On The Super Edge-graceful Spiders of Even Orders

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ABSTRACT

A (p,q)-graph G is said to be **edge graceful** if the edges can be labeled by 1,2,…, q so that the vertex sums are distinct, mod p. It is shown that if a tree T is edge-graceful then its order must be odd. Lee conjectured that all trees of odd orders are edge-graceful. J. Mitchem and A. Simoson [12] introduced the concept of super edge-graceful graphs which is a stronger concept than edge-graceful for some classes of graphs. A graph G=(V,E) of order p and size q is said to be **super edge-graceful** (SEG) if there exists a bijection

$f: E \to \{0, +1,-1,+2,-2,\ldots,(q-1)/2, -(q-1)/2\}$ if q is odd

$f: E \to \{ +1,-1,+2,-2,\ldots,q/2, -q/2\}$ if q is even

such that the induced vertex labeling f* defined by $f^*(u) = \Sigma\{f(u,v): (u,v) \in E\}$ has the property:

$f^*: V \to \{0,+1,-1,\ldots,+(p-1)/2,-(p-1)/2\}$ if p is odd

$f^*: V \to \{+1,-1,\ldots,+p/2,-p/2\}$ if p is even

is a bijection. The conjecture is still unsettled. In this paper we first characterize spiders of even orders which are not SEG. We then exhibit some spiders of even orders which are SEG of diameter at most four. By the concepts of irreducible part of even tree T, we showed infinite number of spiders of even orders are SEG. Finally, we provide some conjectures for further research.

Key words: Edge-graceful, super edge-graceful, trees, spider, tree reduction, irreducible.

1. Introduction.

All graphs in this paper are simple graphs. A graph G is said to be **edge-graceful** if the edges are labeled by 1,2,3,…,q so that the vertex sums are distinct, mod p.

Figure 1 shows a graph with 4 vertices and 5 edges which is edge-graceful.

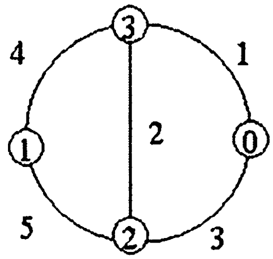


Figure 1.

Figure 2 shows a grid with 12 vertices and 17 edges with two different edge-graceful labelings.

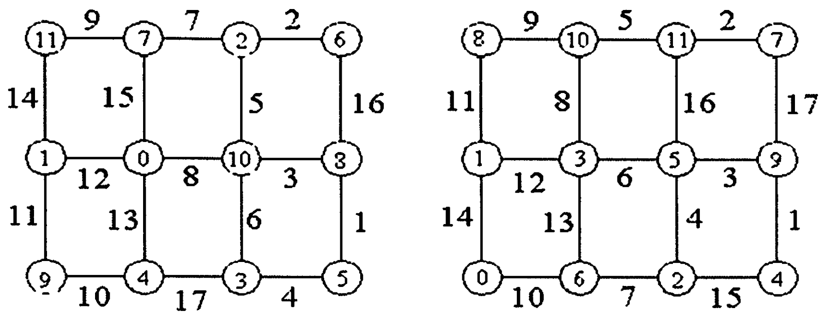


Figure 2.

The concept of edge-graceful graph was introduced by S.P. Lo [11] in 1985. A necessary condition of edge-gracefulness is (Lo [11])

$$q(q+1) \equiv \frac{p(p-1)}{2} \pmod p \qquad (1)$$

The following tantalizing conjecture is proposed in [7].

<u>**Conjecture 1.1.**</u>: The Lo's condition (1) is sufficient for a connected graph to be edge-graceful.

A sub-conjecture of the above (Lee [6]) has also not yet been proved:

<u>**Conjecture 1.2.**</u>: All odd-order trees are edge-graceful.

In [1,9,12,14,15] several classes of trees of odd orders are proved to be edge-graceful. In [9] it is shown that all trees of odd order of diameter at most four are edge-graceful.

J. Mitchem and A. Simoson [12] introduced the concept of super edge-graceful graphs which is a stronger concept than edge-graceful for some classes of graphs.

<u>**Definition 1.1.**</u> A graph G = (V,E) of order p and size q is said to be **super edge-graceful** if there exists a bijection

$f: E \to \{0, +1, -1, +2, -2, \ldots, (q-1)/2, -(q-1)/2\}$ if q is odd
$f: E \to \{+1, -1, +2, -2, \ldots, q/2, -q/2\}$ if q is even
such that the induced vertex labeling f^* defined by $f^*(u) = \{\Sigma f(u,v): (u,v) \in E\}$ has the property:
$f^*: V \to \{0, +1, -1, \ldots, +(p-1)/2, -(p-1)/2\}$ if p is odd
$f^*: V \to \{+1, -1, \ldots, +p/2, -p/2\}$ if p is even
is a bijection.

Let G, H be two graphs, and let G have p vertices. The corona of G with H is the graph obtained by taking one copy of G and p copies of H and then joining the ith vertex of G to each vertex in the ith copy of H, for each i from 1 to p. We will use $G \Theta H$ to denote the corona of G with H.

