Enumerating Optimal Solutions to Special Instances of the Lottery Problem

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Abstract

In this paper a we determine analytically the number of balanced, unlabelled, 3-member covers of an unlabelled finite set, which is then used to find the number of non-isomorphic optimal lottery sets of cardinality three. We also determine numerically the number of non-isomorphic optimal playing sets for lotteries in which a single correct number is required to win a prize.

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AMS Classification: 05B05, 05B40, 05C69, 62K05, 62K10.

1 Introduction

Let $\langle m, n, t; k \rangle$ denote a lottery scheme in which an unordered winning t-set is randomly selected from a universal m-set $\mathcal{U}_m$, and in which a player participates by selecting a playing set of any number of (unordered) n-sets from the universal set prior to the draw. The player is awarded a prize (called a *k-prize*) if k or more elements of the winning t-set occur in at least one of the player's n-sets ($1 \leq k \leq \{n, t\} \leq m$). Let $L(m, n, t; k)$, called the *lottery number*, denote the minimum cardinality of a playing set in $\langle m, n, t; k \rangle$ for which a k-prize is guaranteed and let $\eta(m, n, t; k)$ denote the number of non-isomorphic playing sets of cardinality $L(m, n, t; k)$ in the lottery $\langle m, n, t; k \rangle$.

It is known exactly which combinations of the parameters m, n, t and k render the values 1, 2 or 3 for the lottery number [1]. Furthermore, it is clear that $\eta(m, n, t; k) = 1$ when $L(m, n, t; k) = 1$ and it has been shown that $\eta(m, n, t; k) = \sum_{i=\max\{0, m-2n\}}^{t-2k+1} \zeta_2(m - i, n)$ when $L(m, n, t; k) = 2$,

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where $\zeta_\ell(m,n)$ denotes the number of ways in which $\mathcal{U}_m$ may be covered by means of ℓ distinct unordered subsets of $\mathcal{U}_m$, each of cardinality n [1]. We also showed how to evaluate $\eta(m,n,t;k)$ in terms of $\zeta_3(m,n)$ when $L(m,n,t;k) = 3$ in [1]. Yet, although

$$\zeta_2(m,n) = \begin{cases} 1 & \text{if } n < m \leq 2n \\ 0 & \text{otherwise,} \end{cases}$$

the evaluation of $\zeta_\ell(m,n)$ seems to be a hard problem in general, for $\ell \geq 3$. Our aim in this paper is twofold:

1. To show that $\zeta_3(m,n)$ may be written in terms of the well-known *partition number* $\Pi(r,k)$ of an integer r into k positive parts [8] (thereby rendering the evaluation of $\eta(m,n,t;k)$ tractable by means of a recurrence relation [7] when $L(m,n,t;k) = 3$).

2. To evaluate $\eta(m,n,t;1)$ by means of a generating function and to apply this approach numerically for all combinations of m, n and t within the ranges $1 \leq \{n,t\} \leq m \leq 99$, where $n \leq m/2$ is additionally not allowed to increase above 15 (these are the ranges for which results are published in the online lottery database [2]).

These problems are not only interesting in their own right, *i.e.* from a combinatorial perspective; they are also useful from an application point of view, even though the two lottery classes involved are perhaps the simplest or most trivial classes, because values of η for these classes may be used to establish new lottery numbers outside these classes[1]. Furthermore, an analytic enumeration of non-isomorphic solutions to the two special classes of lottery problems listed above will make it unnecessary to list values numerically in databases such as [2].

For both of the above classes of lotteries values of η are determined by summing together the numbers of covers of appropriate forms[2]. Hence we start our exposition in §2 by recalling, from [3], a well-structured method of viewing and counting set covers, namely by means of sequences of so-called set contractions. This is followed, in §3, by the establishment of a relationship between $\zeta_3(m,n)$ and $\Pi(r,k)$. We then turn our attention, in §4, to the question of evaluating $\eta(m,n,t;1)$.

[1]For example, in [1, Theorem 8] we used the $\eta(17,6,6;3) = 3$ optimal playing structures in a construction technique to show that $L(18,6,6;3) \neq 6$ which, together with the previously known bounds $5 < L(18,6,6;3) \leq 7$ [2], yielded the new result $L(18,6,6;3) = 7$. Another example occurs in [1, Theorem 9], where we used the fact that $\eta(18,6,9;4) = 1$ to establish the new lower bound $L(19,6,9;4) > 6$.

[2]Considerable work has been done on the enumeration of general covers of a finite set (see, for example [4, 5, 6]), yet the enumeration of covers in which all cover members have the same cardinality, called *balanced* covers, seems to be a hard problem. We are not aware of any analytical results on the enumeration of balanced covers of a finite set, except our first steps in this direction contained in [3].

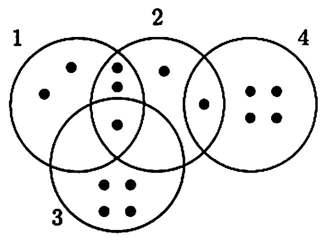


Figure 2.1: A 4-member cover of 15 elements may be formed by means of four contractions from four disjoint sets of 5 elements each.

2 Set covers and contractions

A *cover* of a finite set $\mathcal{U}_m$ of m indistinguishable elements is a family $\mathcal{C}$ of distinct, non-empty, subsets of $\mathcal{U}_m$ whose union is $\mathcal{U}_m$. Any subset of the members of a cover is called a *subcover* of that cover and an element of $\mathcal{U}_m$ is said to be *uniquely covered* by $\mathcal{C}$ if it is an element of exactly one member of $\mathcal{C}$. Furthermore, a cover is said to be *n-balanced* if each of its members has cardinality n, for some $n \in \mathbb{N}$; hence there are $\zeta_\ell(m,n)$ n-balanced, ℓ-member covers of $\mathcal{U}_m$. In the remainder of this paper *all* covers will be assumed to be n-balanced, without mentioning this each time (the symbol n will be reserved for this purpose throughout). Finally, a cover of $\mathcal{U}_m$ of minimum cardinality is called a *minimum cover* of $\mathcal{U}_m$.

