Firefighting on the Triangular Grid

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Abstract

A fire breaks out on a graph G and then f firefighters protect f vertices. At each subsequent interval the fire spreads to all adjacent unprotected vertices and firefighters protect f unburned vertices. Let f_G be the minimum number of firefighters needed to contain a fire on graph G. If the triangular grid goes unprotected to time $t = k$ when f_G firefighters arrive and begin protecting vertices, the fire can be contained by time $t = 18k+3$ with at most $172k^2 + 58k + 5$ vertices burned.

1 Introduction

The firefighting problem was originally introduced at the 24th Manitoba Conference on Combinatorial Mathematics and Computing in 1995 by Hartnell [3]. A fire breaks out at a vertex of a graph and spreads while the firefighters protect vertices in an effort to contain the fire.

The **original burned vertex** is the vertex where the fire initially breaks out. A **burned vertex** is a vertex to which the fire has spread; it is on fire and stays on fire. A **protected** vertex is a vertex which has been visited (protected) by a firefighter. Once a vertex has been protected, it cannot be burned. An **unprotected vertex** is a vertex which has not yet been protected and while the fire can still spread, the vertex may be burned. When the fire can no longer spread, any vertex which has not been burned is a **saved** vertex. Figure 1 illustrates how these types of vertices will be represented in the examples.

Given a graph G and a set of f firefighters, a fire breaks out at vertex $v \in G$ at time $t = 0$. The firefighters protect f vertices, $u_1, u_2, \cdots, u_f$. At time $t = 1$, the fire spreads to the vertices of $N(v)\backslash\{u_1, u_2, \cdots, u_f\}$ then the firefighters protect another f vertices. The clock advances one unit of time, then the fire spreads to all unprotected vertices adjacent to the burned vertices. Time advances by one and both sides move again.

The objective for the firefighters is not to extinguish the burning vertices, *but to contain the fire using the minimum number of firefighters needed.* A fire has been **contained** if the fire has been enclosed by protected vertices and can spread no further. The protected vertices form **fire walls** through which the fire cannot penetrate.

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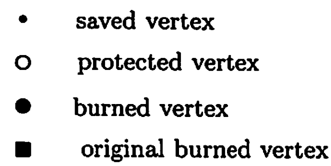
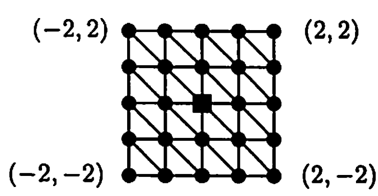


Figure 1: The different states of vertices.

Let $\mathcal{F}(G) = \{n : n$ firefighters can contain any fire on $G\}$ and let $f_G = \min\{n : n \in \mathcal{F}(G)\}$, i.e. *$f_G$ is the minimum number of firefighters needed to contain a fire on G regardless of the starting vertex.* A logical next step is to identify algorithms which minimize the number of vertices burned when using f_G firefighters. Clearly there is a trade off: as the number of firefighters used to contain a fire decreases, the number of vertices burned increases.

In her M.Sc. thesis, Fogarty [1] investigated the firefighting problem on the Cartesian and triangular grids. The **Cartesian grid** C is the resulting infinite rectangular grid from the Cartesian product of two paths of infinite length. The **triangular grid** Tr, also called the *isometric grid*, is another infinite rectangular grid. It is formed by a tessellation: by tiling the plane regularly with equilateral triangles. The orientation of the triangular grid is shown in Figure 2.



4×4 triangular grid

Figure 2: The orientations of the triangular grid.

It was shown in [6] that $f_C = 2$. Fogarty [1] concentrated on the timing of the firefighters and proved that for any $k \in \mathbb{N}$, if a fire breaks out on the Cartesian grid and the grid goes unprotected to time $t = k$, then two firefighters can contain the fire by time $t = 32k + 1$ and leave $318k^2 + 14k + 1$ burned vertices. Fogarty also proved that if x fires break out anywhere on the Cartesian grid, then two firefighters suffice to contain the x fires. Further results using Cartesian grids can be found in [1], [2], [4], and [6].