Definition 1.2. A tree $T = P_n \Theta K_1$, which is the corona of a path with K_1 is called the comb. We will denote it by Comb(n).

In [4], we showed that Comb(2), Comb(3) are not SEG and Comb(4), Comb(5), Comb(6) are SEG.

Example 1. The following tree of order 12, T = Comb(6), is SEG.

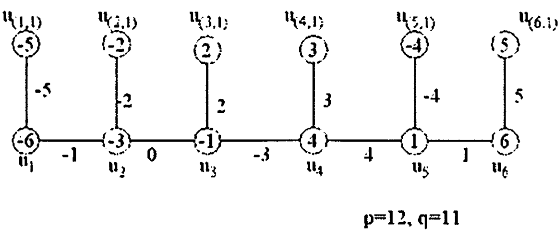


Figure 3.

Mitchem and A. Simoson [12] showed that

Theorem 1.1. If G is a super-edge-graceful graph and $q \equiv -1 \pmod p$, if q is even or $q \equiv 0 \pmod p$, if q is odd, then G is also edge-graceful.

Thus,
Corollary 1.2. If G is super edge-graceful tree of odd order then it is edge-graceful.

A conjecture states that all odd trees are super-edge-graceful which is stronger than the edge-graceful trees conjecture is proposed in [9].

Conjecture 1.3. All trees of odd orders are super-edge-graceful.

In [3,4], we consider infinite many trees of even orders which are SEG. In this paper we studied super-edge graceful labeling for even trees which are spiders.

A tree is called a **spider** if it has a center vertex c of degree k >1 and all the other vertex is either a leaf or with degree 2. Thus, a spider is an amalgamation of k paths with various lengths. If it has x_1's path of length a_1, x_2's path of length a_2, ..., we shall denote the spider by $SP(a_1^{x1}, a_2^{x2}, \ldots, a_m^{xm})$ where $a_1 < a_2 < \ldots < a_m$ and $x_1 + x_2 + \ldots + x_m = k$. (see Figure 4).

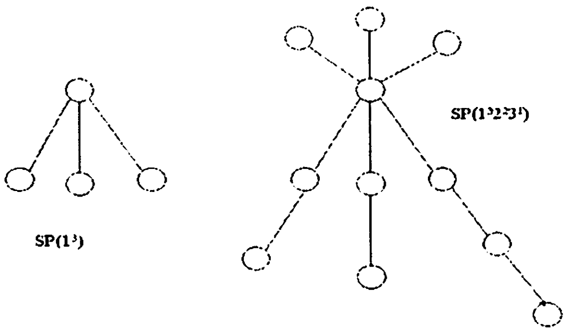


Figure 4.

The concept of SEG graphs is extended to Q(a)P(b)-SEG graphs by Chopra and Lee. For general theory of Q(a)P(b)-super edge-graceful graphs, the reader can refer to [2].

2. <u>Some Even order spider trees which are not super-edge graceful.</u>

<u>**Theorem 2.1.**</u> $SP(1^{2k+1}, 2^t)$ is not SEG, for all $k \geq 0$ and $t \geq 1$.
Proof. For even-order trees (p is even, q = p-1 is odd), edges are labelled by 0, +1, -1, +2, -2,..., (q-1)/2, -(q-1)/2 and the vertices must have induced labels +1, -1,..., +p/2, -p/2. In $SP(1^{2k+1}, 2^t)$, there is no way to place the edge labelled 0.
(a) If the 0 edge label is placed on one of the edge leaves, then that leaf vertex will have vertex sum 0, which is not allowed.
(b) If the 0 edge is placed on one of the non-leaf edges, then there are two vertices will have the same induced vertex sum, which is also not allowed.

<u>**Corollary 2.2.**</u> All even order spiders of diameter at most two are not SEG.

<u>**Example 2.**</u> $SP(1^{3,} 2^2)$ is not SEG.

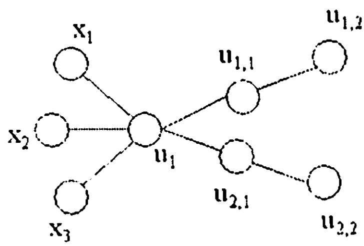


Figure 5.

Remark. One may think that if we add even number of leaves incident with the same vertex in an even order tree $SP(1^{2k+1},2^t)$, for all $k \geq 0$, $t \geq 1$ and we obtain a new tree T^*, then T^* is not SEG. However, this is not true. Consider the non SEG spider of example 2, we append two leaves at X1, we see that the resulting tree is SEG (Figure 6).

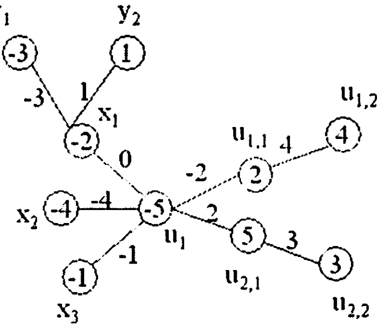


Figure 6.

