On the Covering Numbers $C_2(v, k, t)$, $t \geq 3$

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Abstract

A t-(v,k,λ) covering is a set of blocks of size k such that every t-subset of a set of v points is contained in at least λ blocks. The cardinality of the set of blocks is the size of the covering. The covering number $C_\lambda(v,k,t)$ is the minimum size of a t-(v,k,λ) covering. In this article we find upper bounds on the size of t-$(v,k,2)$ coverings for $t = 3,4$, $k = 5,6$ and $v \leq 18$. Twelve of these bounds are the exact covering numbers.

1 Introduction

A natural upper bound on the covering number $C_\lambda(v,k,t)$ is the following.

$$C_\lambda(v,k,t) \leq \lambda C(v,k,t).$$

It can be easily obtained by taking λ copies of a (v,k,t) covering. In this paper we are naturally interested in obtaining an upper bound on $C_\lambda(v,k,t)$ smaller than $\lambda C(v,k,t)$ in the case when $C(v,k,t)$ is known, and smaller than λu, where u is the best known upper bound on the covering number $C(v,k,t)$, otherwise. All of the coverings found in this paper meet these conditions. References for the best upper bounds on $C(v,k,t)$ are supplied in the proofs. The choice of parameters of interest is justified by the fact that the covering numbers $C_2(v,4,2)$ and $C_2(v,5,2)$ are known [8].

A general lower bound on $C_\lambda(v,k,t)$ is due to Schönheim [11].

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Theorem 1.1

$$C_\lambda(v,k,t) \geq \left\lceil \frac{v}{k} C_\lambda(v-1,k-1,t-1) \right\rceil .$$

By iterating the inequality of Theorem 1.1 we obtain the following.

Corollary 1.2

$$C_\lambda(v,k,t) \geq \left\lceil \frac{v}{k} \left\lceil \frac{v-1}{k-1} \cdots \left\lceil \frac{v-t+1}{k-t+1} \lambda \right\rceil \cdots \right\rceil \right\rceil .$$

Let $D = \{B_1, B_2, ..., B_b\}$ be a collection of k-subsets (called **blocks**) of a v-set $X(v) = \{1, 2, ..., v\}$ (with elements called **points**). Then D is a t-(v,k,λ) **design** if every t-subset of $X(v)$ is contained in exactly λ blocks of D. Clearly, a t-(v,k,λ) design is a t-(v,k,λ) covering of minimum size.

A well-known necessary condition for the existence of a t-(v,k,λ) design D is that the λ_q, $1 \leq q \leq t$, defined by

$$\lambda_0 := b = |D|, \qquad \text{and} \qquad \lambda_q = \frac{k-q+1}{v-q+1} \lambda_{q-1},$$

be integers. Obviously, $\lambda_t = \lambda$ and the number of the blocks of the design is

$$\lambda_0 = b = \frac{v(v-1)...(v-t+1)}{k(k-1)...(k-t+1)} \lambda.$$

Let $D = \{B_1, B_2, ..., B_b\}$ be a t-(v,k,λ) design. It is known [6] that $D_s = \{X(v) \setminus B : B \in D\}$ is a t-$\left(v, v-k, \lambda \binom{v-k}{t} \big/ \binom{k}{t}\right)$ design called the **supplemental design** of D. Given a t-(v,k,λ) design D and a point x, the blocks of D that contain x form a $(t-1)$-$(v-1, k-1, \lambda)$ design on $X \backslash \{x\}$ called the **derived design of D with respect to x**. The blocks of D that do not contain x form a $(t-1)$-$(v-1, k, \lambda_{t-1} - \lambda_t)$ design on $X \backslash \{x\}$ called the **residual design of D with respect to x**.

The next construction is the covering counterpart of a derived design.

Construction 1.3 *Given a t-(v,k,λ) covering design D and a point $x \in X(v)$, the blocks of D that contain x form a $(t-1)$-$(v-1,k-1,\lambda)$ covering on $X(v) \setminus \{x\}$.*

A covering of the smallest size among those obtainable from D is produced by choosing x to be a point that occurs in the fewest blocks of D.

