Almost Resolvable 4-Cycle Systems

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Abstract

A 4-cycle system of order n is said to be almost resolvable provided its 4-cycles can be partitioned into $(n-1)/2$ almost parallel classes ($= (n-1)/4$ vertex disjoint 4-cycles) and a half parallel class ($= (n-1)/8$ vertex disjoint 4-cycles.) We construct an almost resolvable 4-cycle system of every order $n \equiv 1 \pmod 8$ except 9 (for which no such system exists) and possibly 33, 41 and 57.

1 Introduction

A *Steiner triple system* (or triple system) of order n is a pair (S, T), where T is a collection of triangles (or triples) which partitions the edge set of K_n

with vertex set S. It is well-known that the spectrum for triple systems is the set of all $n \equiv 1$ or 3 (mod 6) [2].

If $n \equiv 3$ (mod 6), a *parallel class* is a set of $n/3$ vertex disjoint triples (which necessarily partition S). A *Kirkman triple* system is a triple system of order $n \equiv 3$ (mod 6) whose triples can be *partitioned* into parallel classes and it is well-known that the spectrum for Kirkman triple systems is precisely the set of all $n \equiv 3$ (mod 6) [4].

Clearly a triple system of order $n \equiv 1$ (mod 6) cannot contain a parallel class. The best one can hope for is an *almost parallel class*; $(n-1)/3$ pairwise disjoint triples. A *Hanani triple system* is a triple system of order $n \equiv 1$ (mod 6) whose triples can be partitioned into $(n-1)/2$ almost parallel classes and a single half parallel class consisting of $(n-1)/6$ triples. The spectrum for Hanani triple systems is the set of all $n \equiv 1$ (mod 6), $n \neq 7$ or 13 [5].

A 4-*cycle system* of order n is a pair (X, C), where C is a collection of 4-cycles which partitions the edge set of K_n with vertex set X. It is a well-known Folk Theorem that the spectrum for 4-cycle systems is precisely the set of all $n \equiv 1$ (mod 8) and that if (X, C) is a 4-cycle system of order n, $|C| = n(n-1)/8$. Since the order of a 4-cycle system is 1 (mod 8) it is not possible for a 4-cycle system to contain a parallel class. However, an almost parallel class is possible; i.e., $(n-1)/4$ vertex disjoint 4-cycles. The analogue of a Hanani triple system is also possible; i.e., the partition of the $n(n-1)/8$ 4-cycles into $(n-1)/2$ almost parallel classes and a half parallel class consisting of $(n-1)/8$ 4-cycles. The object of this paper is the construction of an almost resolvable 4-cycle system of every order $n \equiv 1$ (mod 8) ≥ 17 ($n = 9$ is not possible), with the three possible exceptions of 33, 41, and 57.

2 Two examples

The following two examples are crucial for the recursive constructions in sections (3) and (5).

Example 2.1 (almost resolvable 4-cycle system of order 17)
(2, 5, 12, 9)(3, 6, 13, 10)(4, 7, 14, 11)(14, 1, 8, 16)
(4, 5, 13, 14)(8, 11, 12, 15)(6, 9, 16, 7)(2, 10, 0, 3)
(10, 11, 2, 1)(13, 4, 12, 3)(7, 15, 5, 8)(16, 6, 14, 0)
(14, 15, 6, 5)(12, 13, 16, 2)(3, 11, 1, 4)(9, 0, 7, 1)
(3, 15, 9, 14)(7, 2, 13, 1)(11, 6, 0, 5)(4, 16, 10, 8)
(11, 5, 13, 0)(10, 5, 9, 4)(2, 14, 8, 6)(3, 1, 12, 7)
(1, 5, 3, 16)(13, 11, 9, 7)(6, 4, 15, 10)(8, 2, 0, 12)
(0, 4, 2, 15)(9, 13, 8, 3)(7, 5, 16, 11)(14, 12, 6, 1)
(0, 1, 9, 8)(12, 16, 14, 1).

