On the Total Number of Parts in Various Partitions

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ABSTRACT: Let n be a natural number. We obtain convolution-type formulas for the total number of parts in all partitions of n of several different kinds.

Introduction

Let $s(n)$ denote the total number of parts in all partitions of the natural number n. In [3], it was shown that if $d(n)$ is the number of divisors of n, and $p(n)$ is the number of partitions of n, then

$$s(n) = \sum_{k=1}^{n} d(k)p(n-k) \ .$$

Let the integer $r \geq 2$. In this note, we derive analogous identities for the total number of parts in (i) partitions in r colors; (ii) partitions with distinct parts in r colors; (iii) partitions with odd parts in r colors; (iv) overpartitions; (v) overpartitions with odd parts; (vi) partitions into parts not divisible by r; (vii) partitions such that no part occurs r or more times.

Preliminaries

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Definitions

$p_r(n)$ is the number of partitions of n in r distinct colors

$q_r(n)$ is the number of partitions of n with distinct parts in r distinct colors

$q_r'(n)$ is the number of partitions of n with odd parts in r distinct colors

$\overline{p}(n)$ is the number of overpartitions of n

$\overline{q}(n)$ is the number of overpartitions of n with odd parts

$b_r(n)$ is the number of partitions of n such that no part is divisible by r

$b_r^*(n)$ is the number of partitions of n such that no part occurs r or more times.

$d(n)$ is the number of divisors of n

$d_1(n)$ is the number of odd divisors of n

$d_0(n)$ is the number of even divisors of n

$e(n) = d_1(n) - d_0(n)$

$$d^*(n) = \begin{cases} d(n) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Remarks: We define $d(0) = 0$, hence $d_1(0) = d_0(0) = e(0) = d^*(0) = 0$.

Identities

Let $x \in C$, $|x| < 1$.

$$\sum_{n\geq 0} p_r(n)x^n = \prod_{n\geq 1}(1-x^n)^{-r} \tag{1}$$

$$\sum_{n\geq 0} q_r(n)x^n = \prod_{n\geq 1}(1+x^n)^{r} \tag{2}$$

$$\sum_{n\geq 0} q_r'(n)x^n = \prod_{n\geq 1}(1-x^{2n-1})^{-r} \tag{3}$$

$$\sum_{n\geq 0} \overline{p}(n)x^n = \prod_{n\geq 1}\frac{1+x^n}{1-x^n} \tag{4}$$

$$\sum_{n\geq 0} \overline{q}(n)x^n = \prod_{n\geq 1}\frac{1+x^{2n-1}}{1-x^{2n-1}} \tag{5}$$

$$\sum_{n\geq 0} b_r(n)x^n = \prod_{n\geq 1}\frac{1-x^{rn}}{1-x^{rn}} \tag{6}$$

$$\sum_{n\geq 0} b_r^*(n)x^n = \prod_{n\geq 1}(1+x^n+x^{2n}+\cdots+x^{(r-1)n}) \tag{7}$$

$$\sum_{n\geq 0} d(n)x^n = \sum_{j\geq 1} \frac{x^j}{1-x^j} \tag{8}$$

$$\sum_{n\geq 0} e(n)x^n = \sum_{n\geq 1} \frac{x^n}{1+x^n} \tag{9}$$

$$\sum_{n\geq 0} d_1(n)x^n = \sum_{j\geq 1} \frac{x^j}{1-x^{2j}} = \sum_{j\geq 1} \frac{x^{2j-1}}{1-x^{2j-1}} \tag{10}$$

$$\sum_{n\geq 0} d^*(n)x^n = \sum_{j\geq 1} \frac{x^{2j-1}}{1-x^{4j-2}} \tag{11}$$

Remarks: Identities (1) through (7) are well-known generating functions for the partition functions under consideration. (See [1].) Identity (8) is a well-nown Lambert series. Identities (9) through (11) are analogs of (8). Note that $q_r'(n) = q_r(n)$ and that $b_r'(n) = b_r(n)$.

