A Restricted Sarvate-Beam Triple System for $v = 8$

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Abstract

A construction is given for a Restricted Sarvate-Beam Triple System in the case $v = 8$. This is the extremal case, since a Restricted SB Triple System can not exist for $v > 8$.

1 Introduction

Sarvate-Beam Triple Systems (cf. [1] and [2]) are triple systems on v elements in which each pair occurs a different number of times selected from the integers i in the set $\{1, 2, 3, \ldots, v(v-1)/2\}$. In Restricted SB Triple Systems, the pair 12 occurs a single time, say in the triple 123, and all other triples have the form $1xx$ or $2xx$.

In [3], SB Triple Systems were generalized. Only triples are needed if $v \equiv 0$ or $v \equiv 1 \bmod 3$; if $v \equiv 2 \bmod 3$, one uses a single pair (12) along with a collection of triples in which all pairs occur a different number of times selected from the set $\{2, 3, 4, \ldots, v(v-1)/2\}$. The case $v = 5$ was completely enumerated in [3], and it was pointed out that a Restricted SB Triple System could not occur for $v > 8$. Furthermore, for $v = 8$, the frequencies of pairs not involving 1 and 2 had to be (in some order) the integers $2, 3, 4, \ldots, 16$.

2 A Solution for $v = 8$.

In [3], it was pointed out that, if we designate the frequencies of $(13), \ldots, (18)$, as totalling $2X$; the frequencies of $(23), \ldots, (28)$, as totalling $2Y$ (with

$X < Y$); then $X + Y = 135$, and the smallest value of X is 59. We proceed to present one solution for $X = 59$, $Y = 76$.

Let us designate the frequencies of $(34), \ldots, (78)$, in triples $(1xx)$ in some order, by the letters $A, B, C, \ldots$; then the frequencies of $(34), \ldots, (78)$, in the corresponding triples $(2xx)$ are $2-A, 3-B, 4-C, \ldots, 16-P$. Suppose we arbitrarily assign values to these as follows (the reason for this choice will be given shortly).

$$\begin{array}{ll}
A,\ 2-A=2,0 & I,\ 10-I=4,6 \\
B,\ 3-B=2,1 & J,\ 11-J=5,6 \\
C,\ 4-C=4,0 & K,\ 12-K=7,5 \\
D,\ 5-D=3,2 & L,\ 13-L=3,10 \\
E,\ 6-E=3,3 & M,\ 14-M=4,10 \\
F,\ 7-F=4,3 & N,\ 15-N=5,10 \\
G,\ 8-G=4,4 & P,\ 16-P=6,10 \\
H,\ 9-H=3,6 &
\end{array}$$

This guarantees that the frequencies of pairs (ij) with $i < j$, i and j ranging from 3 to 8, are the integers from 2 to 16, in some order. Also, in order to have $X = 59$ and $Y = 76$, the frequencies of (13) to (18), in some order must be 17, 18, 19, 20, 21, 23, and the frequencies of (23) to (28), in some order, must be 22, 24, 25, 26, 27, 28.

Trial and error led only to error, and so I used, as frequencies of 13, 14, 15, 16, 17, 18, in some order,

$$\begin{array}{c}
A+D+G+J+N \\
D+I+C+H+M \\
J+B+H+L+F \\
N+B+E+I+K \\
G+E+C+P+L \\
A+K+M+P+F
\end{array}$$

I assigned all values except E, P, and K, as shown earlier. This led to equations $K + P = 13$, $E + P = 9$, $E + K = 10$, whence $E = 3$, $P = 6$, $K = 7$.

$A + D + G + J + N$ is thus a frequency of some pair $(1x)$. The corresponding frequency of the pair $(2x)$ is then $(2 - A) + (5 - D) + (8 - G) + (11 - J) + (15 - N) = 41 - (A + D + G + J + N)$.

The other corresponding pair frequencies are then:

$$\begin{array}{c}
D+I+C+H+M,\ 42-(D+I+C+H+M); \\
J+B+H+L+F,\ 43-(J+B+H+L+F) \\
N+B+E+I+K,\ 46-(N+B+E+I+K) \\
G+E+C+P+L,\ 47-(G+E+C+P+L) \\
A+K+M+P+F,\ 51-(A+K+M+P+F)
\end{array}$$

Now it is easy to see that values of these pairs can be taken as: (19,22), (18,24), (17,26), (21,25), (20,27), (23,28).

We have thus merely to associate the letters $A, B, \ldots, P$, with appropriate triples. Let us set

$$f(134) = A, f(135) = D, f(136) = G, f(137) = J, f(138) = N.$$

Then we may take

$$f(145) = M, f(146) = P, f(147) = F, f(148) = K.$$

It is easy to complete the assignment as:

$$f(156) = C, f(157) = H, f(158) = I,$$

$$f(167) = L, f(168) = E, f(178) = B.$$

Thus we have constructed an SB Triple System for $v = 8$ as: $(12)+2(134)+0(234)+3(135)+2(235)+4(136)+4(236)+5(137)+6(237)+5(138)+10(238)+4(145)+10(245)+6(146)+10(246)+4(147)+3(247)+7(148)+5(248)+4(156)+0(256)+3(157)+6(257)+4(158)+6(258)+3(167)+10(267)+3(168)+3(268)+2(178)+(278)$.

3 Conclusion

Consequently, we have established the existence of an SB Restricted Triple System in the extremal case $v = 8$. It would appear likely that the number of non-isomorphic systems is large.

References

[1] Dinesh G. Sarvate and William Beam, A New Type of Block Design, *Bulletin of the ICA* **50** (2007), 26-28.

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[3] R.G. Stanton, Sarvate-Beam Triple Systems for $v \equiv 2 \pmod 3$, *J. of Combinatorial Mathematics and Combinatorial Computing* **61** (2007), 129-134.