Example 2.2 (almost resolvable 4-cycle system of order 25)
(0,12,18,6)(24,1,4,14)(7,20,21,2)(3,19,17,11)(9,13,23,5)(8,22,15,10)
(24,0,3,13)(6,19,20,1)(2,17,8,23)(4,12,16,5)(7,21,14,9)(10,18,22,11)
(24,9,0,22)(5,21,19,1)(2,14,20,8)(3,16,17,10),(4,18,23,6)(11,15,13,7)
(0,15,6,21)(24,10,1,23)(2,22,14,3)(4,17,18,11)(5,19,12,7)(8,16,20,9)
(0,13,14,7)(1,15,20,3)(2,18,16,10)(8,12,22,4)(5,17,23,11)(24,6,9,19)
(5,18,19,0)(1,16,7,22)(24,11,2,12)(3,23,15,4)(6,20,13,8)(9,17,21,10)
(0,14,19,2)(1,17,15,9)(7,23,21,3)(4,16,22,10)(24,5,8,18),(11,12,13,6)
(10,12,17,0)(1,21,13,2)(24,3,6,16)(9,22,23,4)(5,20,11,14)(7,15,19,8)
(0,16,14,8)(11,13,18,1)(6,22,20,2)(3,15,21,9)(24,4,7,17)(10,23,12,5)
(4,20,18,0)(1,13,19,7)(24,2,5,15)(8,21,22,3)(10,14,12,6)(9,23,16,11)
(0,20,12,1)(2,15,16,9)(3,17,22,5)(4,19,10,13)(6,14,18,7)(24,8,11,21)
(11,19,23,0)(1,14,15,8)(2,16,21,4)(3,18,9,12)(5,13,17,6)(24,7,10,20)
(12,15,18,21)(13,16,19,22)(14,17,20,23)

3 The $16k+1 \geq 49$ Construction

Let $n = 16k+1 \geq 49$ and $(Q, \circ)$ a commutative quasigroup of order $2k \geq 6$ with holes $H = \{h_1, h_2, h_3, \ldots, h_k\}$ of size 2. (See [3] for example.) Set $S = \{\infty\} \cup (Q \times Z_8)$ and define a collection of 4-cycles C as follows:

(1) For each hole $h_i \in H$, let $\{\infty\} \cup (h_i \times Z_8, C(h_i))$ be an almost resolvable 4-cycle system of order 17 (Example 2.1) and place these 4-cycles in C. (We can do this so that the half parallel class does *not* contain ∞.)

(2) For each a and b belonging to different holes of H, let $(K_{8,8}, b(a,b))$ be a partition of $K_{8,8}$, with parts $\{a\} \times Z_8$ and $\{b\} \times Z_8$, into 4-cycles and place these 4-cycles in C. We can partition $b(a,b)$ into four parallel classes $b_1(a,b), b_2(a,b), b_3(a,b)$ and $b_4(a,b)$.

It is straightforward to see that (S, C) is a 4-cycle systems of order $16k+1$.

Resolution: If $z \in h_i$ let $\pi(z) = \{\{a,b\} \mid \{a \neq b\} \cap h_i = \emptyset$ and $a \circ b = b \circ a = z\}$. Then $\{b_j(a,b) \mid (a,b) \in \pi(z)\}$, $j \in \{1,2,3,4\}$, partitions $S \backslash (\{\infty\} \cup (h_i \times Z_8))$ into 4 parallel classes. Pairing these up with 4 almost parallel classes in $(\{\infty\} \cup (h_i \times Z_8, C(h_i))$ produces 4 almost parallel classes of C. If $h_i = \{x, y\}$, choosing disjoint sets of 4 almost parallel classes in $(\{\infty\} \cup (h_i \times Z_9, C(h_i))$ gives 8 almost parallel classes of C. Running over all holes in H produces $8k$ almost parallel classes, and of course the half parallel classes in each $C(h_i)$ (remember that none of the half parallel classes contains ∞) can be pieced together to form a half parallel class of C.