3. Some Even order SEG spiders of small diameters.

We first show that some even order SEG spiders of diameter at most four.

Theorem 3.1. The spider $SP(1^{2k},2,3)$ is SEG for all $k \geq 0$.
Proof. The spider $SP(1^{2k},2,3)$ has $2k+6$ vertices. Thus we define a labeling $f: E(SP(1^{2k},2,3)) \rightarrow \{0,\pm 1,\ldots,\pm(k+2)\}$ as follows:
$f((x_1,c)) = 2$, $f((x_2,c)) = -2$, $f((x_3,c)) = 3$, $f((x_4,c)) = -3,\ldots$, $f((x_{2k-1},c)) = k+1$, $f((x_{2k},c)) = -(k+1)$, $f((c,u_{1,1})) = 1$, $f((u_{1,1}, u_{1,2})) = k+2$, $f((c,u_{2,1})) = 0$, $f((u_{2,1}, u_{2,2})) = -1$, $f((u_{2,2}, u_{2,3})) = -(k+2)$. It is clear that f is SEG.

Example 3. Figure 7 shows that $SP(1^4,2,3)$ and $SP(1^6,2,3)$ are SEG.

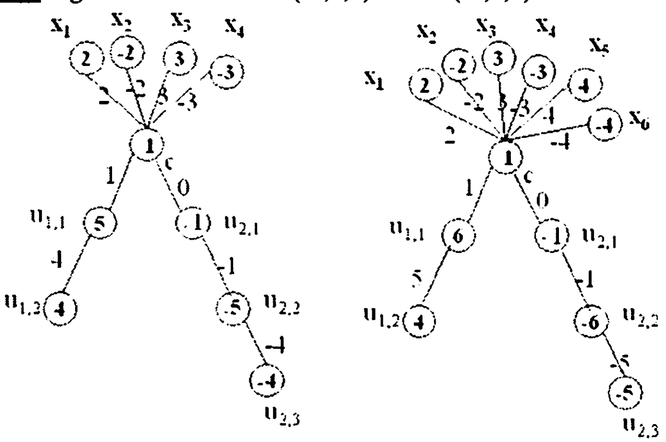


Figure 7.

Remark. By Theorem 2.1, we see that SP(1,2,2) is not SEG. However, we observe the spider SP(1,3,3) is SEG. Figure 8 exhibits a SEG labeling of SP(1,3,3).

Theorem 3.2. The spider $SP(1^{2k+1},3^2)$ is SEG for all $k \geq 0$.
Proof. The spider $SP(1^{2k+1},3^2)$ has 2k+8 vertices. Thus we define a labeling $f: E(SP(1^{2k+1},3^2)) \rightarrow \{0,\pm1,\ldots,\pm(k+3)\}$ as follows:

$f((x_1,c)) = 2,\ \ f((x_2,c)) = -2,\ \ f((x_3,c)) = 3,\ldots,f((x_{2k+1},c)) = k+2,$
$f((c,u_{1,1})) = 0,\ \ f((u_{1,1}, u_{1,2})) = k+3,\ f((u_{1,1}, u_{1,3})) = 1,$
$f((c,u_{2,1})) = -(k+3),\ \ \ f((u_{2,1}, u_{2,2})) = -1,\ \ f((u_{2,2}, u_{2,3})) = -(k+2).$

We can see that f is SEG.

Example 4. Figure 8 shows that the spiders $SP(1^{2k+1},3^2)$ are SEG for k =0,1,2.

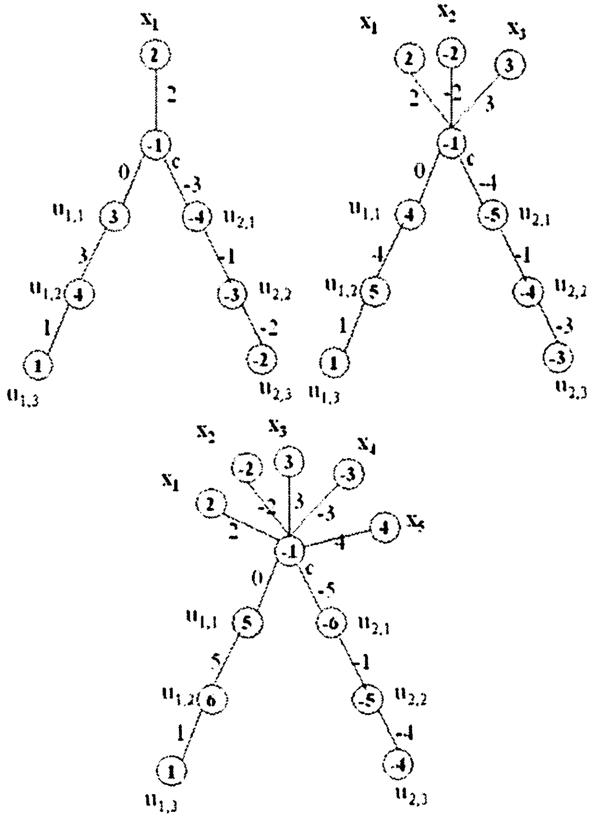
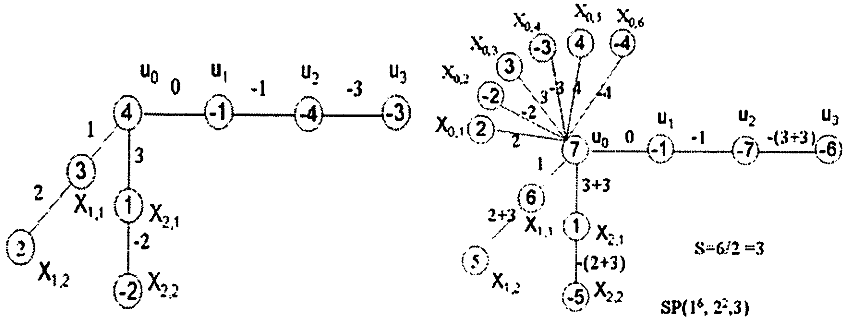