In [3] we showed how the overlapping structure of any ℓ-member cover of $m \leq n\ell$ elements may be described by a so-called *contraction vector*, which has the form

$$\vec{\mathcal{C}} = \left[\left(a_1^{(1)}a_2^{(1)}\cdots a_{z_1+1}^{(1)}\right)\left(a_1^{(2)}a_2^{(2)}\cdots a_{z_2+1}^{(2)}\right)\cdots\left(a_1^{(y)}a_2^{(y)}\cdots a_{z_y+1}^{(y)}\right)\right], \quad (1)$$

where y denotes the number of elements of $\mathcal{U}_m$ not uniquely covered and where each value $a_k^{(j)}$ represents a member of the set $\mathcal{L} = \{1,\ldots,\ell\}$, for all $1 \leq k \leq z_j+1$ and all $1 \leq j \leq y$. For each *element* of $\mathcal{U}_m$ shared exclusively by *members* $a_1^{(j)}, a_2^{(j)}, \ldots, a_{z_j+1}^{(j)}$ of $\mathcal{C}$, an *entry* of the form

$$\left(a_1^{(j)}a_2^{(j)}\cdots a_{z_j+1}^{(j)}\right) \quad (2)$$

is included in the contraction vector $\vec{\mathcal{C}}$. In the remainder of the paper we consistently use the terms element, member and entry in the contexts italicized above.

Because we consider unlabelled covers, any set of m symbols may be used to denote the cover members, but we shall follow the convention of using the natural numbers $1, 2, 3\ldots$ to denote the members. Also, the order of the entries in a contraction vector is irrelevant, but we adopt the convention of sorting the entries first by their length and then by the contents of each entry lexicographically — such a contraction vector is said to be in *standard form*. To decide whether one contraction vector in standard form is *smaller* than another (denoted by means of the symbol $\prec$), we first compare the length of the entries and then their contents. For example, $[(12)(13)(14)] \prec [(12)(13)(15)] \prec [(12)(12)(123)]$. The so-called *canonical form* of a contraction vector is taken to be the smallest contraction vector in standard form that results when all permutations of its member labels are considered. Note that the canonical form of a contraction vector is unique.

Example 1 *The 4-member cover of $\mathcal{U}_{15}$ shown in Figure 2.1 has the contraction vector $\vec{C} = [(12)(12)(24)(123)]$ associated with it, indicating that four members of $\mathcal{U}_{15}$ are not uniquely covered. Moreover, two elements of $\mathcal{U}_{15}$ are shared by cover members 1 and 2, one element of $\mathcal{U}_{15}$ is shared by cover members 1, 2 and 3, and one element of $\mathcal{U}_{15}$ is shared by cover members 2 and 4. The above contraction vector is in standard form, but not in canonical form, because the permutation $\binom{1\ 2\ 3\ 4}{1\ 2\ 4\ 3}$ of the cover member labels yields a smaller contraction vector in standard form (which is then also in canonical form).* ■

If s_i denotes the number of elements that occur in exactly i members of an ℓ-member cover C of $\mathcal{U}_m$ ($i = 1, \ldots, \ell$), then C may be constructed via a series of $\sum_{i=2}^{\ell} s_i$ contractions from ℓ disjoint n-sets. Define the quantity $c_\ell(m,n) = n\ell - m$ as the *contraction number* of the cover. More precisely,

$$c_\ell(m,n) = n\ell - m = \sum_{i=1}^{\ell} is_i - \sum_{i=1}^{\ell} s_i = \sum_{i=2}^{\ell}(i-1)s_i \tag{3}$$

represents the number of elements not uniquely covered by members of C, counting multiplicities (in the sense of counting elements that occur in exactly i members of the cover $i-1$ times), and hence is a measure of the degree of overlap that is present between the members of C. Note that all ℓ-member covers of $\mathcal{U}_m$ have the same contraction number. The number of entries of length i in the contraction vector (1) is given by s_i, so that

$$c_\ell(m,n) = \sum_{i=2}^{\ell}(i-1)s_i = \sum_{j=1}^{y}(z_j+1-1) = \sum_{j=1}^{y} z_j. \tag{4}$$

This implies that each entry of the form (2) contributes a value z_j to the contraction number.

It is clear that two members of a cover cannot share more than $n-1$ elements. If $n > c_\ell(m,n)$, then the size of the cover members is not a restriction and the number of covers are independent of n, so that $\zeta_\ell(m,n) = \zeta'_\ell(c)$ may be defined in terms of ℓ and the contraction number c only. Hence $\zeta'_\ell(c)$ is the number of ℓ-member covers with contraction number c whose members have size larger than c.

In [3] we proved the following characterisation of minimum covers in terms of the contraction number $c_\ell(m,n)$.

Theorem 1 *An ℓ-member cover of $\mathcal{U}_m$ with contraction number $c_\ell(m,n)$ is minimum if and only if $c_\ell(m,n) \leq n-1$.* ■

It is clear, from the above theorem and the definition of $\zeta'_\ell(c)$, that $\zeta'_\ell(c)$ is therefore the number of minimum covers of $\mathcal{U}_m$, where $\ell = \lceil m/n \rceil$ and $c = n\ell - m$. Finally, define a *minimum contraction vector* as a contraction vector for which $c = \sum_{j=1}^{y} z_j \leq n-1$, and let $\mathcal{V}_c$ denote the set of minimum contraction vectors with contraction number c. We apply the results and definitions of this section in the remaining sections of the paper.

