the Tables it is clear that the approach of forcing vertices is an effective technique. Its effect is most notable when simply generating all ovoids without isomorph pruning and without final equivalence check, as can be seen in the first set of timing columns in Table 1. Of course, when a large number of ovoids are generated (such as for $Q(4,9)$ and $H(3,9)$), the final equivalence check will account for most of the total running time, as shown in the second set of columns of Table 1. When isomorph pruning is used, there is still a considerable gain in time when generating the ovoids, as can be seen in the first set of timing columns in Table 2. Again in some cases, e.g. $H(3,9)$, most of the total running time is spent in the final equivalence check, as can be seen in the second set of timing columns in Table 2.

GQ	$\|G\|$	No final equivalence check			With final equivalence check		
		Time			Time		
		No forcing	Forcing	#O	No forcing	Forcing	#O
$Q(4,7)$	400	13 s	1 s	21	16 s	4 s	1
$Q(4,8)$	585	1 722 s	93 s	532	1 865 s	275 s	2
$Q(4,9)$	820	43 355 s	529 s	14 796	58 858 s	16 390 s	2
$H(3,9)$	280	933 s	93 s	196 992	49 260 s	48 435 s	26
$H(4,4)^D$	297	75 s	4 s	0	–	–	–

Table 1: Effect of forcing vertices using look-ahead when searching for all ovoids. No isomorph pruning is done.

GQ	$\|G\|$	No final equivalence check			With final equivalence check		
		Time			Time		
		No forcing	Forcing	#O	No forcing	Forcing	#O
$Q(4,8)$	585	13 s	3 s	12	16 s	6 s	2
$Q(4,9)$	820	47 s	5 s	59	117 s	75 s	2
$Q(4,11)$	1464	/	1 725 s	5	/	1 752 s	1
$H(3,9)$	280	6.1 s	3.6 s	783	225 s	217 s	26
$H(4,4)^D$	297	665 ms	474 ms	0	–	–	–

Table 2: Effect of forcing vertices using look-ahead when searching for all ovoids. For the smaller cases isomorph pruning is done on 5 levels, for $Q(4,11)$ and $H(3,9)$ isomorph pruning is done on 7 levels.

4.2 Effect of span and trace pruning

In this section we present results of the three pruning techniques based on span and trace properties from Section 2.5, for ovoids in $W(q)$, for small q even. Note that $W(q) \cong Q(4,q)$ in these cases.

In Table 3 we compare the running time and number of recursive calls (#c) of the different versions (1.ST, 2.ST and 3.ST) as well as a combination of all three. For each quadrangle we also list the size $|G|$ of its collinearity graph. We conclude that, although the first technique is the least effective regarding the number of recursive calls, it is the best one when comparing the running time.

		1.ST		2.ST		3.ST		all together	
q	$\|G\|$	Time	#c	Time	#c	Time	#c	Time	#c
4	85	20 ms	7	20 ms	6	80 ms	7	80 ms	5
8	585	1.2 s	127	2.2 s	81	115s	66	80 s	47
16	4369	12 h	2 10^6	> 24 h					

Table 3: Effect of span and trace pruning when searching for all ovoids in $W(q)$ without final equivalence check. Isomorph pruning is done on 5 levels.

q	Iso. prun.	Final equiv. ch.	No ST	With 1.ST	$\#\mathcal{O}$
8	No	No	73 s	5.7 s	532
		Yes	180 s	110 s	2
	Yes	No	2.5 s	1.2 s	20
		Yes	6.7 s	5.6 s	2
16	Yes	No	> 4 days	12 h	8
		Yes	> 4 days	12.9 h	1

Table 4: Comparing timing results for $W(8)$ and $W(16)$.

In Table 4 we illustrate the effect of this first version (1.ST) when classifying all non-equivalent ovoids in a generalized quadrangle. We give timings for versions of the program with and without isomorph pruning, and also versions of the program with and without a final filtering of the generated ovoids in order to obtain only the non-equivalent ovoids. We also list the number $\#\mathcal{O}$ of ovoids obtained. We give comparisons only for two examples, $W(8)$ and $W(16)$. Note that the obtained results confirm the earlier results described in Section 4.1. The running time for $q < 8$ is too small to generalize. The generalized quadrangle $W(32)$ with its 33825 vertices is too large for our computer search. Nevertheless it is clear that the approach of span and trace pruning is an effective technique.

4.3 Largest maximal partial ovoids

In Table 5 we present some new results obtained by the exhaustive search algorithms and pruning techniques described in Section 2. For some generalized quadrangles which are known not to have ovoids we determined the size of the largest maximal partial ovoid, as well as a complete classification of all non-equivalent partial ovoids of that size. For the considered quadrangles theoretical upper bounds for the size of a partial ovoid are known (see [6] for a detailed discussion of these bounds), but in all cases the value found by our computer search improves on the best known theoretical bound.

For each quadrangle Table 5 lists the parameters (s,t), the order $|G|$ of the corresponding collinearity graph, the value of $st+1$ (which would be the size of the ovoid) and the value of the best known theoretical upper bound for the size of a maximal partial ovoid . We present the exact value for the size $|\mathcal{O}'|$ of the largest partial ovoid or spread found by the program, as well as the number $\#\mathcal{O}'$ of non-equivalent largest partial ovoids.