The set of all k-subsets of $X(v)$ will be denoted by $X^{(k)}(v)$. (We will use $X^{(k)}$ instead of $X^{(k)}(v)$ whenever the value of v is clear from the context.) Consider the set $X^{(s)}(v) = X^{(s)}$, where $t+1 \leq s \leq \lfloor \frac{v}{2} \rfloor$. The **intersection numbers of an s-subset S of $X(v)$ with respect to the blocks of a t-(v, k, λ) design D** are defined by

$$n_i = n_i(S) = |\{B : B \in D, |B \cap S| = i\}|, \; i = 0, 1, ..., s.$$

So n_i is the number of blocks of D that intersect S in i points. The **intersection equations for S** are given by

$$\sum_{i=m}^{s} \binom{i}{m} n_i = \binom{s}{m} \lambda_m \qquad \text{for} \qquad m = 0, 1, ..., \min(s, t) \;\; [12].$$

The **spectrum of $A \in X^{(s)}$ under D** is the ordered $(m-t)$-tuple

$$Spec_D(A) = (n_{t+1}, n_{t+2}, ..., n_m),$$

where $m = min\{k, s\}$ and n_i, $i = t+1, ..., m$, are the intersection numbers of A with respect to the blocks of the design D.

The **spectral set of $X^{(s)}$ under D** is the collection of all possible spectra of the elements of $X^{(s)}$ under D.

The equivalence relation $\Re$ on $X^{(s)}$ is defined by $A_i \Re A_j$ if and only if $Spec_D(A_i) = Spec_D(A_j)$. Therefore $\Re$ partitions $X^{(s)}$ into equivalence classes $X_1^{(s)}, X_2^{(s)}, ..., X_q^{(s)}$ and we write $Spec_D(A) = Spec_D(X_i^{(s)})$ for all $A \in X_i^{(s)} \subset X^{(s)}$. It turns out that some of these classes, or unions of some of these classes, are t'-designs for some t'.

Let the set X be the disjoint union of the sets $X_1, X_2, ..., X_l$. Then a $(m_1 + m_2 + ... + m_l)$-subset S of X is said to be an **$[m_1, m_2, ...m_l]$-set over $X_1 \cup X_2 \cup ... \cup X_l$** if m_i of the elements of S are in X_i, $i = 1, 2, ..., l$.

It is convenient to represent a covering by a $b \times k$ matrix whose rows are the blocks of the covering. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1q} \\ b_{21} & b_{22} & \cdots & b_{2q} \\ \cdots & \cdots & \cdots & \cdots \\ b_{p1} & b_{p2} & \cdots & b_{pq} \end{pmatrix}$$

be a set of m blocks of size n and a set of p blocks of size q, respectively. We use the notation AB to represent the following set of mp blocks:

$$\{\{a_{i1}, a_{i2}, \ldots, a_{in}, b_{j1}, b_{j2}, \ldots, b_{jq}\} : i = 1, 2, ..., m;\ j = 1, 2, ..., p\}.$$

2 Results

Theorem 2.1 $C_2(10, 5, 3) = 28$.

Proof. We searched for a self-complimentary design, that is, a design which has the same blocks as its suplemental design. The following construction gives a 3-$(10, 5, 2)$ covering design of minimum size. Extend each of the 14 blocks given below by the point 10. The new blocks and their complements form a 3-$(10, 5, 2)$ covering of size 28.

1	2	3	9	1	2	4	8
1	5	7	9	1	3	4	5
1	6	8	9	1	4	6	7
2	4	7	9	2	3	6	8
2	5	6	9	2	5	7	8
3	4	6	9	3	5	6	7
3	7	8	9				
4	5	8	9				

The minimality follows from $C_2(9, 4, 2) = 14$ [8] and the Schönheim Theorem. □.

Theorem 2.2 $C_2(12, 5, 3) = 48$.

Proof. The blocks of a 3-$(12, 5, 2)$ covering of size 48 are listed below. Points given in bold show some structure within the covering; further structure is evident from the grouping of the blocks: Partition $X(12)$ into the three sets $X_1 = \{1, 2, 3, 4\}$, $X_2 = \{5, 6, 7, 8\}$ and $X_3 = \{9, 10, 11, 12\}$. Let A be the set of the first 16 blocks of the first column of blocks in the list, B the set of the remaining 8 blocks in the first column and C the set of blocks in the second column. Then we can observe the following.

Each $[3, 0, 0]$-set is covered exactly four times by the blocks of A.