3 Lotteries $\langle m,n,t;k\rangle$ for which $L(m,n,t;k)=3$

In [1] we gave a characterisation of when $L(m,n,t;k)=3$ and also proved the following result.

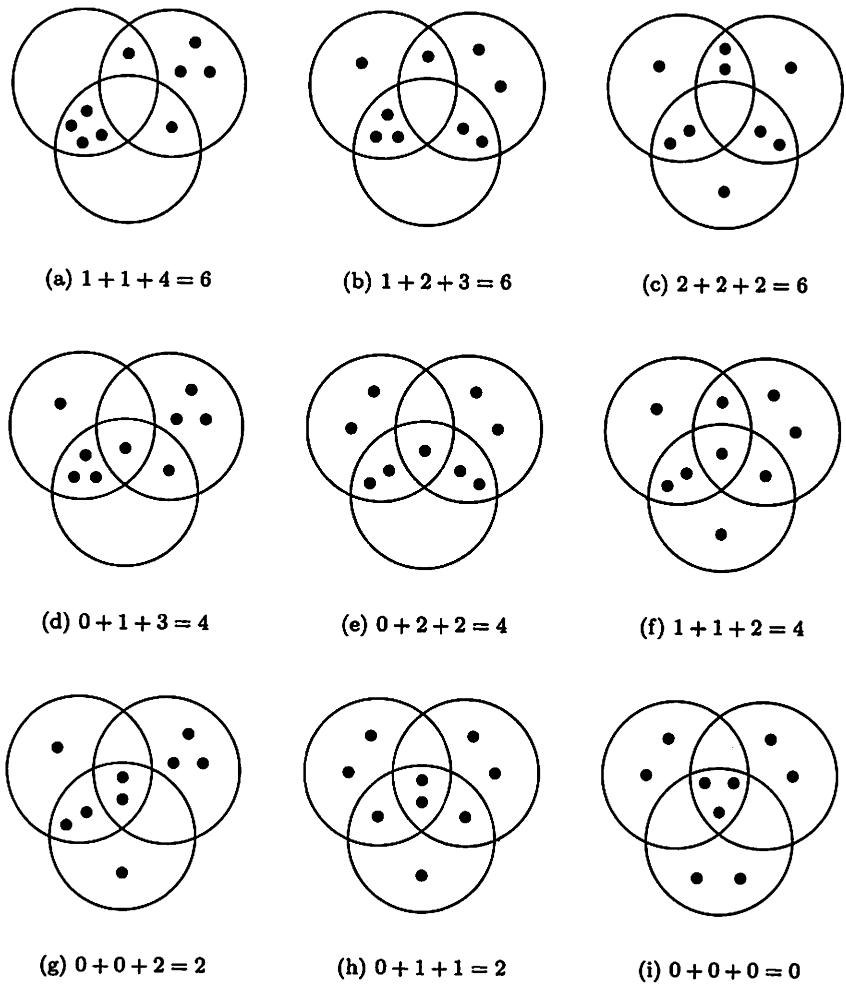
Theorem 2 *When $L(m,n,t;k)=3$,*

$$\eta(m,n,t;k) = \begin{cases} \displaystyle\sum_{i=0}^{t-3k+2} \zeta_3(m-i,n), & \text{if } m \geq 2n \\ \displaystyle\sum_{i=0}^{2t-3k-2m+3n+2} \zeta_3(m-i,m-n), & \text{if } m < 2n. \end{cases} \tag{5}$$

However, the above result was not wholly satisfactory, because we were not able to determine $\zeta_3(\cdot,\cdot)$ analytically — hence we resorted to tabulating values for this parameter numerically (by means of a computationally rather expensive exhaustive search tree enumeration procedure) for small values of its arguments. We are now able to evaluate $\zeta_3(\cdot,\cdot)$ in terms of the well-known partition number $\Pi(r,k)$ of an integer r into k positive parts (which may be found efficiently by means of a recurrence relation [7]), and our main result in this section is the following.

Theorem 3 *If $n < m$, then*

$$\zeta_3(m,n) = \sum_{x=\max\{0,c-n\}}^{\lfloor c/2 \rfloor} \Pi(c-2x+3,3) + \sum_{x=0}^{c-n-1} \Pi(x-c+m+3,3) - \zeta_2(m,n),$$



(a) $1+1+4=6$ (b) $1+2+3=6$ (c) $2+2+2=6$

(d) $0+1+3=4$ (e) $0+2+2=4$ (f) $1+1+2=4$

(g) $0+0+2=2$ (h) $0+1+1=2$ (i) $0+0+0=0$

Figure 3.2: The $\zeta_3(9,5) = 9$ different 5-balanced, 3-member covers of 9 elements, each member being of cardinality 5: (a)–(c) The $\Pi(6,3) = 3$ covers corresponding to the partitions of 6 into three positive parts. (d)–(f) The $\Pi(4+3,3) - 1 = 3$ covers corresponding to the partitions of 4 into three non-negative parts — disregarding the partition $0+0+4 = 4$, because in this case the cover members are not distinct and hence the structure is not a valid cover. (g)–(h) The $\Pi(2+3,3) = 2$ covers corresponding to the partitions of 2 into three non-negative parts. (i) The $\Pi(0+3,3) = 1$ cover corresponding to the partitions of 0 into three non-negative parts.