Figure 8.

Theorem 3.3. The spider $SP(1^{2k},2^2,3)$ is SEG for all $k \geq 0$.
Proof. First we show that for k=0, $SP(2^2,3)$ is SEG.(Figure 8(a)). For an even number 2k, we have the edge labeling set Q= {-3-k, -2-k, …., -1, 0, 1,…., 2+k, 3+k} and the vertex labeling set P= {-4-k, -3-k, ….., -1, 1,…., 3+k, 4+k}.

Thus we define a labeling $f:E(SP(1^{2k},2^2,3)) \rightarrow Q$ as follows:

$f(x_{1,1},u_0) = 1$, $f(x_{1,2}, x_{1,1}) = 2+k$, $f(u_0, x_{2,1}) = 3+k$, $f(x_{2,1}, x_{2,2}) = -2-k$, $f(u_0,u_1) = 0$, $f(u_1,u_2) = -1$, $f(u_2,u_3) = -(k+3)$ and we label the remaining k pair edges incident at node u_o in $SP(1^{2k},2^2,3)$ by 2, -2, 3, -3,…, 1+k, -(1+k). It is clear that f is a bijection function and the labeling is SEG (see Figure 9(b) for k=3).



(a) (b)

Figure 9.

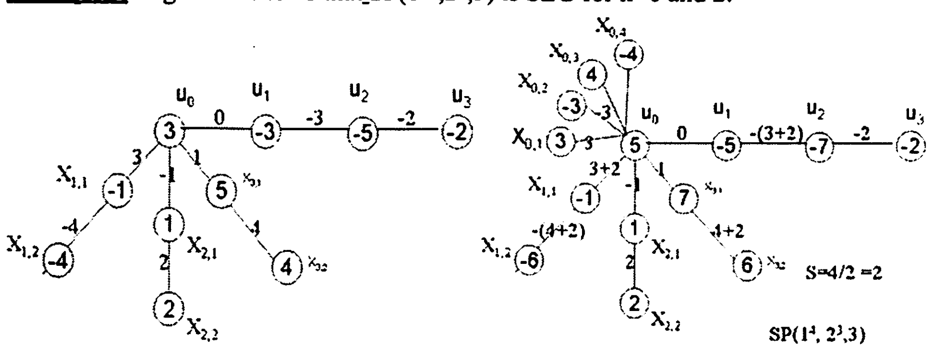
By the similar argument as the proof of Theorem 3.3, we have

Theorem 3.4. The spider $SP(1^{2k},2^3,3)$ is SEG for all $k \geq 0$.

Proof. First we show that for k=0, $SP(2^3,3)$ is SEG.(Figure 9(a)).

For an even number 2k, we have the edge labeling set Q= {-4-k, -3-k, …., -1, 0, 1,…., 3+k, 4+k} and the vertex labeling set P= {-5-k, -4-k, ….., -1, 1,…., 4+k, 5+k}. Thus we define a labeling $f:E(SP(1^{2k},2^2,3)) \rightarrow Q$ as follows: $f(x_{1,1},u_0) = 3+k$, $f(x_{1,2}, x_{1,1}) = -4-k$, $f(u_0, x_{2,1}) = -1$, $f(x_{2,1}, x_{2,2}) = 2$, $f(u_0, x_{3,1}) = 1$, $f(x_{3,1}, x_{3,2}) = 4+k$, $f(u_0,u_1) = 0$, $f(u_1,u_2) = -3-k$, $f(u_2,u_3) = -2$ and we label the remaining k pair edges incident at node u_o in $SP(1^{2k},2^3,3)$ by 3, -3,…, 2+k, -(2+k). It is clear that f is a bijection function and the labeling is SEG (see Figure 10(b) for k=2).

Example 5. Figure 10 shows that $SP(1^{2k},2^3,3)$ is SEG for k=0 and 2.