Each of the $[0, 3, 0]$ and $[0, 0, 3]$-sets is covered exactly four times by the blocks of B.

Each of the $[2,1,0]$ and $[2,0,1]$-sets is covered exactly twice by the blocks of A.

Each of the $[1,2,0]$ and $[1,0,2]$-sets is covered exactly once by a block of B and once by a block of C.

Each of the $[0,1,2]$ and $[0,2,1]$-sets is covered exactly twice by a block of C.

Finally, each $[1,1,1]$-set is covered exactly once by a block of A and once or twice by a block of C.

1	2	3	5	9
1	2	3	6	10
1	2	3	7	11
1	2	3	8	12
1	2	4	5	10
1	2	4	6	9
1	2	4	7	12
1	2	4	8	11
1	3	4	5	12
1	3	4	6	11
1	3	4	7	10
1	3	4	8	9
2	3	4	5	11
2	3	4	6	12
2	3	4	7	9
2	3	4	8	10
1	5	6	7	8
2	5	6	7	8
3	5	6	7	8
4	5	6	7	8
1	9	10	11	12
2	9	10	11	12
3	9	10	11	12
4	9	10	11	12

1	5	6	11	12
1	5	7	9	11
1	5	8	10	12
1	6	7	9	12
1	6	8	9	10
1	7	8	10	11
2	5	6	9	12
2	5	7	10	12
2	5	8	9	11
2	6	7	10	11
2	6	8	11	12
2	7	8	9	10
3	5	6	9	10
3	5	7	11	12
3	5	8	10	11
3	6	7	10	12
3	6	8	9	11
3	7	8	9	12
4	5	6	10	11
4	5	7	9	10
4	5	8	9	12
4	6	7	9	11
4	6	8	10	12
4	7	8	11	12

The result follows from $C_2(11,4,2) = 20$ [8] and the Schönheim Theorem. $\square$.

Proposition 2.3 $C_2(13,5,3) \leq 65$.

Proof. Given below are the representatives of the orbits under the cyclic group of order 13.

1	2	3	6	12
1	2	3	8	11
1	2	4	6	10
1	2	4	9	11
1	2	5	7	8

We note that the best upper bound on $C(13,5,3)$ is 34 [10]. □.

Proposition 2.4 $C_2(18,5,3) \leq 180$.

Proof. Given below are the representatives of the orbits under the cyclic group of order 18.

1	2	3	7	16	1	2	6	14	17
1	2	3	9	10	1	2	10	13	17
1	2	4	5	14	1	3	5	7	13
1	2	4	7	12	1	3	6	11	15
1	2	5	8	15	1	3	7	12	14

We note that $C(18,5,3) = 94$ is known, [9]. □.

Proposition 2.5 $C_2(7,6,4) = 6$.

Proof. Remove one of the 6-sets of $X^{(6)}(7)$. The six remaining 6-sets form a 4-$(7,6,2)$ covering. The result follows from Schönheim Theorem. □.

Proposition 2.6 $C_2(6,5,3) = 5$.

Proof. Remove one of the 5-sets of $X^{(5)}(6)$. The five remaining 5-sets form a 3-$(6,5,2)$ covering. The result follows from Schönheim Theorem. □.

Theorem 2.7 $C_2(8,6,4) = 12$.

Proof. Partition $X(8)$ into the two sets $X_1 = X(4)$ and $X_2 = \{5,6,7,8\}$. Let A and B be the sets of all unordered pairs on X_1 and X_2, respectively. We claim that the collection

$$\begin{array}{c} A\,5678 \\ 1234\,B \end{array}$$

is a $4-(8,6,2)$ covering of size 12.

The [0,4]-sets and the [4,0]-sets are each covered exactly 6 times, the [1,3]-sets and the [3,1]-sets are each covered exactly 3 times and the [2,2]-sets are covered each exactly 2 times. □.

Corollary 2.8 $C_2(7,5,3) = 9$.

Proof. Apply Construction 1.3 to the covering from the preceding theorem. □.

The following theorem has been proved in [4].

Theorem 2.9 *Let* $v \equiv 2$ *or* $4 \pmod 6$ *and* $m = \frac{1}{24}(v-4)(v^2-15v+62)-1$. *Then* $C_m(v, v-4, 4) = \frac{1}{24}v(v-1)(v-2) = C_1(v,4,3)$.