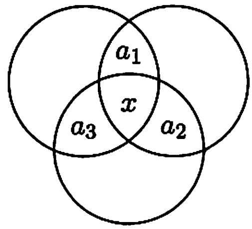


Figure 3.3: A 3-member cover of $\mathcal{U}_m$.

where $c = c_3(m,n) = 3n - m$.

Note that the above theorem holds for *all* (n-balanced) 3-member covers of $\mathcal{U}_m$; not just for minimum covers of $\mathcal{U}_m$. Before proving this theorem, let us first illustrate its result by means of a simple example.

Example 2 *Suppose* $m = 9$ *and* $n = 5$. *Then the contraction number is* $c_3(9,5) = (5 \times 3) - 9 = 6$ *and, by Theorem 3,*

$$\begin{aligned}
\zeta_3(9,5) &= \sum_{x=6-5}^{6/2} \Pi(6-2x+3,3) + \sum_{x=0}^{6-5-1} \Pi(x-6+9+3,3) - \zeta_2(9,5) \\
&= \sum_{x=1}^{3} \Pi(9-2x,3) + \sum_{x=0}^{0} \Pi(x+6,3) - \zeta_2(9,5) \\
&= \Pi(7,3) + \Pi(5,3) + \Pi(3,3) + \Pi(6,3) - \zeta_2(9,5) \\
&= 4+2+1+3-1 \\
&= 9.
\end{aligned}$$

The various partitions enumerated above, as well as their corresponding covers of $\mathcal{U}_9$, *are shown in Figure 3.2.* ■

In order to prove Theorem 3, let a_1, a_2, a_3 and x denote the numbers of elements of $\mathcal{U}_m$ in the lottery set overlapping structure depicted in the 3-member cover of $\mathcal{U}_m$ in Figure 3.3. The contraction number of this cover is given by

$$c = 3n - m = \underbrace{a_1 + a_2 + a_3}_{s_2} + \underbrace{2x}_{2s_3}, \tag{6}$$

where $c = c_3(m,n)$. Assume, without loss of generality, that

$$0 \le a_1 \le a_2 \le a_3. \tag{7}$$

Then, for a fixed non-negative value of x, the number of covers of the form depicted in Figure 3.3 is given by the number of integral solutions to the equation

$$a_1 + a_2 + a_3 = c - 2x \tag{8}$$

subject to the constraints

$$\left.\begin{array}{rrcl} & a_1 + a_2 + x & \leq & n, \\ & a_1 + a_3 + x & \leq & n, \\ \text{and} & a_2 + a_3 + x & \leq & n. \end{array}\right\} \tag{9}$$

Constraint (7) ensures that we count non-isomorphic covers, whilst constraint (8) ensures that the structure is, in fact, a cover of $\mathcal{U}_m$. Finally, constraint (9) ensures that the cover is n-balanced.

If we rewrite the constraint $a_1+a_2+x \leq n$ in (9) as $c-2x-a_3+x \leq n$ by utilisation of (6), then we find that $a_3 \geq c - x - n$, and similarly for a_1 and a_2. Thus we may replace (9) with the system of inequalities

$$a_i \geq c - x - n, \quad i = 1, 2, 3. \tag{10}$$

Therefore the problem of evaluating $\zeta_3(m,n)$ reduces to finding the number of partitions of the integer $c - 2x$ into three parts, each of size at least $c - x - n$. (Note that $c - x - n$ may be negative.)

	n													
	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$m=3$	1	0	0	0	0	0	0	0	0	0	0	0	0	0
$m=4$	2	1	0	0	0	0	0	0	0	0	0	0	0	0
$m=5$	1	3	1	0	0	0	0	0	0	0	0	0	0	0
$m=6$	1	3	4	1	0	0	0	0	0	0	0	0	0	0
$m=7$	0	3	5	4	1	0	0	0	0	0	0	0	0	0
$m=8$	0	1	6	6	4	1	0	0	0	0	0	0	0	0
$m=9$	0	1	4	9	7	4	1	0	0	0	0	0	0	0
$m=10$	0	0	3	8	11	7	4	1	0	0	0	0	0	0
$m=11$	0	0	1	7	12	12	7	4	1	0	0	0	0	0
$m=12$	0	0	1	4	13	15	13	7	4	1	0	0	0	0
$m=13$	0	0	0	3	9	18	17	13	7	4	1	0	0	0
$m=14$	0	0	0	1	7	16	22	18	13	7	4	1	0	0
$m=15$	0	0	0	1	4	14	23	25	19	13	7	4	1	0
$m=16$	0	0	0	0	3	9	23	28	27	19	13	7	4	1
$m=17$	0	0	0	0	1	7	17	31	32	28	19	13	7	4
$m=18$	0	0	0	0	1	4	14	28	38	35	29	19	13	7
$m=19$	0	0	0	0	0	3	9	24	38	43	37	29	19	13
$m=20$	0	0	0	0	0	1	7	17	37	46	47	38	29	19

Table 3.1: Values of $\zeta_3(m,n)$ for $2 \leq n \leq \min\{m, 15\}$ and $3 \leq m \leq 20$.

It is well-known that $\Pi(r+k, k)$ is the number of partitions of the integer $r \in \mathbb{N}$ into k *non-negative* parts [7], where $\Pi(r, k)$ denotes the number of different partitions of r into $k \geq 1$ *positive* parts. Note that the latter quantity may be obtained from the former by "borrowing" k additional units and partitioning total of $r + k$ units into k non-empty parts, and then removing one unit from each part afterwards. Conversely, the former quantity may be obtained from the latter by first placing one unit in each part, and then partitioning the remaining $r - k$ units into k non-negative parts. This basic technique may be extended further: if we require at least j units in each part, then the number of partitions is given by $\Pi(r+k-kj, k)$.

