Integer-Magic Spectra of Functional Extensions of Graphs

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Abstract

For any $k \in \mathbb{N}$, a graph $G = (V, E)$ is said to be $\mathbb{Z}_k$-magic if there exists a labeling $l : E(G) \to \mathbb{Z}_k - \{0\}$ such that the induced vertex set labeling $l^+ : V(G) \to \mathbb{Z}_k$ defined by

$$l^+(v) = \sum_{uv \in E(G)} l(uv)$$

is a constant map. For a given graph G, the set of all $k \in \mathbb{N}$ for which G is $\mathbb{Z}_k$-magic is called the integer-magic spectrum of G and is denoted by $IM(G)$. In this paper we will consider the functional extensions of P_n $(n = 2, 3, 4)$ and will determine their integer-magic spectra.

Key Words: magic, non-magic, functional extension, integer-magic spectrum
AMS Subject Classification: 05C78

1 Introduction

For any abelian group A, written additively, any mapping $l : E(G) \to A - \{0\}$ is called a *labeling*. Given a labeling on the edge set of G one can introduce a vertex set labeling $l^+ : V(G) \to A$ by

$$l^+(v) = \sum_{uv \in E(G)} l(uv).$$

A graph G is said to be *A-magic* if there is a labeling $l : E(G) \to A - \{0\}$ such that for each vertex v, the sum of the labels of the edges incident with v are all equal to the same constant; that is, $l^+(v) = c$ for some fixed $c \in A$. In general, a graph G may admit more than one labeling to become A-magic; for example, if $|A| > 2$ and $l : E(G) \to A - \{0\}$ is a magic labeling of G with sum c, then $l : E(G) \to A - \{0\}$, the inverse labeling of l, defined by $l(uv) = -l(uv)$ is another magic labeling of G with sum $-c$. At present, given an abelian group, no general efficient algorithm is known for finding magic labelings of general graphs. A graph $G = (V, E)$ is called *fully magic* if it is A-magic for every abelian group A. For example, every regular graph is fully magic. Also, a graph $G = (V, E)$ is called *non-magic* if for every abelian group A, the graph is not A-magic. The most obvious example of a non-magic graph is P_n $(n \geq 3)$, the path of order n. As a result, any graph with a pendant path of length $n \geq 3$ would be non-magic. Here is another example of a non-magic graph: Consider the graph H Figure 1. Given any abelian group A, a potential magic labeling of H is illustrated in that figure. Since $l^+(u) = x \neq 0$ and $l^+(v) = 0$, H is not A-magic. This fact can be generalized as follows:

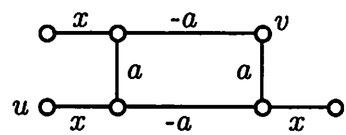

Figure 1: An example of a non-magic graph H for which $IM(H) = \emptyset$.

Observation 1.1. *Every even cycle C_n with $2k+1$ $(< n)$ consecutive pendants is non-magic.*

Observation 1.2. *Every odd cycle C_n with $2k$ $(< n)$ consecutive pendants is non-magic.*

Certain classes of non-magic graphs are presented in [1].
The original concept of an A-magic graph is due to J. Sedlacek [13, 14], who defined it to be a graph with a real-valued edge labeling such that

1. distinct edges have distinct nonnegative labels; and
2. the sum of the labels of the edges incident to a particular vertex is the same for all vertices.

The $\mathbb{Z}$-magic graphs were considered by Stanley [15, 16], who pointed out that the theory of magic labeling can be put into the more general context of linear homogeneous diophantine equations. Doob [2, 3, 4], also considered A-magic graphs where A is an abelian group and determined the wheels that are $\mathbb{Z}$-magic. Also, a graph G is said to be $\mathbb{N}$-magic if there exists a labeling

$l : E(G) \to \mathbb{N}$ such that $l^+(v)$ is a constant, for every $v \in V(G)$. It is well-known that a graph G is $\mathbb{N}$-magic if and only if every edge of G is contained in a 1-factor (a perfect matching) or a $\{1,2\}$-factor [6, 12].
When the group is $\mathbb{Z}_k$, we shall refer to the $\mathbb{Z}_k$-magic graph as *k-magic.* Recently, there has been considerable research articles in graph labeling, interested readers are directed to [5, 17]. For convenience, we will use 1-magic instead of $\mathbb{Z}$-magic. Clearly, if a graph is h-magic, it is not necessarily k-magic $(h \neq k)$. For a given graph G the set of all positive integers h for which G is h-magic is called the *integer-magic spectrum* of G and is denoted by $IM(G)$. Since any regular graph is fully magic, then it is h-magic for all positive integers $h \geq 2$; therefore, $IM(G) = \mathbb{N}$. On the other hand, the graph H, Figure 1, is non magic, hence $IM(H) = \emptyset$. Integer-magic spectra of certain classes of graphs have been studied in [7, 8, 9, 10, 11].