Corollary 2.10 $C_2(10,6,4) = 30$.

Proof. Apply the preceding theorem with $v = 10$. □.

Corollary 2.11 $C_2(9,5,3) = 18$.

Proof. Apply Construction 1.3 to the covering from the preceding theorem. □.

The 4-$(10,6,2)$ covering of size 30 found in [4] is actually a 3-$(10,6,5)$ design D, so that a 3-$(9,5,2)$ covering of size 18 (it is also a 2-$(9,5,5)$ design) can be obtained as the derived design of D.

Proposition 2.12 $C_2(11,6,4) \leq 55$.

Proof. Multiply the following block by 2^i, $i = 0,1,2,3,4$ to generate the representatives of the orbits under the cyclic group of order 11.

$$1 \quad 2 \quad 3 \quad 4 \quad 6 \quad 10$$

We note that $C(11,6,4) = 32$ is known [10]. □.

We use spectral sets for the next result and later on, in proposition 2.20. The exhibited spectral sets can be verified by computer.

Proposition 2.13 $C_2(16,6,4) \leq 280$.

Proof. Let D be the residual design of the unique 3-$(17,5,1)$ design [13]. It is a 2-$(16,5,4)$ design. The spectral set of $X^{(6)}(16)$ under D is

Equivalence class	Spectrum			Size of the class
	n_3	n_4	n_5	
$X_1^{(6)}$	10	0	1	48
$X_2^{(6)}$	8	0	1	480
$X_3^{(6)}$	6	3	0	640
$X_4^{(6)}$	10	2	0	2400
$X_5^{(6)}$	9	2	0	1920
$X_6^{(6)}$	8	2	0	240
$X_7^{(6)}$	13	1	0	960
$X_8^{(6)}$	12	1	0	240
$X_9^{(6)}$	11	1	0	960
$X_{10}^{(6)}$	16	0	0	120

We need the class $X_{10}^{(6)}$. It is a 2-$(16,6,15)$ design. Now, the spectral set of $X^{(6)}(16)$ under $X_{10}^{(6)}$ is

Equivalence class	Spectrum				Size of the class
	n_3	n_4	n_5	n_6	
$Y_1^{(6)}$	40	7	0	1	120
$Y_2^{(6)}$	28	10	2	0	240
$Y_3^{(6)}$	34	9	2	0	960
$Y_4^{(6)}$	40	6	2	0	480
$Y_5^{(6)}$	36	10	1	0	960
$Y_6^{(6)}$	40	9	1	0	960
$Y_7^{(6)}$	38	9	1	0	1920
$Y_8^{(6)}$	32	14	0	0	240
$Y_9^{(6)}$	30	14	0	0	960

$Y_{10}^{(6)}$	32	13	0	0	480
$Y_{11}^{(6)}$	36	12	0	0	160
$Y_{12}^{(6)}$	40	11	0	0	480
$Y_{13}^{(6)}$	40	10	0	0	48

Let D' be the union of the blocks of the classes $Y_1^{(6)}$ and $Y_{11}^{(6)}$. We claim that D' is a 4-$(16,6,2)$ covering design. The easiest way to verify this is by checking the spectral set of $X^{(4)}(16)$ under D'. It is

Equivalence class	Spectrum n_3	n_4	Size of the class
$Z_1^{(4)}$	16	6	20
$Z_2^{(4)}$	28	3	480
$Z_3^{(4)}$	32	2	1320

which shows that the blocks of D' cover each 4-subset of $X(16)$ at least twice. The result now follows from $|D'| = 280$.

We note that the best upper bound on $C(16,6,4)$ is 152 (found in [9]). □.

Proposition 2.14 $C_2(7,6,3) = 5$.

Proof. Remove any two of the 6-sets of $X^{(6)}(7)$. The five remaining 6-sets form a 3-$(7,6,2)$ covering. The result follows from Schönheim Theorem. □.

Theorem 2.15 $C_2(8,6,3) = 7$.

Proof. Partition $X(8)$ into the three sets $X_1 = X(2)$, $X_2 = \{3,4,5\}$ and $X_3 = \{6,7,8\}$. We claim that the collection

$$\begin{array}{ll} 12345i, & i = 6,7,8 \\ 12678j, & j = 3,4,5 \\ 345678 & \end{array}$$

is a 3-$(8,6,2)$ covering of size 7.

