Eternal Security in Graphs of Fixed Independence Number

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Abstract

We show that if the independence number of a graph is α, then the eternal security number of the graph is at most $\binom{\alpha+1}{2}$, solving a problem stated by Goddard, Hedetniemi, and Hedetniemi [*JCMCC*, vol. 52, pp. 160-180].

1 Introduction

Let $G = (V, E)$ be a simple graph with independence number α. The *open neighborhood* of vertex v is denoted by $N(v)$, and its *closed neighborhood* $N(v) \cup \{v\}$ is denoted by $N[v]$. A dominating set of G is a set $D \subseteq V$ such that, for all v, $N[v] \cap D \neq \emptyset$.

Considerable recent interest has been given to problems concerned with protecting the vertices in a graph from a series of one or more attacks, see for example [1, 2, 5]. In such a problem, guards are located at vertices, can protect the vertices at which they are located, and can move to a neighboring vertex to defend an attack there. Under this set of rules, a guard located at each vertex of a dominating set suffices to defend a graph against a single attack. Several variations of this problem have been proposed including Roman Domination [3], Weak Roman Domination [4] and k-secure sets/eternal secure sets [1, 2, 5, 6].

Let R denote a sequence of vertices, with first element $R(1)$ and i^{th} element $R(i)$. The elements of R are interpreted as the locations of a sequence of consecutive attacks at vertices, each of which must be defended by a guard. At most one guard is allowed to move to defend each attack.

Let D_0 be the set of initial locations of the guards and let D_i be the set of locations of the guards after $R(i)$ is defended (so $R(i) \in D_i$). We refer to D_i as a *configuration* of the guards. If $R(i) \notin D_i$, then $D_i = (D_{i-1} \setminus \{v\}) \cup \{R(i)\}$, where $v \in D_{i-1}$ and $R(i) \in N(v)$. We say that the guard at v has *moved* to $R(i)$.

A set D is an *eternal secure set* if, for all possible sequences R, there exists a sequence $D_0, D_1, \ldots$ such that $D_i = D_{i-1} \setminus \{v\} \cup R(i)$ (possibly $v = R(i)$), $R(i) \in N[v]$, and each D_i is a dominating set. The size of a smallest eternal secure set in G is denoted $\gamma_\infty(G)$, or simply γ_∞ [2].

It is not hard to prove that $\gamma_\infty \geq \alpha$ for all graphs G (just imagine a sequence of attacks at independent vertices). Goddard et al. proved that if $\alpha = 2$ then $\gamma_\infty \leq 3$ [5]. They conjectured that there is a constant c such that $\gamma_\infty \leq c$ for all graphs with $\alpha = 3$ [5]. In this paper, we prove that $\gamma_\infty \leq \binom{\alpha+1}{2}$, for all graphs with independence number $\alpha \geq 1$. It is not known if this bound is best possible. We construct connected graphs with $\gamma_\infty = \frac{3}{2}\alpha$.

2 Proofs

Theorem 1 *For any graph G with independence number $\alpha \geq 1$, $\gamma_\infty \leq \binom{\alpha+1}{2}$.*

Proof: The result is clearly true if $|V| \leq \binom{\alpha+1}{2}$, so assume that $|V| > \binom{\alpha+1}{2}$. Define disjoint independent sets $S_\alpha, S_{\alpha-1}, \ldots, S_1$ such that S_α is a maximum independent set of G (so $|S_\alpha| = \alpha$) and, for $t = \alpha-1, \alpha-2, \ldots 1$, the set S_t is either empty or an independent set of size t (not necessarily a maximal independent set). Among all collections of such sets, choose one such that $|\cup_{t=1}^{\alpha} S_t|$ is maximum. Since $|V| > \binom{\alpha+1}{2}$, the set $S_1 \neq \emptyset$.

Start with D_0 equal to the initial location of the guards. Suppose $D_{i-1} = \cup_{t=1}^{\alpha} S_t$, for some $i \geq 1$.