Definition 1.3. *Let G be a graph and $f : V(G) \longrightarrow \mathbb{N}$. The functional extension of G by f, is a graph H with*

$$V(H) = \cup\{u_i : u \in V(G) \text{ and } i = 1,2,\cdots,f(u)\}$$
$$E(H) = \cup\{u_i v_j : uv \in E(G),\ i = 1,2,\cdots,f(u),\ \text{and } j = 1,2,\cdots,f(v)\}$$

We will use $\text{Ext}(G,f)$ to denote the functional extension of graph G by f. Also, when $f(u) = 1$, in $\text{Ext}(G,f)$ we will use u instead of u_1.

Examples 1.4. *Consider P_2, the path of order 2, with vertices a and b.*

(a) If we define the function $f : \{a,b\} \to \mathbb{N}$ by $f(a) = 1$ and $f(b) = n$, then $\text{Ext}(P_2,f)$ would be a *star,* the complete bipartite graph $K(1,n)$, with central vertex a that has n leaves, Figure 2.

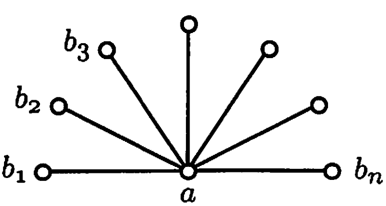

Figure 2: $\text{Ext}(P_2,f)$ is a star with n leaves.

(b) If we define the function $g : \{a,b\} \to \mathbb{N}$ by $f(a) = m$ and $f(b) = n$, then $\text{Ext}(P_2,g)$ would be the complete bipartite graph $K(m,n)$.

Examples 1.5. *Consider P_3, the path of order 3, with vertices a, b and c.*

(a) If we define the function $f : \{a,b,c\} \to \mathbb{N}$ by $f(a) = 1$, $f(b) = 2$, and $f(c) = 3$, then $\text{Ext}(P_3,f)$ is the graph illustrated in Figure 3, which is isomorphic to $K(2,4)$.

(b) If we define the function $g : \{a,b,c\} \to \mathbb{N}$ by $g(a) = g(c) = 1$ and $g(b) = n$, then $\text{Ext}(P_3,g) = K(2,n)$, Figure 4.

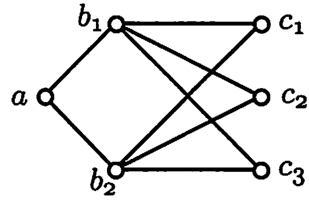

Figure 3: $\mathrm{Ext}(P_3, f)$

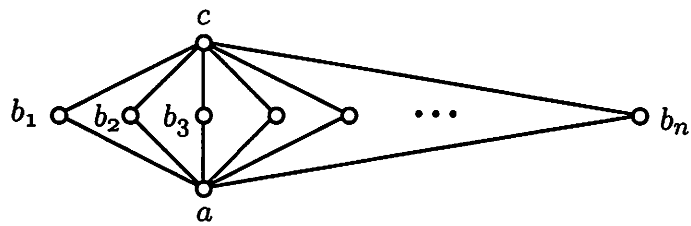

Figure 4: $\mathrm{Ext}(P_3, g) = K(2, n)$

2 Functional Extensions of P_2

As indicated in Example 1.4, the functional extension of P_2 is the complete bipartite graph $K(m, n)$. In this section we will determine the integer-magic spectra of these graphs. Note that $K(1, 1) = P_2$ is a regular graph, fully magic, and $IM(P_2) = \mathbb{N}$. Also, $K(1, 2) = P_3$, is the path of order three, which is non-magic and $IM(P_3) = \emptyset$. When $n > 2$, the complete bipartite graph $K(1, n)$, or star, is denoted by $ST(n)$, for which we have the following theorem [9]:

Theorem 2.1. *Let $n \geq 3$, and $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorization of $n - 1$. Then*

$$IM(\,ST(n)\,) = \bigcup_{i=1}^{k} p_i \mathbb{N}.$$

Examples 2.2.

(a) $IM(\,K(1, 65)\,) = 2\mathbb{N}$; *here* $n - 1 = 64 = 2^6$.