The [0,3,0]-sets, [0,0,3]-sets, [1,2,0]-sets, [2,1,0]-sets, [1,0,2]-sets and [2,0,1]-sets are each covered exactly 3 times, while the [1,1,1]-sets, [0,1,2]-sets and [0,2,1]-sets are each covered exactly 2 times. □.

Theorem 2.16 $C_2(9,6,3) = 11$.

Proof. The following 3-(9,6,2) covering was found by optimization.

1	2	3	4	6	7	1	4	5	6	7	9
1	2	3	5	6	9	2	3	4	5	7	8
1	2	4	5	8	9	2	3	5	7	8	9
1	2	6	7	8	9	2	4	5	6	8	9
1	3	4	7	8	9	3	4	5	6	8	9
1	3	5	6	7	8						

The result follows from $C_2(8,5,2) = 7$ and Schönheim Theorem. □.

Theorem 2.17 $C_2(10,6,3) = 15$.

Proof. Let D be one of the three non-isomorphic 2-(10,4,2) designs [7]. Then D is residual of one of the three symmetric 2-(16,6,2) designs [7]. Therefore, the intersection of any two blocks of D is at most 2. The number of blocks of D is 15. The supplemental design $\overline{D}$ of D is a 2-(10,6,5) design with $\lambda_0 = 15$, $\lambda_1 = 9$ and $\lambda_2 = 5$. We claim that $\overline{D}$ is a 3-(10,6,2) covering of minimum size.

Consider an arbitrary 3-subset S of $X(10)$. The intersection equations of S with respect to $\overline{D}$ are

$$\begin{aligned} n_0 + n_1 + n_2 + n_3 &= 15 \\ n_1 + 2n_2 + 3n_3 &= 27 \\ n_2 + 3n_3 &= 15 \end{aligned}$$

Solving this system for n_0, n_1 and n_2, we obtain

$$\begin{aligned} n_2 &= 15 - n_3 \\ n_1 &= -3 + 3n_3 \\ n_0 &= 3 - n_3 \end{aligned}$$

Since $n_0 \geq 0$ and $n_1 \geq 0$, we have $1 \leq n_3 \leq 3$. If $n_3 = 1$, then $n_0 = 2$, so that there are two different blocks of $\overline{D}$ that do not intersect S. However, every two blocks of $\overline{D}$ have at most 4 points in common, so that they must contain at least 8 points not in S, which is a contradiction, because $|S| = 3$. Therefore, $n_3 \geq 2$, so that $\overline{D}$ is indeed a 3-(10,6,2) covering. The minimality follows from $C_2(9,5,2) = 9$ [8] and Schönheim Theorem. □.

Theorem 2.18 $C_2(13,6,3) = 33$.

Proof. The following 3-(13, 6, 2) covering was found by optimization.

1	2	3	6	8	9
1	2	3	6	11	13
1	2	4	5	6	9
1	2	4	6	9	12
1	2	5	6	7	10
1	2	6	7	10	12
1	2	6	8	11	13
1	3	4	7	9	13
1	3	4	10	11	12
1	3	5	7	8	12
1	3	5	9	10	11
1	4	5	8	10	13
1	4	7	8	9	11
1	5	7	11	12	13
1	8	9	10	12	13
2	3	4	5	8	12
2	3	4	8	11	13
2	3	5	9	12	13
2	3	7	8	10	11
2	3	7	8	10	13
2	4	5	7	9	10
2	4	5	11	12	13
2	4	7	9	10	12
2	5	8	9	11	12
2	7	9	10	11	13
3	4	5	6	7	11
3	4	6	8	9	10
3	5	6	10	12	13
3	6	7	9	11	12
4	6	7	8	12	13
4	6	9	10	11	13
5	6	7	8	9	13
5	6	8	10	11	12

The result follows from $C_2(12, 5, 2) = 15$ and Schönheim Theorem. □.

Proposition 2.19 $C_2(15, 6, 3) \leq 55$.

Proof. Given below are the representatives of the orbits under the cyclic group of order 15.

1	2	3	5	6	13
1	2	3	8	10	12
1	2	4	10	12	13
1	2	6	7	11	12
1	3	6	8	11	13

The last two orbits are short.