Strategy: *Suppose $R(i) = v$. If there is a guard at v, then it is defended* *by the guard located at v. Otherwise, a guard from the set S_j with the smallest subscript among those with a vertex adjacent to v moves to v. Such a set exists because S_α is a dominating set.*

We will show that $D_i = (D_{i-1} \setminus \{v\}) \cup \{R(i)\}$ can be "partitioned" into disjoint independent sets, as above. Suppose $R(i) = v$. If $v \in D_{i-1}$ then $D_i = D_{i-1}$ and the statement is true in this case. If $v \notin D_{i-1}$, then a guard at $g \in S_j$ moves to v according to the above strategy. There are two possibilities. If $(S_j - \{g\}) \cup \{v\}$ is independent, then replacing S_j by $(S_j - \{g\}) \cup \{v\}$ gives another collection of disjoint independent sets with the same maximality properties as in the definition. Otherwise, v is adjacent to at least two vertices in S_j. This implies that $j > 1$. Let k be the greatest subscript less than j such that S_k is non-empty. It must be that $k = j - 1$; otherwise the fact that $S_k \cup \{v\}$ is independent (by definition of j no vertex in S_k is adjacent to v) contradicts the maximality of $| \cup_{k=1}^{\alpha} S_k|$. Replacing S_j by $S_{j-1} \cup \{v\}$ and S_{j-1} by $S_j - \{g\}$ gives another collection of independent sets with the same maximality properties as in the definition. The claim is now proved.

Thus, for all $i \geq 1$, the strategy allows the guards to defend an attack at $R(i)$. Therefore, $\gamma_\infty \leq | \cup_{t=1}^{\alpha} S_t| \leq 1 + 2 + \cdots + \alpha = \binom{\alpha+1}{2}$. This completes the proof. □

Theorem 2 *Let $n \geq 2$ be an integer. There exists a connected graph G with independence number α and eternal security number $\gamma_\infty(G) \geq \frac{3}{2}\alpha$.*

Proof: It is easy to see that C_5 has independence number two and eternal security number three [2]. Let $G_1, G_2, \ldots, G_n$, $n \geq 2$, be disjoint copies of C_5, and let v be a new vertex adjacent to each other vertex. Let G be the graph resulting from this construction. The graph G clearly has independence number $\alpha = 2n$. We claim that it has $\gamma_\infty \geq 3n$.

Suppose G has eternal security number less than $3n$. Then there exists some G_i containing fewer than three guards. Let the vertices of G_i be consecutively numbered around the cycle as $v_1, v_2, \ldots, v_5$ and we assume without loss of generality that G_i contains exactly two guards (the proof will proceed similarly if there are less than two guards in G_i).

According to Burger et al. [2], we can assume that sufficiently many attacks have occurred that no two guards occupy the same vertex. There are two cases depending on the location of the two guards within G_i.

If the guards are on adjacent vertices, assume without loss of generality they are on v_1 and v_2, and in order that the guards induce a dominating set, there must be a guard at v. If there is an attack at v_4, the guard at v must move to v_4. Since there are fewer than $3n$ guards, there is another $G_j, j \neq i$, with at most two guards. Since there is now no guard on v, these guards in G_j are not adjacent, else the guards do not induce a dominating set. If there is then an attack at the vertex on the path of length two joining these two guards, the configuration resulting from defending the attack cannot be a dominating set.

Thus, the guards must be on nonadjacent vertices that are distance two from each other in the subgraph G_i, say v_1 and v_3. We claim that this reduces to the previous case. Attack at v_2. Suppose either a guard at v_1 or v_3 moves to v_2. Then this is exactly the previous case. So suppose a guard at v moves to v_2. Then, there is another $G_j, j \neq i$, with at most two guards and now there is no guard at v, so we may apply the argument above. □

We note that the eternal security number of the graph in Theorem 2 is in fact $3n$. In addition, the technique used proves the following. Let G be a graph. Take n disjoint copies of G. Add a new vertex v and join it to all vertices in these copies. Call this graph H. Then $\alpha(H) = n\alpha(G)$, and $\gamma_\infty(H) = n\gamma_\infty(G)$.

3 Future Directions

Our main question is whether the bounds in Theorem 1 are tight (even in the case $\alpha = 3$); however, we are unable to prove it as of yet and suspect it may require fairly complex or large graphs to prove better lower bounds. It may be worth considering graph with $\alpha = 3$ and, for example, $\Delta = n - 1$. We suspect there is no constant c such that $\gamma_\infty(G) \leq c\alpha$, for all G.

We note that when $\alpha(G) = 2$, $\gamma_\infty(G) \leq 3$, one can determine the eternal security number of such graphs in polynomial time, due to Theorem 5 of [2]. We leave open the complexity of computing the eternal security number of graphs with independence number three (or of any fixed independence number). The general problem of deciding if a set of vertices is an eternal secure set is complete for co-NP^{NP} [7].

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