Lemma 3.1 *There exists an almost resolvable* 4*-cycle system of every order* $16k+1$ *except possibly* 33.

Proof The above construction plus Example 2.1. □

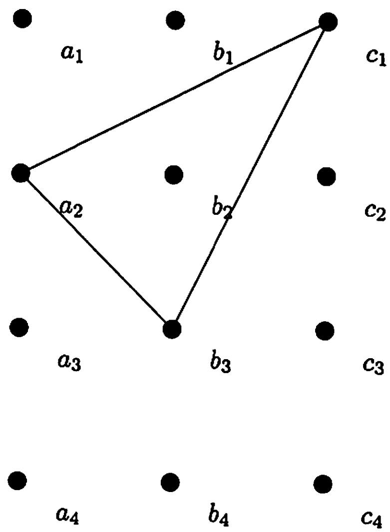
4 Commutative quasigroups with holes

In this section we will construct commutative quasigroups with holes which are necessary for the $16k+9$ Construction in Section 6. (These quasigroups are of interest in their own right.) To this end we collect a few (mostly) well-known ideas and results.

(1) There exists a group divisible design (GDD) with 1 group of size $x \in \{6, 26\}$, u groups of size 4 ($x^1 4^u$), and all blocks of size 3 if and only if (a) $x = 6$, $u \equiv 0$ or 1 (mod 3), and $u \geq 3$; and (b) $x = 26$, $u \equiv 0$ (mod 3) and $u \geq 9$ [1].

(2) If there exists a *GDD* of type $x^1 4^u$ with all blocks of odd size, then there exists a *commutative* quasigroup with 1 hole of size x and u holes of size 4. (Define an idempotent commutative quasigroup on each block.)

(3) A *grid* is a pair $(\{a,b,c\} \times \{1,2,3,4\}, G)$, where $G = \{\{(a,1),(b,2),(c,3)\}, \{(a,1),(b,3),(c,4)\}, \{(a,1),(b,4),(c,2)\}, \{(a,2),(b,1),(c,4)\}, \{(a,2),(b,3),(c,1)\}, \{(a,2),(b,4),(c,3)\}, \{(a,3),(b,1),(c,2)\}, \{(a,3),(b,2),(c,4)\}, \{(a,3),(b,4),(c,1)\}, \{(a,4),(b,3),(c,2)\}, \{(a,4),(b,1),(c,3)\}, \{(a,4),(b,2),(c,1)\}\}$.



Note that these triples do not contain any "vertical" or "horizontal" edges.

Lemma 4.1 *There exists a commutative quasigroup of order* $4t+2$ *with 1 hole of size* 6 *and* $t-1$ *holes of size* 4 *for* $t = 1$ *and all* $t \geq 4$.

Proof We will break the proof into 3 parts: $t \equiv 1, 2, 4,$ or 5 (mod 6); $t \equiv 3$ (mod 6); and $t \equiv 0$ (mod 6).

$t \equiv 1, 2, 4,$ or 5 (mod 6). This follows immediately from (1a) and (2) above by letting $u = t-1$.

$t \equiv 3 \pmod 6$. Let $X = \{\infty_1, \infty_2\} \cup (\{1, 2, \ldots, t\} \times \{1, 2, 3, 4\})$ and define a $GDD(X, G, B)$ as follows:

$G = \{\{\infty_1, \infty_2, (1,1), (1,2), (1,3), (1,4)\}\} \cup \{\{(i,1), (i,2), (i,3), (i,4)\} \mid i \in 2, 3, \ldots, t\}\}$.

Now let $(\{\infty_1, \infty_2\} \cup (\{1, 2, \ldots, t\} \times \{i\}), B_i)$ be a *PBD* of order 5 (mod 6) with one block of size 5 and the remaining blocks of size 3 (see [3] for example) with the proviso that $\{\infty, \infty_2, (1,i)\} \in B_i$. Put the blocks of $B_i \backslash \{\infty_1, \infty_2, (1,i)\}$ in B. Let $(\{1, 2, \ldots, t\}, T)$ be a Steiner triple system of order t. For each triple $\{a, b, c\} \in T$, put the 12 triples of the grid in (3) in B.