We note that the best upper bound on $C(15, 6, 3)$ is 31 (folklore). □.

Proposition 2.20 $C_2(18, 6, 3) \leq 88$.

Proof. Let D be the affine plane of order 4. Hence D is a 2-(16, 4, 1) design. The spectral set of $X^{(6)}(16)$ under D is

Equivalence class	Spectrum n_3	n_4	Size of the class
$X_1^{(6)}$	0	0	48
$X_2^{(6)}$	2	0	3520
$X_3^{(6)}$	3	0	2880
$X_4^{(6)}$	4	0	240
$X_5^{(6)}$	0	1	360
$X_6^{(6)}$	1	1	960

The class $X_1^{(6)}$ is a 2-(16, 6, 6) design. Now consider the spectral sets of $X^{(3)}(16)$ and $X^{(4)}(16)$ under the design $X_1^{(6)}$. These are

Equivalence class	Spectrum n_3	Size of the class
$X_1^{(3)}$	0	80
$X_2^{(3)}$	2	480

and

Equivalence class	Spectrum n_3	n_4	Size of the class
$X_1^{(4)}$	0	0	20
$X_2^{(4)}$	6	0	960
$X_3^{(4)}$	8	0	120
$X_4^{(4)}$	4	1	720

They show that the blocks of $X_1^{(6)}$ cover exactly twice every triple in $X^{(3)}(16)$ except for the triples contained in the blocks of D. Therefore, the set of blocks

$$D\,17\,18\ \text{(twice)}$$
$$X_1^{(6)}$$

is a 3-(18, 6, 2) covering of size 88.

The best upper bound on $C(18, 6, 3)$ is 48 [5]. □.

The next table summarizes the current knowledge on $C_2(v, k, t)$ for pairs (k, t) in $\{(5, 3), (6, 4), (6, 3)\}$. Columns 2 and 5 are given to illustrate the computation of the lower bounds in columns 3 and 6 via Schönheim Theorem. The dot after an entry indicates that it is the covering number. The starred entries are covering numbers found in this paper. The remaining entries in columns 3,4 and 6 are presented in the form $a - b - c$, where a is the lower bound obtained by Schönheim Theorem, c is equal to $2u$, where u is the best known upper bound on $C(v, k, t)$ (see [1] and [3] for the best known upper bounds on $C(v, k, t)$) and b is the upper bound on $C_2(v, k, t)$ found in this paper (except for $C_2(v, 6, 4)$, $v = 14, 17, 18$, for which no bound better than $2u$ was found).

v	$C_2(v,4,2)$	$C_2(v,5,3)$	$C_2(v,6,4)$	$C_2(v,5,2)$	$C_2(v,6,3)$
5	4.				
6	6.	$5.^*$		4.	
7	7.	$9.^*$	$6.^*$	5.	$5.^*$
8	10.	$12 - 14^o - 16$	$12.^*$	7.	$7.^*$
9	14.	$18.^*$	$18 - 22^o - 24$	9.	$11.^*$
10	15.	$28.^*$	30.	10.	$15.^*$
11	20.	$33 - 36^o - 40$	$52 - 55^c - 64$	11.	$19 - 20^o - 22$
12	24.	$48.^*$	$66 - 72^o - 82$	15.	$22.^d$
13	26.	$63 - 65^c - 68$	$104 - 118^o - 132$	17.	$33.^*$
14	32.	$73 - 78^o - 86$	$147 - 160 - 160$	19.	$40 - 41^o - 50$
15	38.	$96 - 101^o - 112$	$183 - 224^o - 234$	22.	$48 - 55^c - 62$
16	40.	$122 - 125^o - 130$	$256 - 280^s - 304$	26.	$59 - 68^o - 76$
17	47.	$136.^d$	$346 - 376 - 376$	28.	$74 - 82^o - 88$
18	54.	$170 - 180^c - 188$	$408 - 472 - 472$	33.	$84 - 88^s - 96$

Key to the tables:

c— cyclic covering

d— there exist a design with the parameters of the covering

o— covering found by optimization

s— covering found by studying spectra

The coverings indicated by "o" are given in the Appendix.

References

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Appendix

The appendix contains the coverings obtained by optimization.