(b) $IM(\,K(1, 7)\,) = 2\mathbb{N} \cup 3\mathbb{N}$; *here* $n - 1 = 6 = 2 \times 3$.

(c) $IM(\,K(1, 151)\,) = 2\mathbb{N} \cup 3\mathbb{N} \cup 5\mathbb{N}$; *here* $n - 1 = 150 = 2 \times 3 \times 5^2$.

Theorem 2.3. *Let $m, n \geq 2$. Then*

$$IM(K(m, n)) = \begin{cases} \mathbb{N} & \text{if } m + n \text{ is even;} \\ \mathbb{N} - \{2\} & \text{if } m + n \text{ is odd.} \end{cases}$$

Moreover, in each case the labeling can be done so that the sum is 0.

Proof. Let $S = \{u_1, u_2, \cdots, u_m\}$ and $T = \{v_1, v_2, \cdots, v_n\}$ be the partite sets. In labeling of edges $u_i v_j$, with elements of $\mathbb{Z}_h$ $(h \geq 3)$, we will consider three cases:

Case I. m, n are both even. We label the edges by $l(u_i v_j) = (-1)^{i+j}$. This will result in $l^+ \equiv 0$.

Case II. m is even and n is odd. We label the edges by

$$l(u_i v_j) = \begin{cases} 2(-1)^{i-1} & \text{if } j = 1 \\ (-1)^i & \text{if } j = 2, 3 \\ (-1)^{i+j} & \text{otherwise} \end{cases}$$

This labeling is illustrated in table (2.1).

	v_1	v_2	v_3	v_4	$\dots$	v_n
u_1	2	-1	-1	-1	$\dots$	1
u_2	-2	1	1	1	$\dots$	-1
u_3	2	-1	-1	-1	$\dots$	1
u_4	-2	1	1	1	$\dots$	-1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
u_{m-1}	2	-1	-1	-1	$\dots$	1
u_m	-2	1	1	1	$\dots$	-1

(2.1)

Case III. m, n are both odd. We label the edges by using the following table (2.2):

	v_1	v_2	v_3	v_4	v_5	$\dots$	v_n
u_1	2	-1	-1	2	-2	$\dots$	-2
u_2	-1	2	-1	-1	1	$\dots$	1
u_3	-1	-1	2	-1	1	$\dots$	1
u_4	2	-1	-1	$\ddots$			
u_5	-2	1	1				
$\vdots$	$\vdots$	$\vdots$	$\vdots$			$(-1)^{i+j}$	
u_m	2	-1	-1				$\ddots$

(2.2)

Finally, we observe that if m, n have different parities, the graph would not be 2-magic. □

3 Functional Extensions of P_3

Consider P_3, the path of order 3, with vertices a, b and c, the middle vertex being b. Also, let $f : \{a, b, c\} \to \mathbb{N}$ be defined by $f(a) = r$, $f(b) = s$, and $f(c) = t$. Then $\text{Ext}(P_3, f) = K(r + t, s)$ and its integer-magic spectrum has been determined in Theorem 2.3.

4 Functional Extensions of P_4

In this section we will consider P_4, the path of order 4, with vertices a, b, c, and d, Figure 5. The integer-magic spectra of $G_f = \text{Ext}(P_4, f)$, with different functions $f : \{a, b, c, d\} \to I\!N$, will be determined. Note that if $f(a) = f(b) = f(c) = 1$, then $\text{Ext}(P_4, f)$ would be non-magic and $IM(G_f) = \emptyset$. Depending on the function f, there are 8 other non-isomorphic cases which will be considered.

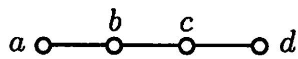

Figure 5:

Theorem 4.1. *Using the above notation, let $f(a) = f(b) = f(d) = 1$, and $f(c) = n$. Also, let $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorization of $n-1$. Then*

$$IM(G_f) = \begin{cases} \emptyset & \text{if } n = 1, 2; \\ \cup p_i I\!N - \{2,3\} & \text{if } 2|n \text{ or } n \equiv 1 \pmod 4; \\ \cup p_i I\!N - \{2,3,4\} & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

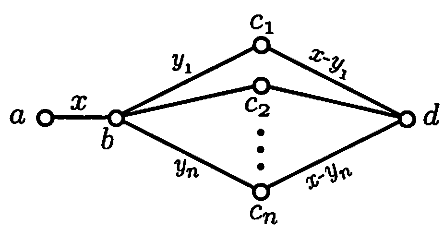

Figure 6: A typical magic labeling of G_f.