Then (X, G, B) is a *GDD* of type $6^1 4^{t-1}$ with blocks of size 3 and 5. The statement of the lemma follows from (2).

$t \equiv 0 \pmod 6$. When $t = 6$ we have the following example.

				23	24	17	18	13	14	21	22	19	20	25	11	5	6	26	9	15	16	7	8	10	12
				24	23	18	17	14	13	22	21	20	19	12	25	6	5	10	26	16	15	8	7	9	11
				9	10	21	22	23	24	17	18	26	8	5	6	25	16	11	12	13	14	19	20	15	7
				10	9	22	21	24	23	18	17	7	26	6	5	15	25	12	11	14	13	20	19	16	8
23	24	9	10					25	20	3	4	21	22	1	2	26	14	15	16	17	18	11	12	19	13
24	23	10	9					19	25	4	3	22	21	2	1	13	26	16	15	18	17	12	11	20	14
17	18	21	22					15	16	26	1	9	10	23	24	11	12	25	3	19	20	13	14	2	4
18	17	22	21					16	15	2	26	10	9	24	23	12	11	4	25	20	19	14	13	1	3
13	14	23	24	25	19	15	16					3	4	26	17	7	8	21	22	5	6	1	2	18	20
14	13	24	23	20	25	16	15					4	3	18	26	8	7	22	21	6	5	2	1	17	19
21	22	17	18	3	4	26	2					25	6	19	20	23	24	13	14	7	8	15	16	5	1
22	21	18	17	4	3	1	26					5	25	20	19	24	23	14	13	8	7	16	15	6	2
19	20	26	7	21	22	9	10	3	4	25	5					1	2	23	24	11	12	17	18	8	6
20	19	8	26	22	21	10	9	4	3	6	25					2	1	24	23	12	11	18	17	7	5
25	12	5	6	1	2	23	24	26	18	19	20					21	22	7	8	9	10	3	4	11	17
11	25	6	5	2	1	24	23	17	26	20	19					22	21	8	7	10	9	4	3	12	18
5	6	25	15	26	13	11	12	7	8	23	24	1	2	21	22					3	4	9	10	14	16
6	5	16	25	14	26	12	11	8	7	24	23	2	1	22	21					4	3	10	9	13	15
26	10	11	12	15	16	25	4	21	22	13	14	23	24	7	8					1	2	5	6	3	9
9	26	12	11	16	15	3	25	22	21	14	13	24	23	8	7					2	1	6	5	4	10
15	16	13	14	17	18	19	20	5	6	7	8	11	12	9	10	3	4	1	2						
16	15	14	13	18	17	20	19	6	5	8	7	12	11	10	9	4	3	2	1						
7	8	19	20	11	12	13	14	1	2	15	16	17	18	3	4	9	10	5	6						
8	7	20	19	12	11	14	13	2	1	16	15	18	17	4	3	10	9	6	5						
10	9	15	16	19	20	2	1	18	17	5	6	8	7	11	12	14	13	3	4						
12	11	7	8	13	14	4	3	20	19	1	2	6	5	17	18	16	15	9	10						

When $t \geq 18$ there exists a *GDD* of type $26^1 4^{t-6}$ with all blocks of size 3 (1*b*). This gives a commutative quasigroup with one hole of size 26 and $t - 6$ holes of size 4. Filling in the hole of size 26 with the above example gives a commutative quasigroup with one hole of sizer 6 and the remaining $t - 1$ holes of size 4.