The triangular grid was also examined in [1] where it was shown that that

$f_{Tr} = 3$ and for any $k \in \mathbb{N}$, if the grid goes unprotected to time $t = k$ when three firefighters begin protecting vertices, they can contain the fire by time $t = 44k - 7$ and leave $1100k^2 - 372k + 27$ burned vertices. A simple corollary of this result from [1] is a follows:

Corollary 1 *If x fires break out anywhere in the triangular grid Tr, three firefighters suffice to contain the fire.*

Proof: Assume x fires break out anywhere on the triangular grid Tr. Label the fires f_1, f_2, f_3, $\cdots$, f_x. Let κ =max $d(f_i,f_j)$ over all i,j, where $i \neq j$, $1 \leq i,j \leq x$.

There must exist some hexagon of vertices with the six corners being $(-k',k')$, $(0,k')$, $(k',0)$, $(k',-k')$, $(0,-k')$, and $(-k',0)$ which contains all the fires. If κ is even, let $k' = \frac{\kappa}{2}$ and if κ is odd, let $k' = \frac{\kappa+1}{2}$. If possible, position the hexagon on the triangular grid such that all x fires are contained within the hexagon. If this is not possible, increase k' by one and try again. When all x fires are contained within the hexagon, then the distance from the origin to any corner of the hexagon is relabeled as k. Theorem 3 can then be applied with the appropriate k value. Thus, if x fires break out anywhere on the triangular grid, three firefighters suffice to contain the fire. ∎

It was conjectured in [1] that for any $k \in \mathbb{N}$, if a fire breaks out on Tr and the grid remains unprotected to time $t = k$ when three firefighters begin protecting vertices, the minimum time by which they can contain the fire is $t = 18k + 5$ with a minimum of $193k^2 + 47k + 20$ burned vertices. It will be shown that if the triangular grid goes unprotected to time $t = k$ when three firefighters begin protecting vertices, the fire can be contained by time $t = 18k + 3$ with a maximum of $172k^2 + 58k + 5$ vertices burned. This greatly improves the results from [1] and disproves the conjecture of [1].

2 Results

It was shown in [1] that if a fire breaks out on the triangular grid, two firefighters do not suffice to contain the fire. Figure 3 illustrates that three firefighters suffice to contain a fire on the triangular grid by at most time $t = 5$ and with at most 17 vertices burned. Note that the number next to a protected vertex indicates the time at which the vertex was protected.

Conjecture 2 *If a fire breaks on Tr, then the minimum time by which two firefighters can contain the fire is $t = 5$, leaving a minimum of 17 vertices burned.*

Theorem 3 *For any $k \in \mathbb{N}$, if a fire breaks out on Tr and the grid goes unprotected to time $t = k$ when three firefighters begin protecting vertices, then*
a) they can contain the fire by time $t = 18k + 3$
b) they can leave at most $172k^2 + 58k + 5$ vertices burned.

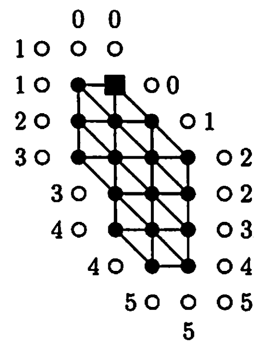


Figure 3: The triangular grid protected with three firefighters.

Proof of a): An algorithmic approach will be used and the pattern of protecting vertices produced will be referred to as **Algorithm A**. The algorithm builds fire walls beginning on the western side of the fire (Phase 1); then creates a northern fire wall (Phase 2); a south-west fire wall (Phase 3 and Phase 4); and finally a north-east fire wall (Phase 5 and Phase 6). An example of the triangular grid with $k = 1$ protected by Algorithm A is shown in Figure 4. The vertices burned at time $t = k$ are coloured grey.

Assume a fire breaks out at $(0, 0)$ on a triangular grid. Let the grid go unprotected to time k. At time $t = k$, all vertices $v \in N_k(0, 0)$ are burned: a hexagon of vertices. Three firefighters begin protecting vertices at $t = k$ in the following manner:

Phase 1:
The firefighters begin by creating a vertical fire wall to the west of the initial burned vertex.