If we now split our problem of evaluating $\zeta_3(m,n)$ into two cases (depending on the value of x), namely (i) $c - x - n \leq 0$ where parts should merely be non-negative (*i.e.* $j = 0$), and (ii) $c - x - n \geq 1$ where parts should, in fact, be positive (*i.e.* $j = c - x - n \geq 1$), then it is clear that

$$\zeta_3(m,n) = \underbrace{\sum_{x=\max\{0,c-n\}}^{\lfloor c/2 \rfloor} \Pi(c-2x+3,3)}_{\text{(i)}} + \underbrace{\sum_{x=0}^{c-n-1} \Pi(x-c+m+3,3)}_{\text{(ii)}} - \underbrace{\zeta_2(m,n)}_{\text{(iii)}}, \tag{11}$$

which proves Theorem 3. The term (iii) above is subtracted to correct for covers counted in (i) and (ii) in which two members coincide exactly. ■

Values for $\zeta_3(m,n)$ in (11) within the parameter ranges $2 \leq n \leq \min\{m, 15\}$ and $3 \leq m \leq 20$ are tabulated in Table 3.1.

The following corollary follows directly from Theorem 3.

Corollary 1 *When* $L(m,n,t;k) = 3$,

$$\eta(m,n,t;k) = \begin{cases} \displaystyle\sum_{i=0}^{t-3k+2} \sum_{x=0}^{\lfloor \frac{3n-m+i}{2} \rfloor} \Pi(3n-m+i-2x+3,3) & \text{if } m \geq 2n \\ \displaystyle\sum_{i=0}^{\substack{2t-3k-2m \\ +3n+2}} \sum_{x=0}^{\lfloor \frac{2m-3n+i}{2} \rfloor} \Pi(2m-3n+i-2x+3,3) & \text{if } m < 2n. \end{cases} \tag{12}$$

Proof: For minimum covers $n \geq c+1$ by Theorem 1. Hence the result of Theorem 3 simplifies to

$$\zeta_3(m,n) = \sum_{x=0}^{\lfloor c/2 \rfloor} \Pi(c-2x+3,3),$$

because the terms (ii) and (iii) in (11) are zero in this case. Substitution of the above expression into (5) yields the desired result. ■

4 Lotteries of the form $\langle m, n, t; 1\rangle$

Lotteries of the form $\langle m, n, t; 1\rangle$ constitute the simplest class of lotteries, because when $k = 1$ the lottery problem reduces to the problem of covering the universal sets $\mathcal{U}_{m-t+1}, \ldots, \mathcal{U}_{\max\{n\ell, m\}}$ where $\ell = L(m, n, t; 1)$, as will be described in this section. The following result is well-known and seems to be folklore.

Proposition 1 $L(m, n, t; 1) = \lceil \frac{m-t+1}{n} \rceil$.

Proof: The set $\mathcal{U}_{m-t+1}$ may be covered by a collection $\mathcal{C}_1$ of $\lceil \frac{m-t+1}{n} \rceil$ n-subsets of $\mathcal{U}_{m-t+1}$, of which $\lfloor \frac{m-t+1}{n} \rfloor$ are mutually disjoint, in which case any t-subset w of $\mathcal{U}_m$ coincides with at least one element in at least one of the members of $\mathcal{C}_1$. This shows that $L(m, n, t; 1) \leq |\mathcal{C}_1| = \lceil \frac{m-t+1}{n} \rceil$.

However, a collection $\mathcal{C}_2 = \{S_1, \ldots, S_{\lceil \frac{m-t+1}{n} \rceil - 1}\}$ of n-subsets of $\mathcal{U}_m$ can cover at most $n\left(\lceil \frac{m-t+1}{n} \rceil - 1\right) < n\left(\frac{m-t+1}{n}\right) = m - t + 1$ elements of $\mathcal{U}_m$, namely when the members of $\mathcal{C}_2$ are mutually disjoint. Hence $|\mathcal{U}_m \backslash \cup S_i| > m - (m - t + 1) = t - 1$, so that there is an empty intersection between any t-subset $w \in \mathcal{U}_m \backslash \cup S_i$ and all members of $\mathcal{C}_2$. This shows that $L(m, n, t; 1) > |\mathcal{C}_2| = \lceil \frac{m-t+1}{n} \rceil - 1$. ■

In [3] we determined the number, $\hat{\xi}(m, n)$, of minimum (n-balanced) *labelled* covers of $\mathcal{U}_m$ in terms of the contraction number, where $\ell = \lceil m/n \rceil$ is the minimum number of n-sets required to cover $\mathcal{U}_m$. If this result is generalised to *unlabelled* covers, then the function $\zeta_\ell(m, n)$ is clearly obtained. The value of $\eta(m, n, t; 1)$ may then be computed by means of the following result.

Theorem 4 $\eta(m, n, t; 1) = \sum_{i=m-t+1}^{\min\{n\ell, m\}} \zeta_\ell(i, n) = \sum_{c=\max\{0, n\ell-m\}}^{n\ell-m+t-1} \zeta'_\ell(c)$, *where* $\ell = L(m, n, t; 1)$.