3. The Main Results

If $f(n)$ counts the number of partitions of n of a certain type, let its generating function be

$$F(x) = \sum_{n\geq 0} f(n)x^n$$

where $x \in C$, $|x| < 1$. Let $f(n,k)$ denote the number of that type of partition with k parts, where $1 \leq k \leq n$. Let $s(n)$ denote the total number of parts in all partitions of n of that type, so that

$$s(n) = \sum_{k=1}^{n} kf(n,k) \ .$$

Let the generating function of $s(n)$ be

$$S(x) = \sum_{n\geq 0} s(n)x^n \ .$$

Consider the bivariate generating function

$$F(x,u) = \sum_{k\geq 0}\sum_{n\geq 0} f(n,k)u^k x^n \ .$$

Then we have

$$S(x) = \frac{\partial F}{\partial u}\bigg|_{u=1} \tag{12}$$

(For more details concerning bivariate generating functions, see [5], p. 133-137.)

Often, the right member of (12) has the form $D(x)F(x)$, where $D(x)$ is the generating function of a divisor function $\delta(n)$, that is,

$$D(x) = \sum_{n\geq 0} \delta(n)x^n .$$

Thus we have

$$\sum_{n\geq 0} s(n)x^n = (\sum_{n\geq 0} \delta(n)x^n)(\sum_{n\geq 0} f(n)x^n) .$$

This implies that $s(n)$ is the convolution of $\delta(n)$ and $f(n)$, that is,

$$s(n) = \sum_{k=1}^{n} \delta(k)f(n-k) .$$

(For more details concerning convolutions, see [2], p. 31.)

We begin by considering partitions in r distinct colors.

Theorem 1 If $r \geq 2$, let $s_r(n)$ denote the total number of parts in all partitions of n in r distinct colors. Then

$$s_r(n) = r\sum_{k=1}^{n} d(k)p_r(n-k) .$$

Proof: If $1 \leq k \leq n$, let $p_r(n,k)$ denote the number of partitions of n into k parts in r distinct colors. Then

$$s_r(n) = \sum_{k=1}^{n} kp_r(n,k) \quad .$$

Let

$$P_r(x) = \sum_{n\geq 0} p_r(n)x^n = \prod_{n\geq 1}(1-x^n)^{-r} .$$

Consider the bivariate generating function

$$P_r(x,u) = \prod_{i\geq 1}(1-ux^i)^{-r} .$$

Then we have

$$\frac{\partial}{\partial u}(P_r(x,u)) = \sum_{j\geq 1} -r(1-ux^j)^{-r-1}(-x^j)\prod_{i\neq j}(1-ux^i)^{-r} =$$

$$\sum_{j\geq 1}\frac{rx^j}{1-ux^j}\prod_{i\geq 1}(1-ux^i)^{-r} .$$

Let

$$S_r(x) = \sum_{n\geq 0} s_r(n)x^n .$$

Then

$$S_r(x) = \frac{\partial}{\partial u}(P_r(x,u))\Bigg|_{u=1} = r\sum_{j\geq 1}\frac{x^j}{1-x^j}\prod_{i\geq 1}(1-x^i)^{-r} .$$

Invoking (8) and (1), we have

$$S_r(x) = r(\sum_{n\geq 0} d(n)x^n)(\sum_{n\geq 0} p_r(n)x^n) .$$

The conclusion now follows if we match coefficients of like powers of x. ∎

Next, we consider partitions into distinct parts in r distinct colors.

Theorem 2 If $r \geq 2$, let $s_r^*(n)$ denote the total number of parts in all partitions of n into distinct parts in r distinct colors. Then

$$s_r^*(n) = r\sum_{k=1}^{n} e(k)q_r(n-k) .$$

Proof: If $1 \leq k \leq n$, let $q_r(n,k)$ denote the number of partitions of n into k distinct parts in r distinct colors. Then

$$s_r^*(n) = \sum_{k=1}^{n} kq_r(n,k) .$$

Let

$$Q_r(x) = \sum_{n\geq 0} q_r(n)x^n = \prod_{n\geq 1}(1+x^n)^r .$$

Consider the bivariate generating function

$$Q_r(x,u) = \prod_{i\geq 1}(1+ux^i)^r .$$

Then we have

$$\frac{\partial}{\partial u}(Q_r(x,u)) = \sum_{j\geq 1} r(1+ux^j)^{r-1}x^j\prod_{i\neq j}(1+ux^i)^r =$$

$$\sum_{j\geq 1}\frac{rx^j}{1+ux^j}\prod_{i\geq 1}(1+ux^i)^r .$$

Let

$$S_r^*(x) = \sum_{n\geq 0} s_r^*(n)x^n$$

Then

$$S_r^*(x) = \frac{\partial}{\partial u}(Q_r(x,u))\Big|_{u=1} = r\sum_{j\geq 1}\frac{x^j}{1+x^j}\prod_{i\geq 1}(1+x^i)^r \ .$$

Invoking (9) and (2), we have

$$S_r^*(x) = r(\sum_{n\geq 0} e(n)x^n)(\sum_{n\geq 0} q_r(n)x^n) \ .$$

The conclusion now follows if we match coefficients of like powers of x. ∎

Next, we consider partitions with odd parts in r distinct colors.