We use the following compressed notation [10]. Suppose the k-subsets of $X(v)$ are arranged in lexicographical order (for example, let $v = 4$, $k = 3$, then the order is 123, 124, 134, 234). We present the blocks of a design by a sequence $a_1, a_2, \ldots, a_b$, such that the n-th block of the design is the $(\sum_{i=1}^{n} a_i)$-th k-set from the lexicographical arrangement of $X^{(k)}(v)$, where $1 \leq n \leq b$.

$C_2(8,5,3) \leq 14$:
4, 2, 3, 2, 6, 5, 6, 6, 1, 4, 5, 1, 3, 4.

$C_2(11,5,3) \leq 36$:
16, 12, 3, 4, 22, 25, 8, 4, 17, 20, 39, 2, 8, 2, 10, 9, 13, 8, 13, 24, 29, 10, 2, 8, 8, 9, 35, 3, 9, 19, 3, 8, 6, 4, 10, 39.

$C_2(14,5,3) \leq 78$:
21, 22, 14, 28, 20, 69, 38, 3, 25, 22, 20, 6, 39, 23, 59, 32, 4, 29, 38, 50, 27, 7, 19, 88, 24, 26, 20, 23, 22, 18, 11, 51, 23, 62, 15, 19, 39, 12, 34, 16, 18, 9, 3, 54, 16, 17, 12, 61, 13, 23, 9, 6, 72, 63, 26, 4, 7, 30, 15, 16, 16, 43, 19, 19, 12, 14, 40, 30, 49, 1, 25, 12, 21, 19, 27, 46.

$C_2(15,5,3) \leq 101$:
22, 26, 11, 31, 19, 15, 18, 88, 13, 13, 64, 10, 43, 12, 12, 36, 82, 17, 23, 28, 5, 43, 20, 62, 8, 67, 19, 10, 13, 22, 72, 11, 46, 7, 21, 32, 17, 93, 23, 46, 17, 51, 13, 34, 7, 69, 33, 34, 15, 36, 26, 6, 30, 14, 20, 70, 13, 23, 38, 8, 67, 9, 3, 18, 36, 26, 15, 19, 67, 26, 20, 4, 15, 29, 70, 22, 59, 25, 11, 45, 48, 15, 12, 27, 71, 18, 3, 59, 12, 56, 28, 4, 40, 8, 29, 53, 39, 25, 56, 3, 14.

$C_2(16,5,3) \leq 125$:
41, 12, 24, 5, 16, 100, 15, 48, 21, 49, 71, 4, 29, 34, 18, 100, 31, 13, 41, 19, 51, 10, 31, 65, 28, 28, 15, 55, 11, 113, 3, 31, 55, 34, 8, 15, 70, 13, 36, 11, 4, 2, 97, 57, 79, 7, 72, 55, 16, 18, 19, 59, 40, 11, 14, 24, 5, 36, 37, 23, 3, 62, 29, 39, 97, 8, 24, 31, 66, 64, 23, 37, 44, 23, 42, 11, 40, 22, 55, 20, 47, 30, 2, 72, 10, 9, 28, 63, 38, 17, 37, 26, 49, 25, 30, 29, 10, 15, 73, 64, 29, 38, 29, 24, 60, 94, 67, 59, 22, 14, 14, 50, 8, 43, 12, 79, 19, 19, 43, 4, 52, 28, 11, 26, 11.

$C_2(9,6,4) \leq 22$:
4, 2, 8, 4, 5, 2, 1, 1, 6, 3, 9, 3, 5, 3, 2, 6, 2, 2, 3, 1, 9, 1.

$C_2(12,6,4) \leq 72$:
1, 1, 7, 5, 9, 2, 1, 2, 126, 7, 7, 1, 11, 5, 2, 13, 77, 13, 2, 5, 11, 1, 7, 7, 25, 1, 7, 5, 21, 5, 7, 1, 133, 5, 7, 1, 25, 1, 7, 5, 20, 13, 2, 5, 11, 1, 7, 7, 83, 7, 7, 1, 11, 5, 2, 13, 137, 5, 7, 1.