Proof: To determine $\eta(m, n, t; 1)$ it is necessary to add together the number of distinct unlabelled $L(m, n, t; 1)$-member covers $\zeta_{L(m,n,t;1)}(i, n)$ of $\mathcal{U}_i$, where $i \leq m$ denotes the number of elements from $\mathcal{U}_m$ that are utilised in such valid minimum covers. Furthermore, i is clearly at least $m-t+1$ and at most $n\ell$. The second sum is obtained by changing the summation index by means of the substitution $c = n\ell - i$, where c represents the contraction number. ■

Our aim is to evaluate $\zeta'_\ell(c)$ for small values of ℓ and c, from which values of $\eta(m, n, t; 1)$ may then be deduced via Theorem 4 for values of m, n and t within the ranges mentioned in the introduction. The evaluation of $\zeta'_\ell(c)$ hinges on the following result.

Theorem 5 $\xi'_\ell(c) = |\mathcal{V}_c|$.

Proof: It is clear, from the definition of a contraction vector in canonical form and by Theorem 1, that there exists for each minimum cover of $\mathcal{U}_m$ a unique element $v \in \mathcal{V}_p$, so that we have $\xi'_\ell(c) \leq |\mathcal{V}_p|$. The proof is completed by showing that for each contraction vector $v \in \mathcal{V}_p$ there exists a unique corresponding minimum cover of $\mathcal{U}_m$, and hence that $\xi'_\ell(c) \geq |\mathcal{V}_p|$. From (4) it may be seen that there can be at most $c_\ell(m,n) \leq n-1$ entries in a contraction vector of the form (1) satisfying (3), because $y \leq \sum_{j=1}^{y} i_j = c_\ell(m,n) \leq n-1$. Thus there can be at most $n-1$ contraction vector entries, which means that any member of an overlapping set structure corresponding to the contraction vector may be involved in at most $n-1$ contractions. As a result all members of the corresponding overlapping set structure are distinct (and have cardinality n). Hence the overlapping set structure is indeed a cover of $\mathcal{U}_m$. Furhermore, it is a minimum cover of $\mathcal{U}_m$ by Theorem 1, and it is clear that this minimum cover is unique. ■

A cover is said to be *disconnected* if the entries of the corresponding contraction vector may be partitioned into a number of parts such that no element of $\mathcal{U}_m$ occurs in more than one part; otherwise it is called *connected.* A subcover is called a *component* of the cover if it is a maximal connected subcover (in the sense that the addition of any cover member to the subcover would render the new subcover disconnected). A component of a cover is called a *trivial component* if it comprises exactly one member of the cover; otherwise it is called a *non-trivial component.*

In order to reduce the computational complexity of evaluating $\zeta'_\ell(c)$, we only count the number of contraction vectors of connected covers and then use a generating function to accommodate disconnected covers. Our algorithm for evaluating the number of connected covers (given in pseudocode in Algorithm 1) generates the contraction vectors of all connected covers sequentially and determines their canonical form. If the canonical form of a vector is smaller than one previously considered, it is discarded (because the corresponding cover has already been considered and hence counted); otherwise the cover is included in the count. We represent the total number of connected covers with contraction number c by means of the polynomial

$$C_j(x) = \sum_{i=2}^{j+1} a(i,j)x^i, \tag{13}$$

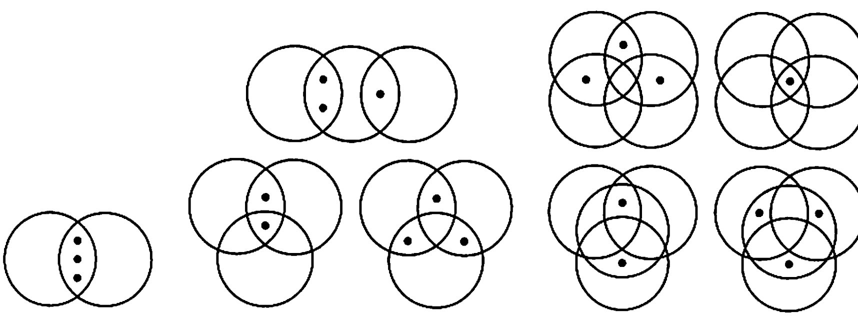
where the coefficient $a(i,j)$ denotes the number of connected i-member covers with contraction number j.

Algorithm 1: Counting connected covers with contraction number c

Input: The contraction number, j, of set structures to be considered.
Output: The coefficients $a(i,j)$ in (13).

1. **for** $i = 2, 3 \ldots, j+1$ **do** $a(i,j) \leftarrow 0$
2. **for** each partition p of j **do**
   - 2.1 **for** each contraction vector $\vec{C}$ corresponding to partition p **do**
     - 2.1.1 **if** $\vec{C}$ is canonical and its structure is connected **then**
       - 2.1.1.1 $M \leftarrow$ largest cover member label in $\vec{C}$
       - 2.1.1.2 $a(M,j) \leftarrow a(M,j) + 1$
3. **output** $a(2,j), a(3,j), \ldots, a(j+1,j)$

The number of unlabelled, connected, i-member covers with contraction number j, namely $a(i,j)$, is tabulated in Table 4.2 for $2 \le i \le 11$ and $1 \le j \le 14$ and illustrated graphically in Figure 4.4 for the cases $i = 2, 3, 4$ and $j = 3$.



$a(2,3) = 1$ $\quad$ $a(3,3) = 3$ $\quad$ $a(4,3) = 4$

Figure 4.4: The number of unlabelled, connected, i-member covers with contraction number j, for $i = 2, 3, 4$ and $j = 3$.

Let $C_j^{\alpha_j}(x)$ be a generating function for the number of covers of $\mathcal{U}_m$ comprising α_j non-trivial components, each with contraction number j. If $\alpha_j = 1$ then $C_j^{\alpha_j}(x) = C_j(x)$ as described in (13). We describe a method of computing this generating function later in this section.