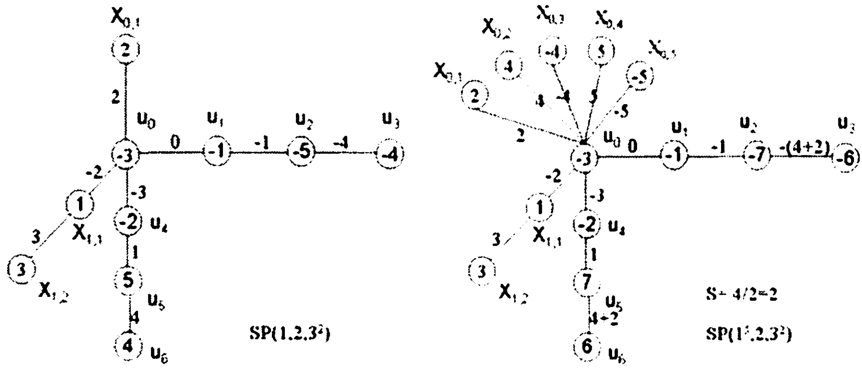
(a) (b)

Figure 10.

Theorem 3.5. The spider $SP(1^{2k+1},2,3^2)$ is SEG for all $k \geq 0$.

Proof. First we show that for k=0, SP(1,2,3^2) is SEG.(Figure 10(a)).
For an even number 2k, we have the edge labeling set Q= {-4-k, -3-k, …., -1, 0, 1,…., 3+k, 4+k} and the vertex labeling set P= {-5-k, -4-k, ….., -1, 1,…., 4+k, 5+k}.

Thus we define a labeling f:E(SP(1^{2k+1},2,3^2)) → Q as follows:
$f(x_{1,1},u_0) = -2$, $f(x_{1,2}, x_{1,1}) = 3$, $f(u_4,u_0) = -3$, $f(u_5,u_4) = 1$, $f(u_6,u_5) = 4+k$, $f(u_0,u_1) = 0$, $f(u_1,u_2) = -1$, $f(u_2,u_3) = -(4+k)$ $f(u_0, x_{0,1}) = 2$ and we label the remaining k pair edges incident at node u_o in SP(1^{2k+1},2,3^2) by 4, -4, …., 3+k, -(3+k). It is clear that f is a bijection and the labeling is SEG (see Figure 11(b) for k=2).

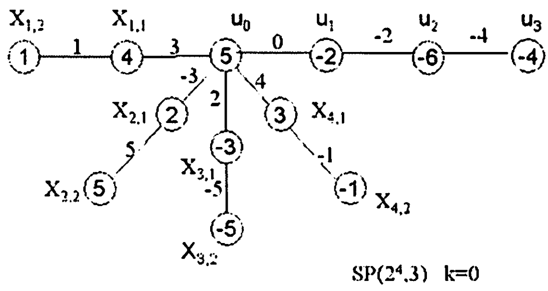


(a) (b)

Figure 11.

__Theorem 3.6.__ The spider SP(1^{2k},2^4,3) is SEG for all $k \geq 0$.
The spider SP(1^{2k},2^4,3) is SEG for all $k \geq 0$.
Proof. For an even number 2k, we have the edge labeling set Q= {-5-k, -4-k, …., -1, 0, 1,…., 4+k, 5+k} and the vertex labeling set P= {-6-k, -5-k, ….., -1, 1,…., 5+k, 6+k}.

Thus we define a labeling f:E(SP(1^{2k},2^4,3)) → Q as follows:
$f(x_{1,1},u_0) = 3+k$, $f(x_{1,2}, x_{1,1}) = 1$, $f(x_{2,1},u_0) = -3-k$, $f(x_{2,2}, x_{2,1}) = 5+k$, $f(x_{3,1},u_0) = 2$, $f(x_{3,2}, x_{3,1}) = -5-k$, $f(x_{4,1},u_0) = 4+k$, $f(x_{4,2}, x_{4,1}) = -1$, $f(u_0,u_1) = 0$, $f(u_1,u_2) = -2$, $f(u_2,u_3) = -4-k$ and we label the remaining k pair edges incident at node u_o in SP(1^{2k},2^4,3) by 3, -3,…, 3+k, -(3+k). It is clear that f is a bijection and the labeling is SEG (see Figure 12 (a) for k=0, and Figure 12 (b) for k=2).



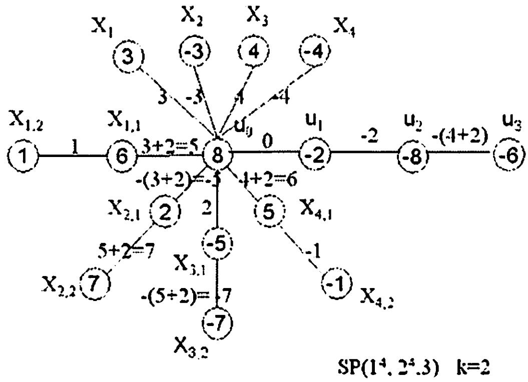


Figure 12.