This leaves only the case where $t = 12$. So, let $X = \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6\} \cup (\{1, 2, 3, \ldots, 11\} \times \{1, 2, 3, 4\})$ and define a *GDD* (X, G, B) of type $6^1 4^{11}$ as follows:

$$G = \{\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6\}\} \cup \{\{(i,1),(i,2),(i,3),(i,4)\} \mid i \in \{1,2,3,\ldots,11\}\}.$$

Define $\{1,2,3,\ldots,11\} \times \{i\}, i \in \{1,2,3,4\}$, to be blocks and place these blocks in B. The remaining blocks in B are the following: $\{\{\infty_1,(1,i),(2,i+9)\}, \{\infty_1,(3,i),(4,i+5)\}, \{\infty_2,(1,i),(2,i+10), \{\infty_2,(3,i),(4,i+10)\}, \{\infty_3,(1,i),(3,i+2)\}, \{\infty_3,(2,i),(4,i+1)\}, \{\infty_4,(1,i),(3,i+4)\}, \{\infty_4,(2,i),(4,i+10)\}, \{\infty_5,(1,i),(4,i+6)\}, \{\infty_5,(2,i),(3,i+9)\}, \{\infty_6,(1,i),(4,i+10)\}, \{\infty_6,(2,i),(3,i+10)\} \mid i \in Z_{11}\} \cup \{\{(1,i),(2,i+j),(3,i+2j)\} \mid j \in \{5,6,7,8\}, i \in Z_{11}\} \cup \{\{(2,i),(3,i+j),(4,i+2j)\} \mid j \in \{1,2,3,4\}, i \in Z_{11}\} \cup \{\{(1,i),(3,i+j),(4,i+2j)\} \mid j \in \{6,7,8,9\}, i \in Z_{11}\} \cup \{\{(1,i),(2,i+1),(4,i+4)\}, \{(1,i),(2,i+2),(4,i+9)\}, \{(1,i),(2,i+3),(4,i+8)\}, \{(1,i),(2,i+4),(4,i+2) \mid i \in Z_{11}\}$. The statement of the lemma follows from (2).

Combining all of the above cases completes the proof. □

5 $n = 9$

Lemma 5.1 *There does not exist an almost resolvable* 4*-cycle system of order* 9.

Proof Suppose there exists an almost resolvable 4-cycle system of order 9 on the vertex set $\{1,2,\ldots,9\}$. We can assume one almost parallel class is $\pi_1 = \{(1,3,2,4),(5,7,6,8)\}$. Vertex 9 must appear in each of the 3 remaining almost parallel classes π_2, π_3 and π_4, and in the half parallel class π_5. We can assume that the half parallel class is

(1) $(9,5,1,6)$,

(2) $(9,5,1,8)$, or

(3) $(9,2,1,5)$.

(π_5 either contains an edge joining two vertices in the same 4-cycle in π_1 (Case (3)) or π_5 doesn't use such an edge and has two vertices which are non-adjacent or adjacent (Cases (1) and (2) respectively) in the second cycle in π_1).

We consider each case in turn. Note that the vertices missing from the almost parallel classes are the vertices in π_5. We refer to the edges $\{1,2\}, \{3,4\}, \{5,6\}$, and $\{7,8\}$ as pure edges. It is useful to note that each pure edge must be in a 4-cycle containing either 9 or another pure edge.

Case (1) $\pi_5 = (9,5,1,6)$.

The edge $\{5,6\}$ must be in a 4-cycle in, say, π_2 avoiding 9, so (by renaming vertices 3 and 4 if necessary) we can assume it is $(5,6,4,3)$ since it must contain another pure edge. Since π_2 must avoid vertex 1, we can assume (by renaming 7 and 8 if necessary) $\pi_2 = \{5,6,4,3),(2,8,7,9)\}$ or $\{(5,6,4,3),(7,2,8,9)\}$. In the former case, only 1 pure edge $\{1,2\}$ remains, so must be in π_3 in a 4-cycle that includes 9, but no such 4-cycle exists (the only remaining neighbor of 2 are 5, 6 and 7). In the latter case no 4-cycle remains containing the pure edge $\{7,8\}$.