The number of different ways in which components that have *different* contraction numbers may be combined to form disconnected covers may be achieved via a straight forward multiplication of the corresponding polynomials. For example, $C_r^{\alpha_r}(x) \times C_s^{\alpha_s}(x)$ is a polynomial representing all covers comprising $\alpha_r + \alpha_s$ components with corresponding contraction numbers r and s so as to form a cover with overall contraction number $r\alpha_r + s\alpha_s$. However, we need to be careful not to count covers more than once when components that have the *same* contraction number are combined. To count the total number of covers with contraction number c and no trivial components, we consider all partitions of c, where each part represents a component. Thus, for a partition $\pi(j) = 1^{\alpha_1}2^{\alpha_2}\cdots j^{\alpha_j}$ we may

	c													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\ell = 2$	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\ell = 3$	0	2	3	6	8	13	16	23	28	37	44	56	65	80
$\ell = 4$	0	0	4	11	30	70	142	281	521	928	1 597	2 681	4 352	6 942
$\ell = 5$	0	0	0	9	37	149	474	1 401	3 783	9 750	23 815	56 128	127 337	280 366
$\ell = 6$	0	0	0	0	22	139	714	3 102	12 033	43 335	146 851	474 117	1 466 423	4 370 145
$\ell = 7$	0	0	0	0	0	59	505	3 395	18 799	93 319	426 898	1 837 421	7 513 562	29 419 804
$\ell = 8$	0	0	0	0	0	0	165	1 913	15 879	109 187	667 192	3 755 661	19 839 530	—
$\ell = 9$	0	0	0	0	0	0	0	496	7 255	73 502	609 502	—	—	—
$\ell = 10$	0	0	0	0	0	0	0	0	1 540	27 917	—	—	—	—
$\ell = 11$	0	0	0	0	0	0	0	0	0	4 960	—	—	—	—
Time	0	0	0	0	0	< 1	4	68	1 431	39 935	53 277	58 614	317 051	131 009

Table 4.2: The number, $a(i, j)$, of unlabelled, connected, ℓ-member covers with contraction number $j = c_\ell(m, n)$, where $3 \leq \ell \leq 11$ and $1 \leq c_\ell(m, n) \leq 14$, as obtained via Algorithm 1. Note that $a(i, j) = 1$ if $i \in \{1, 2\}$, for all $j \geq 1$. Entries denoted by '—' were not computed due to the considerable computation times involved and because they are not required to evaluate $\eta(m, n, t; 1)$ within the parameter ranges mentioned in the introduction. The row labelled 'Time' contains the times (in seconds) it took to compute a column of $a(i, j)$ values on an 3.2GHz processor with 512MB of memory.

compute the generating function $C_1^{\alpha_1}(x)C_2^{\alpha_2}(x)\cdots C_j^{\alpha_j}(x)$ to represent all covers associated with the concerned partition. The number of covers with contraction number j (but with no trivial components) is then given by the generating function[3]

$$D_c'(x) = \sum_{1\alpha_1+\ldots+c\alpha_c=c} \prod_{j=1}^{i} C_j^{\alpha_j}(x) = \sum_{i=2}^{2c} b(i,c)x^i \quad \text{(say)}. \tag{14}$$

To compute $C_j^{\alpha_j}(x)$ we count the number of ways in which a total of α_j components may be selected from the various possible connected subcover structures with contraction number j. We do this by selecting (with replacement) n_i components from the $a(i,j)$ components with i members and contraction number j for all $i = 2, \ldots, j+1$ in such a way that $\sum_{i=2}^{j+1} n_i = \alpha_j$. Thus we have

$$C_j^{\alpha_j}(x) = \sum_{\substack{n_2+\ldots+n_{j+1}=\alpha_j \\ 0\le n_i\le\alpha_j}} \prod_{i=2}^{j+1} \binom{a(i,j)-1+n_i}{n_i} x^{in_i}. \tag{15}$$

Note that an arbitrary number of trivial components may be added to a subcover without affecting the contraction number of the cover. Thus, to find the *total* number of covers comprising ℓ members (including covers with trivial components), we have to add $\ell - i$ trivial components to subcovers consisting of i members, for all $1 \le i < \ell$. Therefore a generating function, $D_c(x)$, for the total number of covers (including covers with trivial components) may be found by taking a cumulative sum of the coefficients of the generating function $D_c'(x)$, that is

$$D_c(x) = \sum_{i=2}^{\infty} \underbrace{\left(\sum_{i'=2}^{i} b(i',c) \right)}_{\zeta_\ell'(c)} x^i. \tag{16}$$

The first nineteen coefficients $\zeta_\ell'(c)$ of the polynomial $D_c(x)$ are listed as columns in Table 4.3 for values of the contraction number within the range $2 \le c \le 10$.

[3]The above process of partitioning the contraction number may be achieved automatically by means of the generating function

$$\mathcal{P}(t,x) = \prod_{j=1}^{\infty} \sum_{q=0}^{\infty} (t^q)^j C_j^q(x)$$

in which the coefficient of $x^\ell t^c$ is the number, $\zeta_\ell'(c)$, of ℓ-member covers with contraction number c (and with no trivial components). Thus, to obtain *all* ℓ-member covers with contraction number c (including covers with trivial components), we need to add together the coefficients of $x^i t^c$ for which $i \le \ell$.