$C_2(13,6,4) \leq 118$:
25, 10, 8, 4, 13, 16, 10, 35, 15, 3, 16, 19, 16, 53, 10, 20, 7, 5, 21, 8, 15, 10, 16, 14, 2, 19, 29, 10, 8, 22, 2, 23, 7, 30, 8, 11, 8, 44, 1, 12, 1, 17, 7, 9, 17, 24, 2, 13, 16, 2, 25, 5, 45, 7, 25, 1, 18, 12, 18, 16, 21, 7, 31, 17, 12, 2, 1, 20, 2, 25, 6, 9, 2, 45, 6, 16, 9, 14, 6, 24, 5, 26, 2, 20, 11, 33, 23, 13, 8, 6, 4, 14,

5, 20, 29, 26, 8, 16, 11, 21, 1, 7, 4, 36, 1, 26, 23, 11, 36, 7, 10, 13, 29, 9, 8, 16, 6, 8.

$C_2(15, 6, 4) \leq 224$:
13, 18, 14, 20, 22, 23, 19, 31, 10, 7, 31, 24, 24, 24, 15, 34, 28, 25, 5, 11, 50, 22, 8, 9, 29, 40, 32, 13, 21, 7, 20, 63, 21, 20, 11, 29, 29, 40, 39, 2, 11, 14, 8, 57, 12, 89, 5, 9, 30, 12, 21, 4, 4, 8, 35, 13, 1, 4, 29, 29, 59, 6, 4, 49, 2, 19, 45, 6, 37, 2, 16, 91, 45, 15, 8, 4, 10, 16, 29, 6, 20, 13, 7, 44, 1, 17, 47, 11, 46, 27, 9, 37, 32, 10, 17, 13, 25, 12, 16, 77, 6, 5, 24, 2, 20, 28, 18, 25, 19, 3, 85, 15, 48, 16, 13, 18, 39, 42, 15, 37, 22, 19, 24, 5, 24, 32, 6, 7, 30, 7, 1, 50, 8, 15, 4, 11, 48, 37, 17, 29, 15, 46, 5, 5, 41, 49, 3, 26, 27, 11, 14, 8, 6, 39, 16, 16, 9, 43, 27, 25, 44, 1, 21, 7, 10, 6, 7, 90, 3, 37, 7, 26, 12, 8, 56, 10, 17, 18, 9, 3, 11, 63, 2, 11, 61, 44, 46, 1, 8, 48, 26, 17, 7, 12, 27, 14, 18, 2, 26, 10, 45, 38, 7, 30, 48, 56, 11, 3, 16, 21, 6, 6, 33, 8, 83, 18, 15, 48, 1, 43, 26, 2, 27, 9.

$C_2(11, 6, 3) \leq 20$:
17, 19, 23, 39, 5, 12, 12, 30, 2, 10, 79, 30, 13, 24, 22, 15, 32, 40, 25, 3.

$C_2(14, 6, 3) \leq 41$:
11, 112, 31, 45, 105, 131, 35, 121, 31, 29, 20, 184, 92, 23, 62, 120, 22, 51, 95, 68, 52, 104, 174, 64, 24, 30, 86, 61, 12, 101, 24, 59, 36, 201, 82, 32, 46, 61, 203, 58, 93.

$C_2(16, 6, 3) \leq 68$:
5, 224, 112, 196, 23, 203, 32, 176, 14, 171, 81, 52, 172, 125, 73, 143, 106, 17, 149, 75, 389, 16, 59, 102, 180, 159, 39, 5, 79, 170, 70, 99, 513, 45, 43, 54, 101, 75, 149, 43, 52, 41, 520, 19, 180, 248, 103, 5, 234, 181, 109, 42, 30, 9, 227, 292, 141, 43, 18, 184, 9, 89, 7, 209, 92, 66, 161, 22.

$C_2(17, 6, 3) \leq 82$:
51, 175, 174, 151, 163, 241, 46, 232, 83, 196, 124, 42, 171, 78, 257, 170, 30, 154, 93, 353, 328, 175, 140, 32, 227, 36, 283, 8, 34, 181, 277, 122, 54, 215, 78, 164, 241, 109, 25, 51, 109, 167, 72, 297, 42, 35, 225, 289, 280, 43, 104, 240, 13, 297, 94, 262, 88, 230, 208, 126, 126, 14, 168, 111, 33, 48, 179, 638, 76, 79, 66, 330, 24, 222, 97, 216, 59, 269, 110, 56, 38, 37.