Case 2 $\pi_5 = (9,5,1,8)$

Since this case has 4 pure edges to allocate to π_2, π_3 and π_4, we can assume π_2 has 2 pure edges; by symmetry we consider just the cases where π_2 contains $(1,2,5,6)$ and $(3,4,7,8)$ in turn.

In the first case, $\pi_2 = \{(1,2,5,6),(3,7,4,9)\}$ or $\{(1,2,5,6),(7,3,4,9)\}$ since it must miss 8 (and 3 and 4 can be interchanged). In the former situation, the only 4-cycle containing 1 is $(1,7,2,9)$, so $\pi_3 = \{(1,7,2,9),(3,6,4,8)\}$ (since it must avoid 5), leaving $\pi_4 = \{(3,4,5),(6,2,8,7,9)\}$. In the latter case, $(1,7,2,9)$ is the only 4-cycle left containing 1, so $\pi_3 = \{(1,7,2,9),(3,6,4,8)\}$, leaving $\pi_4 = \{(6,2,8,7,4,5,3,9)\}$.

In the second case $\pi_2 = \{(3,4,7,8),(2,5,6,9)$ or $\{(3,4,7,8),(2,1,6,9)\}$ or $\{(3,4,7,8),(1,2,6,9)\}$ or $\{(3,4,7,8),(1,6,2,9)\}$. In the first case, the only 4-cycle containing vertex 8 is $(2,6,4,8)$, so $\pi_3 = \{(2,6,4,8\},(1,7,3,9)\}$, leaving $\pi_4 = \{(4,5,3,6,1,2,7,9)\}$. In the second case, the only 4-cycle containing 1 is $(1,7,3,9)$, so $\pi_3 = \{(1,7,3,9),(2,5,4,6)\}$ (since it must miss 8 and include 5), leaving $\pi_4 = \{(3,5,6),(4,8,2,7,9)\}$. In the third case the only 4-cycle containing 1 is $(1,6,3,7)$, so $\pi_3 = \{(1,6,3,7),(2,5,4,9)\}$, leaving $\pi_4 = \{(3,5,6,4,8,2,7,9)\}$. In the last case, the only 4-cycle containing 8 is $(8,2,5,4)$, so $\pi_3 = \{(8,2,5,4),(6,3,7,9)\}$, leaving $\pi_4 = \{(4,6,5,3,9),(1,2,7)\}$.

Case 3 $\pi_5 = (9,2,1,5)$

The edge $\{2,5\}$ must be in a 4-cycle avoiding 9; we can assume it is the 4-cycle $(2,5,3,7)$ or $(2,5,3,6) \in \pi_2$. Since π_2 must miss 1, $\pi_2 = \{(2,5,3,7),(6,4,8,9)\}$, $\{(2,5,3,6),(4,8,7,9)\}$ or $\{(2,5,3,6),(7,4,8,9)\}$. In the first case, the only remaining 4-cycle containing 5 is $(3,4,5,6)$; completing this to an almost parallel class avoiding vertex 2 requires $\pi_3 = \{(3,4,5,6),(1,8,7,9)\}$. But the remaining edges form the 8-cycle $(1,6,2,8,3,9,4,7)$. In the second case, no 4-cycle remains containing 5. In the last case, the only 4-cycle containing 5 is $(4,5,6,9)$, so $\pi_3 = \{(4,5,6,9),(1,7,3,8)\}$ (since it avoids 1), leaving $\pi_4 = \{(1,6,4,3,9),(2,7,8)\}$. □

6 The $16k + 9 \geq 73$ Construction

In this section we show, except possibly for 41 and 57, the existence of an almost resolvable 4-cycle system of every order $n \equiv 9 \pmod 6$ except for $n = 9$ for which no such system exists.

The $16k + 9 \geq 73$ Construction. This construction is identical to the $16k + 1$ Construction except for the use of quasigroups with holes of size 6 and 4 instead of holes of size 2.