$t = k$
while $(t \leq 2k - 1)$ {
  protect $(-2k, 3t - 3k), (-2k, 3t - 3k + 1), (-2k, 3t - 3k + 2)$
  $t := t + 1$
}
$t = 2k$: protect $(-2k, 3t - 3k), (-2k, 3t - 3k + 1), (-2k, -1)$

At time $t = 2k$ the firefighters have completed the western vertical fire wall at $(-2k, y)$ with $-1 \leq y \leq 3k + 1$ while the fire has spread as far north as $y = 2k$ and south to $(-2k + 1, -1)$.

Phase 2:
The firefighters now maintain the appropriate length of the western fire wall to prevent the fire from spreading around it and create a northern fire wall.

$t = 2k + 1$

```
i = −2
while (t ≤ 3k) {
    protect (2t − 6k − 1, 3k + 1), (2t − 6k, 3k + 1), (−2k, i)
    t := t + 1
    i := i − 1
}
```

Beginning at time $t = 2k + 1$ and ending at time $t = 3k$, the appropriate length of the western fire wall is maintained and the northern horizontal fire wall is created at $(x, 3k + 1)$ with $-2k \leq x \leq k - 1$. At time $t = 3k$ the fire has spread as far north as $y = 3k$.

Phase 3:
The firefighters now maintain the appropriate length of the northern fire wall, while creating a south-west fire wall extending from the western fire wall.

```
t = 3k + 1
i = −k − 2
while (t ≤ 5k) {
    protect (t − 5k − 1, i), (t − 5k, i − 1), (t − 3k, 3k + 1)
    t := t + 1
    i := i − 2
}
```

From time $t = 3k+1$ to $t = 5k-1$ the fire was one step ahead of the south-west fire wall. At time $t = 5k$, the south-west fire wall has caught up to the fire: it is one more step to the south than the fire. At $x = 0$, the south-west wall extends as far south as $y = -5k - 1$ while the fire is as far south as $y = -5k$. The northern fire wall is at $(x, 3k + 1)$ with $-2k \leq x \leq 5k$.

Phase 4:
From time $t = 5k$ to $t = 10k + 1$ the firefighters complete the south-west fire wall while maintaining the appropriate length of the northern fire wall.

```
t = 5k + 1
while (t ≤ 10k + 1) {
    protect (2t − 10k − 1, −t − 1), (2t − 10k, −t − 1), (t − 3k, 3k + 1)
    t := t + 1
}
```

At time $t = 10k + 1$ the south-west fire wall is complete. The fire has spread as far south as $y = -10k - 1$ while the fire wall bounds the fire at $y = -10k - 2$. The fire has spread as far east as $x = 10k + 1$ while the south-west fire wall is as far east as $x = 10k + 2$. The northern fire wall is at $(x, 3k + 1)$ with $-2k \leq x \leq 10k + 1$.

Phase 5:
The firefighters now create a north-east fire wall to completely contain the fire by extending the northern fire wall down to the south-west fire wall. They also maintain the appropriate length of the southwest fire wall.

```
t = 10k + 2
```

```
i = 3k + 1
while (t ≤ 13k + 2) {
    protect (2t − 13k − 2, i), (2t − 13k − 1, i − 1), (t + 1, −10k − 2)
    t := t + 1
    i := i − 1
}
```

From time $t = 10k+2$ to $t = 13k+2$ the fire was one step ahead of the noth-east fire wall. At time $t = 13k + 2$, the north-east fire wall has caught up to the fire: it is one more step to the east than the fire. The north-east wall extends as far south as $(13k + 3, 0)$ while the fire is as far east as $(13k + 2, 0)$. The south-west fire wall is as far east as $x = 13k + 2$.