GQ	(s,t)	$\lvert G\rvert$	$st+1$	bound	$\lvert\mathcal{O}'\rvert$	$\#\mathcal{O}'$
$W(5)$	(5,5)	156	26	21 [23]	**18**	**2**
$W(7)$	(7,7)	400	50	43 [23]	**33**	**1**
$Q^-(5,4)$	(4,16)	325	65	37 [2]	**25**	**3**
$H(4,4)$	(4,8)	165	33	25 [13]	**21**	**1**
$H(4,4)^D$	(8,4)	297	33	32 [3]	**29**	**6**

Table 5: Largest maximal partial ovoids in some generalized quadrangles, obtained by exhaustive search.

4.4 Maximal partial ovoids and spreads in $H(4,q^2)$

In this section we present some new results, found by exhaustive and heuristic searches, for maximal partial ovoids and spreads in the Hermitian variety $H(4,q^2)$. As mentioned before, $H(4,q^2)$ is known to have no ovoids, while the question whether it has spreads remains open, except for the smallest case $H(4,4)$ [3].

For the size of a maximal partial ovoid in $H(4,q^2)$ there are theoretical upper bounds by Moorhouse and by Govaerts:

Theorem 4.1 (Moorhouse [13]) *If K is a k-cap of a Hermitian variety in* $\mathrm{PG}(n,q^2)$*, with $q=p^h$ and p prime, then*

$$k \leq \left[\binom{p+n-1}{n}^2 - \binom{p+n-2}{n}^2\right]^h + 1.$$

Theorem 4.2 (Govaerts [9]) *If O' is a partial ovoid of $H(4,q^2)$, then* $|O'| < q^5 - (4q-1)/3$.

A lower bound for the size of a maximal partial ovoid in $H(4,q^2)$ is given by Hirschfeld and Korchmáros:

Theorem 4.3 (Hirschfeld and Korchmáros [11]) *The size k of a complete cap of a Hermitian variety $\mathcal{U}_n$ in* $\mathrm{PG}(n,q^2)$ *satisfies* $k \geq q^2+1$.

GQ	$\|G\|$	LB	UB	Spectrum found
$H(4,4)$	165	5	25	**9,11..17,19,21**
$H(4,9)$	2440	10	201	**28,31,34..97,99..100,105**
$H(4,16)$	17425	17	577	**65,69,73,77,81,85..287,289**

Table 6: Spectrum of sizes for maximal partial ovoids of $H(4,q^2)$, for small values of q, obtained by exhaustive and/or heuristic search.

GQ	$\|G\|$	$st+1$	Spectrum found
$H(4,4)$	297	33	**11,15..29**
$H(4,9)$	6832	244	**86,88..162**
$H(4,16)$	66625	1025	**303,307..494**

Table 7: Spectrum of sizes for maximal partial spreads of $H(4,q^2)$, for small values of q, obtained by exhaustive and/or heuristic search.

Hirschfeld and Korchmáros also construct a complete cap of size q^3+1 for any q, which is currently the smallest known complete cap of the Hermitian variety:

Theorem 4.4 (Hirschfeld and Korchmáros [11]) *Let α be a plane of $\mathrm{PG}(n,q^2)$ which meets the Hermitian variety $\mathcal{U}_n$ in an non-degenerate Hermitian curve $\mathcal{U}_2$. Then $\mathcal{U}_2$ is a complete cap of $\mathcal{U}_n$ of size q^3+1.*

To the best of our knowledge, no theoretical bounds for the size of maximal partial spreads in $H(4,q^2)$ are known.

In Table 6 we give results for maximal partial ovoids in $H(4,q^2)$, while Table 7 gives results for maximal partial spreads in $H(4,q^2)$. Recall that a (partial) spread in $H(4,q^2)$ is a (partial) ovoid in $H(4,q^2)^D$. For each value of q we give the order $|G|$ of the corresponding collinearity graph. Also listed are the values of the best known lower (LB) and upper (UB) bound for the size of maximal partial ovoids in $H(4,q^2)$, resp. the value of $st+1$ which would be the size of a spread in $H(4,q^2)$. Finally the last column lists the sizes for which our program found maximal partial ovoids, resp. spreads, of that given size. The notation $a..b$ means that for all values in the interval $[a,b]$ a maximal partial ovoid or spread of that size has been found.

For $H(4,4)$ the largest values found are indeed the size of the largest maximal partial ovoid, resp. spread; this was confirmed by exhaustive search, as described above. Exhaustive search also shows that the maximal partial ovoid of size 21 in $H(4,4)$ is unique up to equivalence, while $H(4,4)$ has 6 non-equivalent maximal partial spreads of size 29.

Moreover, for maximal partial ovoids in $H(4,4)$, we confirmed by exhaustive search that the spectrum found is complete, i.e. exhaustive search confirmed that no maximal partial ovoids with size less than 9, or with sizes 10, 18 or 20 exist.

For maximal partial spreads in $H(4,4)$, we confirmed by exhaustive search that no maximal partial spreads with size less than 11 exist; for sizes 12, 13, 14 it remains open whether such maximal partial spreads exist.

Our searches confirm the existence of maximal partial ovoids of size $q^3 + 1$ in $H(4, q^2)$, for small q. No maximal partial ovoids with size smaller than $q^3 + 1$ were found, and for $q = 2$ exhaustive search excludes the existence of maximal partial ovoids with size smaller than $q^3 + 1 = 9$. We also observe the existence of maximal partial ovoids of size $q^3 + 1 + iq$ for small values of $i \geq 1$.

For results on other generalized quadrangles, obtained by exhaustive and heuristic searches, we refer to [5] and [6].

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