Let $n = 16k + 9 \geq 73$ and let $(Q, \circ)$ be a commutative quasigroup of order $4k + 2 \geq 18$ with holes $H = \{h_1^*, h_2, h_3, \cdots, h_k\}$, where h_1^* is a hole of size 6 and the remaining holes have size 4. (Section 4.) Let $S = \{\infty\} \cup (Q \times Z_4)$ and define a collection of 4-cycles C as follows:

(1) Let $(\{\infty\} \cup (h_1^* \times Z_4), C(h_1^*))$ be an almost resolvable 4-cycle system of order 25 (Example 2.2) and place these 4-cycles in C. (We can do this so that the half parallel class does *not* contain ∞.)

(2) For each $i \geq 2$, let $(\{\infty\} \cup (h_i \times Z_4), C(h_i))$ be an almost resolvable 4-cycle system of order 16 (Example 2.1) and place these 4-cycles in C. (We can do this so that the half parallel class does *not* contain ∞.)

(3) For each a and b belonging to different holes of H, let $(K_{4,4}, b(a, b))$ be a partition of $K_{4,4}$ with parts $\{a\} \times Z_4$ and $\{b\} \times Z_4$ into 4-cycles and place these 4-cycles in C. We can partition $b(a, b)$ into 2 parallel classes $b_1(a, b)$ and $b_2(a, b)$.

Then (S, C) is a 4-cycle system of order $16k + 9$.

Resolution: For each $z \in h_1^*$ (see Section 3) $\{b_i(a, b) \mid (a, b) \in \pi(z)\}$, $i \in \{1, 2\}$, partitions $S \backslash (\{\infty\} \cup (h_1^* \times Z_4))$ into 2 parallel classes. Pairing these up with 2 almost parallel classes in $(\{\infty\} \cup (h_1^* \times Z_4), C(h_1^*))$ produces 2 almost parallel classes of C. Since $|h_1^*| = 6$, choosing disjoint sets of 2 almost parallel classes in $(\{\infty\} \cup (h_1^* \times Z_4), C(h_1^*))$ gives 12 almost parallel classes of C. An analogous argument for each of the holes of size 4 gives 8 almost parallel classes of C. This gives a total of $12 + 8(k - 1) = 8k + 4$ almost parallel classes. Since none of the half parallel classes in the holes contain ∞ they can be pieced together to form a half parallel class of C. We have the following lemma.

Lemma 6.1 *There exists an almost resolvable 4-cycle system of every order $16k + 9$ except 9 (for which no such system exists) and possibly 41 and 57.* □

7 Summary

Combining Lemmas 3.1, 5.1 and 6.1 gives the following theorem.

Theorem 7.1 *There exists an almost resolvable 4-cycle system of every order* $n \equiv 1 \pmod 8 \geq 17$, *except possibly for* 33, 41, *and* 57. *There does not exist an almost resolvable 4-cycle system of order* 9. □

The problem of constructing almost resolvable 4-cycle systems of orders 33, 41, and 57 seems a difficult problem. The authors have struggled valiantly with this problem, so far without success.

References

[1] C. J. Colbourn, D. G. Hoffman, and R. S. Rees, A new class of group divisible designs with block size 3, J. Combin. Theory (A), 59 (1992), 73-89.

[2] T. P. Kirkman, *On a problem in combinations*, Cambridge and Dublin Math. J., 2 (1847), 191-204.

[3] C. C. Lindner and C. A. Rodger, *Design Theory*, CRC Press, (1997), 208 pages.

[4] D. K. Ray-Chaudhuri and R. M. Wilson, *Solution of Kirkman's Schoolgirl problem*, Proc. Symp. Pure Math., 19, Amer. Math. Soc., Providence, RI, (1971), 187-203.

[5] S. A. Vanstone, D. R. Stinson, P. J. Schellenberg, A. Rosa, R. Rees, C. J. Colbourn, M. J. Carter, and J. E. Carter, *Hanani triple systems*, Israel J. Math., 83 (1993), 305-319.