We now show that some even order SEG spiders of diameter six.

<u>Theorem 3.7.</u> The spider SP(1^{2k+1},2,4) is SEG for all $k \geq 0$.
Proof. First we show that for k=0, SP(1,2,4) is SEG. (Figure 13(a)).
For odd number 2k+1, we have the edge labeling set Q= {-3-k, -2-k, ..., -1, 0, 1,...., 2+k, 3+k} and the vertex labeling set P= { -4-k, -3-k, ..., -1, 1,...., 3+k, 4+k}.

Thus we define a labeling f:E(SP(1^{2k+1},2,4)) → Q as follows:
$f(x_{0,1},u_0) = -(3+k)$, $f(x_{1,2}, x_{1,1}) = 2$, $f(u_0, x_{1,1}) = -1$, $f(u_0, u_1) = 0$, $f(u_1, u_2) = 3+k$, $f(u_2, u_3) = 1$, $f(u_3,u_4) = -2$ and we label the remaining k pair edges incident at node u_o in SP(1^{2k+1},2,4) by 3,-3,...,2+k,-(2+k). It is clear that f is a bijection and the labeling is SEG (see Figure 13(b) for k=2).

<u>Example 6</u>. Figure 13 illustrates the SEG labeling scheme for SP(1^{2k+1},2,4) where k= 0 and 2.

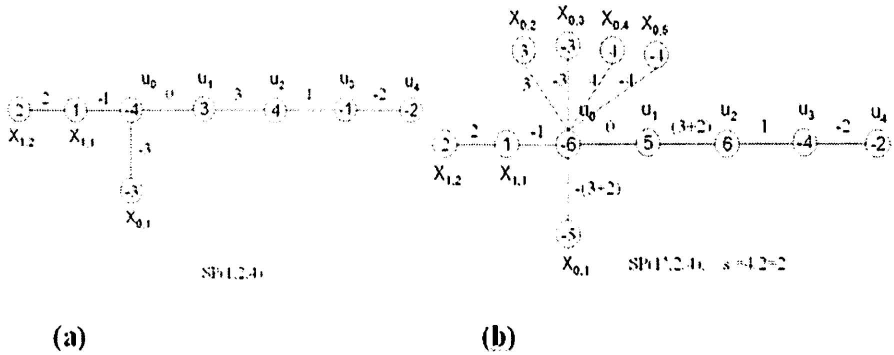


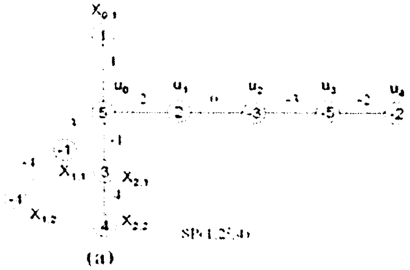
Figure 13.

By the similar argument as Theorem 3.7, we can show that
<u>Theorem 3.8.</u> The spider SP(1^{2k+1},2^2,4) is SEG for all $k \geq 0$.

Proof. First we show that for k=0, SP(1,2^2,4) is SEG. (Figure 13(a)).
For odd number 2k+1, we have the edge labeling set Q= {-4-k, -3-k, …., -1, 0, 1,…., 3+k, 4+k} and the vertex labeling set P= { -5-k, -4-k, ….., -1, 1,…., 4+k, 5+k}.

Thus we define a labeling f:E(SP(1^{2k+1},2^2,4)) → Q as follows:
$f(u_0, x_{1,1}) = 3+k$, $f(x_{1,2}, x_{1,1}) = -4-k$, $f(u_0, x_{2,1}) = -1$, $f(x_{2,2}, x_{2,1}) = 4+k$, $f(u_0, u_1) = 2$, $f(u_1, u_2) = 0$, $f(u_2, u_3) = -3-k$, $f(u_3,u_4) = -2$, $f(x_{0,1},u_0) = 1$ and we label the remaining k pair edges incident at node u_o in SP(1^{2k+1},2^2,4) by 3,-3,…,2+k, -(2+k). It is clear that f is a bijection and the labeling is SEG (see Figure 13(b) for k=2).

__Example 7.__ Figure 14 illustrates the labeling scheme for SP(1,2^2,4) and SP(1^5,2^2,4).



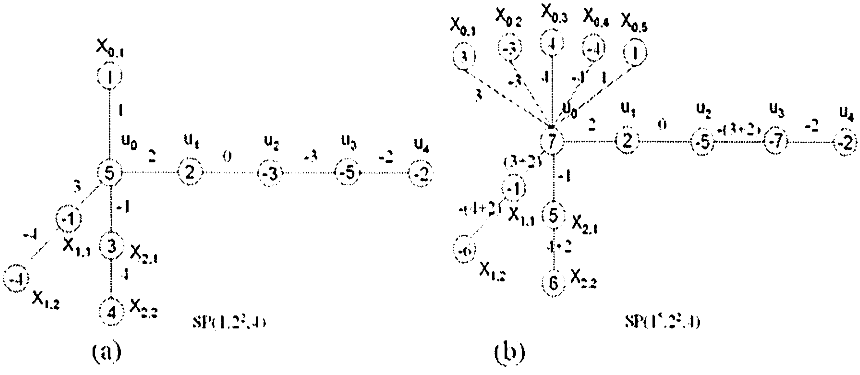


Figure 14.