Phase 6:
From time $t = 13k + 3$ to $t = 18k + 3$ the firefighters finish the north-east fire wall while maintaining the south-west fire wall.

```
t = 13k + 3
i = −1
while (t ≤ 18k + 2) {
    protect (t + 1, i), (t + 1, i − 1), (t + 1, −10k − 2)
    t := t + 1
    i := i − 2
}
t = 18k + 3:     protect (18k + 4, −10k − 2), (18k + 4, −10k − 1)
```

From time $t = 13k + 3$ to $t = 18k + 3$ the firefighters finished the north-east fire wall while maintaining the south-west fire wall. The fire has spread as far east as $x = 18k + 3$ while the north-east fire wall bounds the fire at $x = 18k + 4$. The north-east and south-west fire walls meet at time $t = 18k + 3$.
Three firefighters began protecting vertices at time $t = k$ and have contained the fire by time $t = 18k + 3$. ∎

Proof of b): After using Algorithm A to contain a fire, the firefighters have protected vertices as far west as $x = -2k - 1$, as far east as $x = 18k + 4$, as far north as $y = 3k + 1$, and as far south as $y = -10k - 2$. Thus, by observing the rectangle of vertices with dimensions $(20k + 3) \times (13k + 2)$, the number of vertices saved when the fire is contained can simply be counted. There are two large areas of vertices, as shown in Figure 5 for $k = 1$, and each area is subdivided into two smaller regions.

Region R_1 is formed by the vertices protected in Phase 3 and Region R_2 is formed by the vertices protected in Phase 4. Region R_3 is formed by the vertices protected in Phase 5 and Region R_4 is formed by the vertices protected in Phase 6. Each region is bound by vertices protected in Phase $(i + 2)$, the range of the x-coordinates of the vertices protected in Phase $(i + 2)$, and by one of the rectangles boundaries. For each region R_i, let h_i denote the length of the vertical height of the region in terms of the number of vertices.

Region R_1 is a polygon bounded by $y = -10k - 1$, and the vertices protected in Phase 3. The vertices protected in Phase 3 range from $x = -2k$ to $x = 0$. The

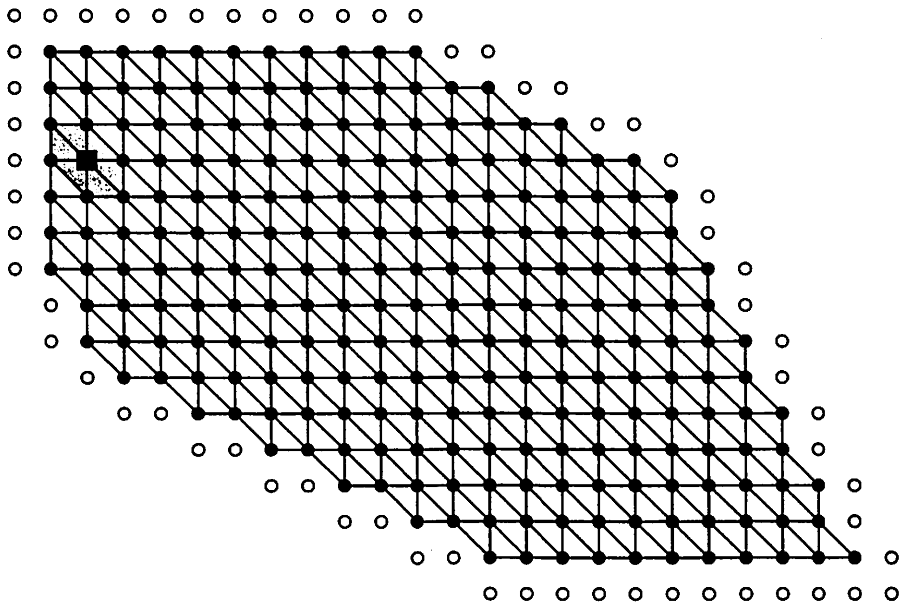


Figure 4: The triangular grid with $k = 1$ protected by Algorithm A.