Proof. We observe that for $n = 1$ or 2, the functional extension G_f is non-magic. Therefore, assume that $n \geq 3$. A typical magic labeling of G_f requires at least two distinct nonzero elements x, y_1, as illustrated in Figure 6. The graph G_f is not 2-magic, because the vertices a and c_1 have degrees of different parity. The two conditions $l^+(a) = l^+(b)$ and $l^+(a) = l^+(d)$ will provide two equations

$$y_1 + y_2 + \cdots + y_n \equiv 0 \pmod h; \tag{4.1}$$

$$(n-1)x \equiv 0 \pmod h. \tag{4.2}$$

The graph G_f is not 3-magic, because the above system of equations becomes

$$\begin{cases} ny & \equiv 0 \pmod 3; \\ (n-1)x & \equiv 0 \pmod 3), \end{cases}$$

which does not have non-zero solutions for $x, y \in \mathbb{Z}_3$.
Now, assume that G_f is h-magic. From equation (4.2) we realize that $\gcd(h, n-1) > 1$. Therefore, $h \in \cup p_i\mathbb{N} - \{2,3\}$.
Conversely, let $h \in \cup p_i\mathbb{N} - \{2,3\}$. To find a magic labeling for G_f that satisfies the above equations (4.1) and (4.2), we will consider the following cases:
Case I. n is even, as a result h is odd and $h \geq 5$. Let p be a prime factor of $\gcd(h, n-1)$. With the choice of $x = h/p$ the equation (4.2) is automatically satisfied. Now, choose $a \in \mathbb{Z}_h - \{0, x, -x\}$ and let $y_i = (-1)^i a$. Then $y_1 + y_2 + \cdots + y_n \equiv 0 \pmod h$.
Case II. n is odd, as a result h is even, assume that $h \geq 6$. Let $h = 2r$ $(r \geq 3)$ and label the edges of G_f by $x = r$ and

$$y_i = \begin{cases} 2 & \text{if} \quad i = 1; \\ -1 & \text{if} \quad i = 2, 3; \\ (-1)^i & \text{if} \quad i \geq 4. \end{cases}$$

This labeling satisfies the equations (4.1) and (4.2).
Finally let $h = 4$. If $n = 4k+1$, then G_f is 4-magic; Because, the choices of $x = 1$ and

$$y_i = \begin{cases} 3 & \text{if} \quad i = 1, 2; \\ 2 & \text{if} \quad 3 \leq i \leq n, \end{cases}$$

would satisfy the above equations ($h = 4$).
But if $n = 4k+3$, the only non-zero solution for $(n-1)x = (4k+2)x \equiv 0 \pmod 4$ in $\mathbb{Z}_4$ would be $x = 2$. Then the equation $y_1 + y_2 + \cdots + y_n \equiv 0 \pmod 4$ does not have solutions in $\mathbb{Z}_4 - \{0, x\} = \{1, 3\}$. Therefore, in this case, G_f would not be 4-magic. □

Using the notation of Figure 5, let $f(b) = f(c) = 1$, $f(a) = m$ and $f(d) = n$ $(m, n \geq 2)$. The resulting graphs $\mathrm{Ext}(G, f)$ are trees of diameter three, also known as double-stars. The integer-magic spectra of double-stars are examined in [9]. For the sake of completeness of our discussion, we will mention the major results. For the proof of theorems 4.2, 4.3, 4.4, and corollary 4.5, we refer the interested readers to [9].

Theorem 4.2. *The graph $DS(m,n)$ is $\mathbb{Z}$-magic (or 1-magic) if and only if $m = n$.*

Theorem 4.3. $IM(DS(m,m)) = \mathbb{N} - \{h \in \mathbb{N} : h > 1 \text{ and } h|(m-2)\}$.

Theorem 4.4. *Let $m - n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ and $n - 2 = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$ be the prime factorizations of these two numbers. Then $IM(DS(m,n)) = \cup_{i=1}^{k} A_i$, where*

$$A_i = \begin{cases} p_i^{1+\beta_i} \mathbb{N} & \text{if } \alpha_i > \beta_i \geq 0; \\ \emptyset & \text{if } \beta_i \geq \alpha_i \geq 0. \end{cases}$$

Corollary 4.5. *If $m-n$ is a divisor of $n-2$, then $IM(DS(m,n)) = \emptyset$.*

Examples 4.6.

(a) $IM(DS(14,14)) = \mathbb{N} - \{2,3,4,6,12\}$; here $m-2 = 12$, and we need to exclude its divisors that are bigger than one: namely, $2,3,4,6,12$.