Theorem 3 If $r \geq 2$, let $s_r'(n)$ denote the total number of parts in all partitions of n into odd parts in r distinct colors. Then

$$s_r'(n) = r\sum_{k=1}^{n} d_1(k)q_r(n-k) \ .$$

Proof: If $1 \leq k \leq n$, let $q_r'(n,k)$ denote the number of partitions of n into k odd parts in r distinct colors. Then

$$s_r'(n) = \sum_{k=1}^{n} kq_r'(n,k) \quad .$$

Let

$$Q_r'(x) = \sum_{n\geq 0} q_r'(n)x^n = \prod_{n\geq 1}(1-x^{2n-1})^{-r} \ .$$

Consider the bivariate generating function

$$Q_r'(x,u) = \prod_{i\geq 1}(1-ux^{2i-1})^{-r} \ .$$

Then we have

$$\frac{\partial}{\partial u}(Q_r'(x,u)) = \sum_{j\geq 1} -r(1-ux^{2j-1})^{-r-1}(-x^{2j-1})\prod_{i\neq j}(1-ux^{2i-1})^{-r} =$$

$$r\sum_{j\geq 1}\frac{x^{2j-1}}{1-ux^{2j-1}}\prod_{i\geq 1}(1-ux^{2i-1})^{-r} \ .$$

Let

$$S'_r(x) = \sum_{n\geq 0} s'_r(n)x^n .$$

Then

$$S'_r(x) = \frac{\partial}{\partial u}(Q'_r(x,u))\Big|_{u=1} = r\sum_{j\geq 1}\frac{x^{2j-1}}{1-x^{2j-1}}\prod_{i\geq 1}(1-x^{2i-1})^{-r} .$$

Invoking (10) and (3), we have

$$\sum_{n\geq 0} s'_r(n)x^n = r(\sum_{n\geq 0} d_1(n)x^n)(\sum_{n\geq 0} q'_r(n)x^n) .$$

The conclusion now follows if we match coefficients of like powers of x. ■

We now consider overpartitions. (An overpartition of n is a partition such that one part of each size may be overlined. See [4].) If $\overline{p}(n)$ denotes the number of overpartitions of n, then the generating function for $\overline{p}(n)$ is given by:

$$\overline{P}(x) = \sum_{n\geq 0}\overline{p}(n)x^n = \prod_{n\geq 1}\frac{1+x^n}{1-x^n} .$$

The next theorem is a formula for $\overline{s}(n)$, which denotes the total number of parts in all overpartitions of n.

Theorem 4

$$\overline{s}(n) = 2\sum_{k=1}^{n} d_1(k)\overline{p}(n-k) .$$

Proof: If $1 \leq k \leq n$, let $\overline{p}(n,k)$ denote the number of overpartitions of n into k parts . Then

$$\overline{s}(n) = \sum_{k=1}^{n} k\overline{p}(n,k) .$$

Consider the bivariate generating function

$$\overline{P}(x,u) = \prod_{i\geq 1}\frac{1+ux^i}{1-ux^i} .$$

Then we have

$$\frac{\partial}{\partial u}(\overline{P}(x,u)) = \sum_{j\geq 1}\frac{(1-ux^j)x^j-(1+ux^j)(-x^j)}{(1-ux^j)^2}\prod_{i\neq j}\frac{1+ux^i}{1-ux^i} =$$

$$\sum_{j\geq 1}\frac{2x^j}{1-u^2x^{2j}}\prod_{i\geq 1}\frac{1+ux^i}{1-ux^i}\,.$$

Let

$$\overline{S}(x)=\sum_{n\geq 0}\overline{s}(n)x^n\,.$$

Then

$$\overline{S}(x)=\frac{\partial}{\partial u}(\overline{P}(x,u))\Big|_{u=1}=\sum_{j\geq 1}\frac{2x^j}{1-x^{2j}}\prod_{i\geq 1}\frac{1+x^i}{1-x^i}\,.$$