__Theorem 3.9__. The spider SP(1^{2k+1},2^3,4) is SEG, for all $k \geq 0$.
__Proof:__ For odd number 2k+1, we could label the spider tree T = SP(1^{2k+1},2^3,4) as follows: we have the edge labeling set Q(T) = {-5-k, -4-k, …., -1, 0, 1,…., 4+k, 5+k} and the vertex labeling set P(T) = { -6-k, -5-k, ….., -1, 1,…., -5-k, -6-k}. Thus we define a labeling f:E(T) → Q(T) as follows:
$f(x_1,u_0) = -5-k$, $f(u_0, u_1) = 3+k$, $f(u_1,u_2) = 0$, $f(u_2,u_3) = -4-k$, $f(u_3,u_4) = -2$, $f(u_0, x_{1,1}) = 5+k$, $f(x_{1,1}, x_{1,2}) = -3-k$, $f(u_0, x_{2,1}) = 2$, $f(x_{2,1}, x_{2,2}) = -1$, $f(u_0, x_{3,1}) = 1$, $f(x_{3,1},x_{3,2}) = 4+k$ and we label the remaining s pair edges incident at node u_o in T by the following way: for two edges e_i, e_j incident at u_0, we label

e_i, e_j such that $f(e_i) = - f(e_j)$. It is clear that f is a bijection function and therefore the labeling is SEG. (see Figure 15 for k=0).

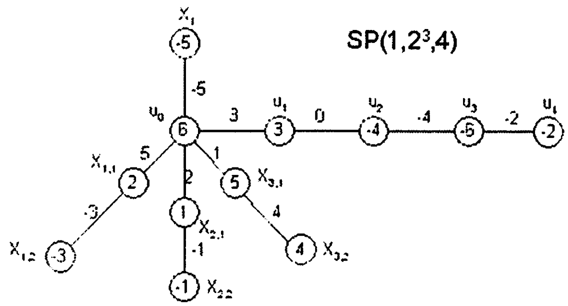
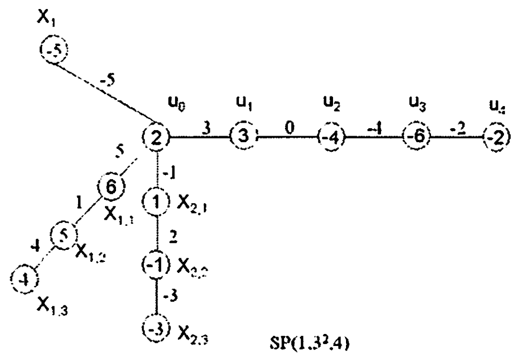


Figure 15.

<u>Theorem 3.10</u>. The spider $SP(1^{2k+1},3^2,4)$ is SEG, for all $k \geq 0$.
Proof. We have the edge labeling set Q(T) = {-5-k, …., -1, 0, 1,…., 5+k} and the vertex labeling set P(T) = { -6-k, ….., -1, 1,…., 6+k}. Thus we define
f: Q(T) → P(T) as follows:
$f(x_1, u_0) = -5-k$, $f(u_0, u_1) = 3$, $f(u_1, u_2) = 0$, $f(u_2, u_3) = -4-k$, $f(u_3, u_4) = -2$,
$f(u_0, x_{1,1}) = 5+k$, $f(x_{1,1}, x_{1,2}) = 1$, $f(x_{1,3}, x_{1,2}) = 4+k$
$f(u_0, x_{2,1}) = -1$, $f(x_{2,1}, x_{2,2}) = 2$, $f(x_{2,2}, x_{2,3}) = -3$ and then we label the remaining k pair edges incident at node u_o in T by the following way: for two edges e_i, e_j incident at u_0, we label e_i, e_j such that $f(e_i) = - f(e_j)$ from the remaining 2k numbers. It is clear that f is a bijection and therefore the labeling is SEG.

<u>Example 8</u>. Figure 16 illustrates the labeling scheme for $SP(1^{2k+1},3^2,4)$ where k=0, and 2.



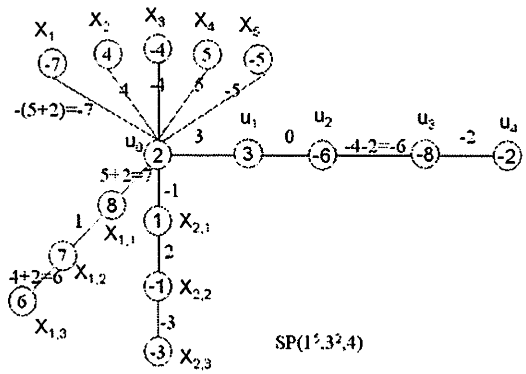


Figure 16.