	$c_\ell(m,n)$								
	2	3	4	5	6	7	8	9	10
$\ell=2$	1	1	1	1	1	1	1	1	1
$\ell=3$	3	4	7	9	14	17	24	29	38
$\ell=4$	4	9	20	41	87	162	309	554	971
$\ell=5$	4	11	34	89	255	668	1758	4408	10820
$\ell=6$	4	12	42	134	460	1535	5193	17105	55430
$\ell=7$	4	12	44	156	612	2376	9630	38860	156573
$\ell=8$	4	12	45	164	688	2926	13263	61459	290032
$\ell=9$	4	12	45	166	714	3175	15333	77245	404748
$\ell=10$	4	12	45	167	722	3263	16222	85337	479803
$\ell=11$	4	12	45	167	724	3289	16522	88555	507361
$\ell=12$	4	12	45	167	725	3297	16615	89626	519543
$\ell=13$	4	12	45	167	725	3299	16641	89942	523425
$\ell=14$	4	12	45	167	725	3300	16649	90035	524567
$\ell=15$	4	12	45	167	725	3300	16651	90061	524889
$\ell=16$	4	12	45	167	725	3300	16652	90069	524982
$\ell=17$	4	12	45	167	725	3300	16652	90071	525008
$\ell=18$	4	12	45	167	725	3300	16652	90072	525016
$\ell=19$	4	12	45	167	725	3300	16652	90072	525018
$\ell=20$	4	12	45	167	725	3300	16652	90072	525019

Table 4.3: Values of $\zeta'_\ell(c_\ell(m,n))$ for $2 \le \ell \le 20$ and $2 \le c_\ell(m,n) \le 10$.

	n									
	1	2	3	4	5	6	7	8	9	10
$m=1$	1	—	—	—	—	—	—	—	—	—
$m=2$	1	1	—	—	—	—	—	—	—	—
$m=3$	1	1	1	—	—	—	—	—	—	—
$m=4$	1	2	1	1	—	—	—	—	—	—
$m=5$	1	1	1	1	1	—	—	—	—	—
$m=6$	1	2	3	1	1	1	—	—	—	—
$m=7$	1	1	2	1	1	1	1	—	—	—
$m=8$	1	2	1	4	1	1	1	1	—	—
$m=9$	1	1	5	3	1	1	1	1	1	—
$m=10$	1	2	2	2	5	1	1	1	1	1
$m=11$	1	1	1	1	4	1	1	1	1	1
$m=12$	1	2	6	9	3	6	1	1	1	1
$m=13$	1	1	2	5	2	5	1	1	1	1
$m=14$	1	2	1	2	1	4	7	1	1	1
$m=15$	1	1	6	1	16	3	6	1	1	1
$m=16$	1	2	2	15	9	2	5	8	1	1
$m=17$	1	1	1	6	5	1	4	7	1	1
$m=18$	1	2	6	2	2	25	3	6	9	1
$m=19$	1	1	2	1	1	16	2	5	8	1
$m=20$	1	2	1	17	35	9	1	4	7	10

Table 4.4: Values of $\eta(m,n,n,1)$ for $1 \le n \le m \le 20$ and $n \le 10$.

We illustrate the above process by means of a simple example.

Example 3 *It follows by (14) that*

$$\begin{aligned}D_6'(x) &= C_1^6(x) + C_1^3(x)C_1(x)C_2(x) + C_1^3(x)C_3(x) + C_1^2(x)C_2^2(x)\\ &\quad + C_1^2(x)C_4(x) + C_1(x)C_2(x)C_3(x) + C_1(x)C_5(x) + C_2^3(x)\\ &\quad + C_2(x)C_4(x) + C_3^2(x) + C_6(x),\end{aligned}$$

where the coefficients of $C_1(x), \ldots, C_6(x)$ are obtained via Algorithm 1. Note that $C_1(x) = x^2$, so that $C_1^r(x) = \sum_{n_2=r} \binom{a(2,1)-1+r}{r} x^{2r} = x^{2r}$. Furthermore,

$$\begin{aligned}C_3^2(x) &= \sum_{n_2+n_3+n_4=2} \prod_{i=2}^{4} \binom{a(i,3)-1+n_i}{n_i} x^{in_i}\\ &= \binom{1-1+2}{2}x^{2\cdot 2} + \binom{3-1+2}{2}x^{3\cdot 2} + \binom{4-1+2}{2}x^{4\cdot 2} + \binom{1-1+1}{1}x^{2\cdot 1}\binom{3-1+1}{1}x^{3\cdot 1}\\ &\quad + \binom{1-1+1}{1}x^{2\cdot 1}\binom{4-1+1}{1}x^{4\cdot 1} + \binom{3-1+1}{1}x^{3\cdot 1}\binom{4-1+1}{1}x^{4\cdot 1}\\ &= x^4 + 3x^5 + 10x^6 + 12x^7 + 10x^8.\end{aligned}$$

Hence there are, for example, twelve 7-member covers consisting of two non-trivial compontents, each with contraction number 3. The generating functions $C_2^2(x) = x^4 + 2x^5 + 3x^6$ and $C_2^3(x) = x^6 + 2x^7 + 3x^8 + 4x^9$ are computed similarly. Finally, the generating function

$$\begin{aligned}D_6(x) &= x^2 + 14x^3 + 87x^4 + 255x^5 + 460x^6 + 612x^7 + 688x^8 + 714x^9\\ &\quad + 722x^{10} + 724x^{11} + 725\sum_{i=12}^{\infty} x^i\end{aligned}$$

may be obtained from $D_6'(x)$ by taking the cumulative sum in (16). Hence there are, for example, six hundred and twelve 7-member covers with contraction number 6 (including covers with trivial components). ■

Finally, Table 4.3 and Theorem 4 may be used to compute values of $\eta(m, n, t, 1)$. As an example, values of $\eta(m, n, n, 1)$ are given for $1 \leq n \leq m \leq 20$ and $n \leq 10$ in Table 4.4.

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