Invoking (10) and (4), we have

$$\sum_{n\geq 0}\overline{s}(n)x^n=2\Big(\sum_{n\geq 0}d_1(n)x^n\Big)\Big(\sum_{n\geq 0}\overline{p}(n)x^n\Big)\,.$$

The conclusion now follows if we match coefficients of like powers of x. ■

Next, we consider overpartitions of n with odd parts. If $\overline{q}(n)$ denotes the number of overpartitions of n with odd parts, then the generating function for $\overline{q}(n)$ is given by:

$$\overline{Q}(x)=\sum_{n\geq 0}\overline{q}(n)x^n=\prod_{n\geq 1}\frac{1+x^{2n-1}}{1-x^{2n-1}}\,.$$

The next theorem is a formula for $\overline{s}'(n)$, which denotes the total number of parts in all overpartitions of n into odd parts.

Theorem 5

$$\overline{s}'(n)=2\sum_{k\geq 0}d^*(2k+1)\overline{q}(n-2k-1)\,.$$

Proof: If $1\leq k\leq n$, let $\overline{q}(n,k)$ denote the number of overpartitions of n with k odd parts. Then

$$\overline{s}'(n)=\sum_{k=1}^{n}k\overline{q}(n,k)\quad.$$

Consider the bivariate generating function

$$\overline{Q}(x,u)=\prod_{i\geq 1}\frac{1+ux^{2i-1}}{1-ux^{2i-1}}\quad.$$

Then we have

$$\frac{\partial}{\partial u}(\overline{Q}(x,u)) = \sum_{j\geq 1} \frac{(1-ux^{2j-1})x^{2j-1} - (1+ux^{2j-1})(-x^{2j-1})}{(1-ux^{2j-1})^2} \prod_{i\neq j} \frac{1+ux^{2i-1}}{1-ux^{2i-1}} =$$

$$\sum_{j\geq 1} \frac{2x^{2j-1}}{(1-ux^{2j-1})^2} \prod_{i\neq j} \frac{1+ux^{2i-1}}{1-ux^{2i-1}} = 2\sum_{j\geq 1} \frac{x^{2j-1}}{1-u^2x^{4j-2}} \prod_{i\geq 1} \frac{1+ux^{2i-1}}{1-ux^{2i-1}} .$$

Let

$$\overline{S}'(x) = \sum_{n\geq 0} \overline{s}'(n)x^n .$$

Then

$$\overline{S}'(x) = \frac{\partial}{\partial u}(Q(x,u))\bigg|_{u=1} = 2\sum_{j\geq 1} \frac{x^{2j-1}}{1-x^{4j-2}} \prod_{i\geq 1} \frac{1+x^{2i-1}}{1-x^{2i-1}} .$$

Invoking (10) and (5), we have

$$\sum_{n\geq 0} \overline{s}'(n)x^n = 2(\sum_{n\geq 0} d(2n+1)x^{2n+1})(\sum_{n\geq 0} \overline{q}(n)x^n) .$$

The conclusion now follows if we match coefficients of like powers of x. ■

Next, we consider partitions such that no part is divisible by r. Let $b_r(n)$ denote the number of partitions of n such that no part is divisible by r. The generating function for $b_r(n)$ is given by:

$$B_r(x) = \sum_{n\geq 0} b_r(n)x^n = \prod_{n\geq 1} \frac{1-x^{rn}}{1-x^n} .$$

The following theorem gives a formula for $s^{(r)}(n)$, the total number of parts in all partitions of n into parts not divisible by r.

Theorem 6

$$s^{(r)}(n) = \sum_{k=1}^{n} (d(k) - d(\frac{k}{r}))b_r(n-k) .$$

Proof: If $1 \leq k \leq n$, let $b_r(n,k)$ denote the number of partitions of n into k parts, none divisible by r. Then

$$s^{(r)}(n) = \sum_{k=1}^{n} kb_r(n,k) .$$

Consider the bivariate generating function

$$B_r(x,u) = \prod_{i\geq 1} \frac{1-ux^{ri}}{1-ux^i} \quad .$$

Then

$$\frac{\partial B_r(x,u)}{\partial u} = \sum_{j\geq 1} \frac{(1-ux^j)(-x^{rj})-(1-ux^{rj})(-x^j)}{(1-ux^j)^2} \prod_{i\neq j} \frac{1-ux^{ri}}{1-ux^i} =$$

$$\sum_{j\geq 1} \left(\frac{-x^{rj}}{1-ux^{rj}} + \frac{x^j}{1-ux^j}\right) \prod_{i\geq 1} \frac{1-ux^{ri}}{1-ux^i} \; .$$