4. Applications.

One can obtain many SEG graphs by the following construction. Let (G_1,u) and (G_2,v) be two graphs with fixed vertices u,v respectively. The amalgamation of (G_1,u) and (G_2,v) is the graph which is the disjoint union of G_1 and G_2 with u and v identified. We will denote the resulting graph as $Amal(G_1, G_2,\{u,v\})$. It is obvious that u is a cut-vertex of the amalgamation.

For a tree T we delete all sets of even number of leaves incident with the same vertex and generate a new tree T*. Continue with the deletion process until no such sets of even number of leaves can be found. The final tree is said to be **irreducible part** of T and will denoted by irr(T).

Definition 4.1. A tree T is irreducible if irr(T) = T.

We see that for a tree T of odd order if irr(T) is super edge-graceful then T is super edge-graceful.

Theorem 4.1. If a tree T has even order such that irr(T) = SP(2,3) then T is SEG.

Proof. Suppose $V(T) - V(irr(T)) = \{ e_1, ..., e_{2y}\}$. Then we see that $P(T) = 2y+6$. Thus we define a labeling f: $E(SP(1^{2k},2,3)) \rightarrow \{0,\pm1,..., \pm(y+2) \}$ as follows:
$f((c,u_{1,1})) = 1$, $f((u_{1,1}, u_{1,2})) = y+2$, $f((c,u_{2,1})) = 0$,
$f((u_{2,1}, u_{2,2})) = -1$, $f((u_{2,2}, u_{2,3})) = -(y+2)$ and
$f{:}E(T) - E(irr(T)))) \rightarrow \{\pm2,..., \pm(y+1) \}$.
It is clear that f is a bijection function and the labeling is SEG.

Example 8. Figure 17 illustrates a tree of order 16 with a SEG labeling.

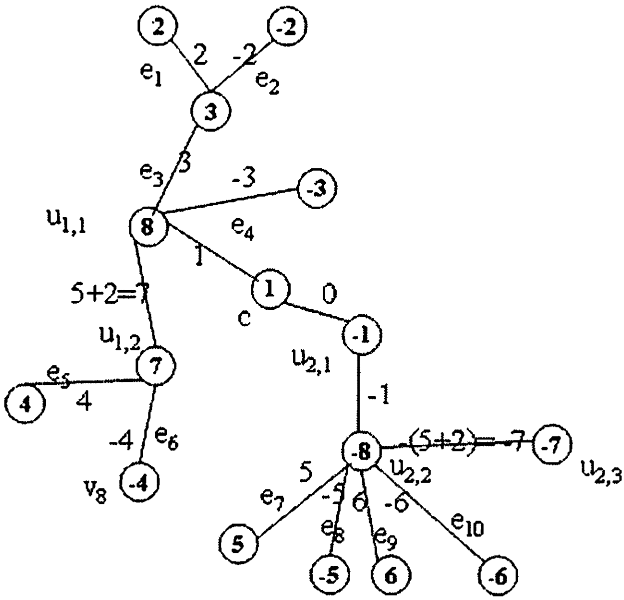


Figure 17.

By the similar argument as Theorem 4.1, we have the following

__**Theorem 4.2.**__ If a tree T has even order such that $irr(T) = SP(1,3^2)$ then T is SEG.

__**Theorem 4.3.**__ If a tree T has even order such that $irr(T) = SP(2^2,3)$ then T is SEG.

__**Theorem 4.4.**__ If a tree T has even order such that $irr(T) = SP(2^3,3)$ then T is SEG.

__**Theorem 4.5.**__ If a tree T has even order such that $irr(T) = SP(1,2,3^2)$ then T is SEG.

__**Theorem 4.6.**__ If a tree T has even order such that $irr(T) = SP(2^4,3)$ then T is SEG.

__**Theorem 4.7.**__ If a tree T has even order such that $irr(T) = SP(1,2,4)$ then T is SEG.

__**Theorem 4.8.**__ If a tree T has even order such that $irr(T) = SP(1,2^2,4)$ then T is SEG.

Theorem 4.9. If a tree T has even order such that $irr(T) = SP(1,2^3,4)$ then T is SEG.

Theorem 4.10. If a tree T has even order such that $irr(T) = SP(1^{2k+1},3^2,4)$ for all $k \geq 0$ then T is SEG.

5.Direction for further research.

We propose the following problem and conjectures for further research.

Problem. Characterize spiders of even orders of diameter k which are not SEG, where $k > 2$.

Conjecture 5.1. For $2 < k \leq 4$, all even order spiders of diameter k are SEG.

Another conjecture listed below is for even order tree Comb(n).

Conjecture 5.2. For any $n \geq 7$, Comb(n) is SEG.

Conjecture 5.3. Every tree of even order is an induced subtree of a SEG tree of even order.

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