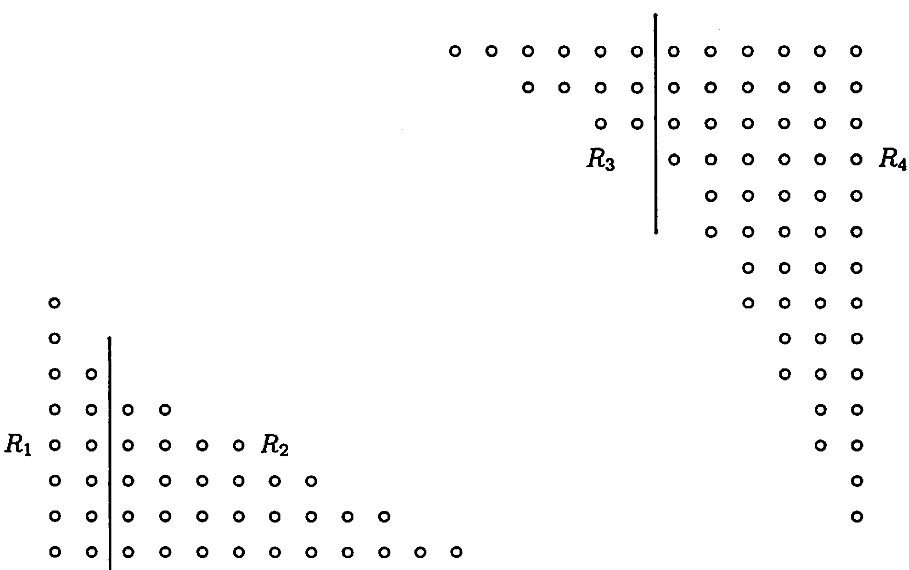


Figure 5: Counting saved vertices on the triangular grid with $k = 1$ protected with Algorithm A.

height of R_1 stretches from $y = -10k - 1$ to $y = -k - 3$, so $h_1 = 9k - 1$. Region R_1 contains $h_1 + (h_1 - 2) + (h_1 - 4) + \cdots + (h_1 - (4k - 2)) = 2kh_1 - 2\frac{2k(2k-1)}{2} = 14k^2 + 10k - 4$ vertices.

Region R_2 is a polygon bounded by $y = -10k - 1$ and the vertices protected in Phase 4. The vertices protected in Phase 4 range from $x = 1$ to $x = 10k$. The height of R_2 stretches from $y = -10k - 1$ to $y = -5k - 2$, so $h_2 = 5k$. Region R_2 contains $2h_2 + 2(h_2 - 1) + 2(h_2 - 2) + \cdots + 2(2) + 2(1) = 2h_2(5k) - 2\frac{5k(5k+1)}{2} = 5k(5k + 1) = 25k^2 + 5k$ vertices.

Region R_3 is a polygon bounded by $y = 3k$ and the vertices protected in Phase 5. The vertices protected in Phase 5 range from $x = 7k+3$ to $x = 14k+1$. The height of R_3 stretches from $y = 3k$ down to $y = 1$, so $h_3 = 3k$. Region R_3 contains $2h_3 + 2(h_3 - 1) + \cdots + 2(2) + 2(1) = 2h_3(3k) - 2\frac{3k(3k-1)}{2} = 3k(3k+1) = 9k^2 + 3k$ vertices.

Region R_4 is a polygon bounded by $y = 3k$ and the vertices protected in Phase 6. The vertices protected in Phase 6 range from $x = 14k+2$ to $x = 18k+3$. The height of R_4 stretches from $y = 3k$ down to $y = -10k$, so $h_4 = 13k + 1$. Region R_4 contains $h_4 + (h_4 - 2) + (h_4 - 4) + \cdots + (h_4 - 10k) = 40k^2 + 13k + 1$ vertices.

The total number of vertices in the rectangle is $260k^2 + 79k + 6$. Thus, the total number of vertices burned is $172k^2 + 58k + 5$. ∎

Conjecture 4 *For any $k \in \mathbb{N}$, if a fire breaks out on the triangular grid Tr and the grid remains unprotected to time $t = k$ when three firefighters begin protecting vertices, the minimum time by which they can contain the fire is $t = 18k + 3$.*

It was proven in [1] that a fire on the triangular grid can be contained by three firefighters leaving at most $1100k^2 - 372k + 27$ vertices burned. It was also conjectured that a minimum of $193k^2 + 47k + 20$ vertices are burned. Theorem 3 proved a significantly stronger result than that of [1].

References

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