Let

$$S^{(r)}(x) = \sum_{n\geq 0} s^{(r)}(n) x^n \; .$$

Then

$$S^{(r)}(x) = \left.\frac{\partial B_r(x,u)}{\partial u}\right|_{u=1} = \sum_{j\geq 1} \left(\frac{x^j}{1-x^j} - \frac{x^{rj}}{1-x^{rj}}\right) \prod_{i\geq 1} \frac{1-x^{ri}}{1-x^i} \; .$$

Invoking (8) and (6), we have

$$\sum_{n\geq 0} s^{(r)}(n) x^n = \sum_{n\geq 0} (d(n) - d(\frac{n}{r})) \sum_{n\geq 0} b_r(n) x^n \; .$$

(Note that $d(\alpha) = 0$ if α is not a positive integer.) The conclusion now follows if we match coefficients of like powers of x. ■

Finally, we consider partitions such that no term appears r or more times. The number of such partitions of n is $b_r^*(n) = b_r(n)$. The following theorem gives a formula for $s_*^{(r)}(n)$, the total number of parts in all partitions of n such that no part occurs r or more times.

Theorem 7

$$s_*^{(r)}(n) = \sum_{k=1}^{n} (d(k) - rd(\frac{k}{r})) b_r^*(n-k) \; .$$

Proof: If $1 \leq k \leq n$, let $b_r^*(n,k)$ denote the number of partitions of n into k parts such that no part occurs r or more times. Then

$$s_*^{(r)}(n) = \sum_{k=1}^{n} k b_r^*(n,k) \; .$$

Let

$$B_r^*(x) = \sum_{n\geq 0} b_r^*(n)x^n = \prod_{n\geq 1}(1 + x^n + x^{2n} + \cdots + x^{(r-1)n}) .$$

Consider the bivariate generating function

$$B_r^*(x,u) = \prod_{i\geq 1}(1 + ux^i + u^2x^{2i} + \cdots + u^{r-1}x^{(r-1)i}) \quad .$$

Let $f_i(u,x,r) = 1 + ux^i + u^2x^{2i} + \cdots + u^{r-1}x^{(r-1)i}$. Then we have

$$B_r^*(x,u) = \prod_{i\geq 1} f_i(u,x,r) \text{ and } B_r(x,1) = \prod_{i\geq 1} f_i(1,x,r) = B_r^*(x) .$$

Then

$$\frac{\partial B_r^*(x,u)}{\partial u} =$$

$$\sum_{j\geq 1}(x^j + 2ux^{2j} + 3u^2x^{3j} + \cdots + (r-1)u^{r-2}x^{(r-1)j})\prod_{i\neq j} f_i(u,x,r) =$$

$$\sum_{j\geq 1}\frac{x^j + 2ux^{2j} + 3u^2x^{3j} + \cdots + (r-1)u^{r-2}x^{(r-1)j}}{1 + ux^j + u^2x^{2j} + \cdots + u^{r-1}x^{(r-1)j}}\prod_{i\geq 1} f_i(u,x,r) .$$

Let

$$S_*^{(r)}(x) = \sum_{n\geq 0} s_*^{(r)}(n)x^n \quad .$$

Then

$$S_*^{(r)}(x) = \left.\frac{\partial B_r^*(x,u)}{\partial u}\right|_{u=1} =$$

$$\sum_{j\geq 1}\frac{x^j + 2x^{2j} + 3x^{3j} + \cdots + (r-1)x^{(r-1)j}}{1 + x^j + x^{2j} + \ldots + x^{(r-1)j}}\prod_{i\geq 1} f_i(1,x,r) =$$

$$\sum_{j\geq 1}\frac{x^j + x^{2j} = x^{3j} + \cdots + x^{(r-1)j} - (r-1)x^{rj}}{1 - x^{rj}}B_r^*(x) =$$

$$\sum_{j\geq 1}\left(\frac{x^j}{1-x^j} - \frac{rx^{rj}}{1-x^{rj}}\right)B_r^*(x) .$$

Therefore, invoking (8), we have

$$\sum_{n\geq 0} s_*^{(r)}(n)x^n = (\sum_{n\geq 0}(d(n) - rd(\frac{n}{r}))(\sum_{n\geq 0} b_r^*(n)x^n) .$$

The conclusion now follows if we match coefficients of like powers of x. ■

4. References

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