(b) $IM(DS(18,10)) = \emptyset$; here $m-n = 8$ is a divisor of $n-2 = 8$.

(c) $IM(DS(28,8)) = 4\mathbb{N} \cup 5\mathbb{N}$; here $m-n = 20 = 2^2 \times 5$, while $n-2 = 6 = 2 \times 3$.

Theorem 4.7. *Using the notation of Figure 5, let $f(b) = f(d) = 1$, $f(a) = m$ and $f(c) = n$ $(m, n \geq 2)$. Also, let $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorization of $m+n-2$. Then*

$$\cup p_i \mathbb{N} - \{2,3,4\} \subset IM(G_f) \subset \cup p_i \mathbb{N} - \{2\}.$$

Moreover,

(a) G_f is 3-magic if and only if $m \equiv 0 \pmod 3$ and $n \equiv 2 \pmod 3$.

(b) G_f is 4-magic if and only if m, n are both even or m, n are both odd and $m \equiv n \pmod 4$.

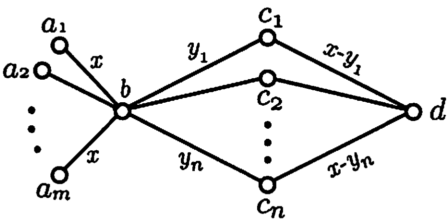

Figure 7: A typical magic labeling of G_f.

Proof. A typical magic labeling of G_f requires at least two distinct nonzero elements x, y, as illustrated in Figure 7. The graph G_f is not 2-magic, because the vertices a_1 and c_1 have degrees of different parities. The two conditions $l^+(a_1) = l^+(b)$ and $l^+(a_1) = l^+(d)$ will translate to equations

$$(m-1)x + y_1 + y_2 + \cdots + y_n \equiv 0 \pmod h; \quad (4.3)$$

$$(n-1)x - (y_1 + y_2 + \cdots + y_n) \equiv 0 \pmod h, \quad (4.4)$$

which will provide

$$(m+n-2)x \equiv 0 \pmod{h}. \tag{4.5}$$

Now, assume that G_f is h-magic. Then from equation (4.5) we realize that $\gcd(h, m+n-2) > 1$. Therefore, $h \in \cup p_i \mathbb{N} - \{2\}$.

Conversely, let $h \in \cup p_i \mathbb{N} - \{2,3,4\}$. To find a magic labeling for G_f that satisfies the above equations (4.3) and (4.4), we will consider the following cases:

Case I. h is odd ($h \geq 5$). Let p be a prime factor of $\gcd(h, m+n-2)$ and $x = h/p$, which implies that $2x \not\equiv 0 \pmod{h}$. Also, let $a \in \mathbb{Z}_h - \{0, x, -x, 2x\}$. If n is even, we choose

$$y_i = \begin{cases} 2x - a & \text{if} \quad i = 1; \\ x + (-1)ia & \text{if} \quad 2 \leq i \leq n-2; \\ (-1)ia & \text{if} \quad i = n-1, n. \end{cases}$$

If n is odd, we choose

$$y_i = \begin{cases} 2x & \text{if} \quad i = 1; \\ x + (-1)ia & \text{if} \quad 2 \leq i \leq n-2; \\ (-1)ia & \text{if} \quad i = n-1, n. \end{cases}$$

These labelings will satisfy the equations (4.3) and (4.4).

Case II. $\gcd(h, m+n-2) = 2^r q$, where q is odd and greater than 1, and $r \geq 1$. Let $x = \begin{cases} 2 & \text{if} \quad r = 1; \\ q & \text{if} \quad r \geq 2, \end{cases}$ which guarantees that $2x \not\equiv 0 \pmod{h}$, and the labelings for y_i described in Case I, will work.

This proves that $\cup p_i \mathbb{N} - \{2,3,4\} \subset IM(G_f) \subset \cup p_i \mathbb{N} - \{2\}$.

The graph G_f is 3-magic if and only if $x = 1$, $y_i = 2$ or $x = 2$, $y_i = 1$. Either choices will convert the equations (4.3) and (4.4) to $\begin{cases} m & \equiv & 0 \pmod{3}; \\ n & \equiv & 2 \pmod{3}. \end{cases}$

For G_f to be 4-magic, we need $\gcd(m+n-2, 4) = 2$ or 4. Therefore, m, n must have the same parity.

Suppose m, n are both even. Let $x = 2$. If $n \equiv 2 \pmod{4}$, we choose $y_i = 1$. If $n \equiv 0 \pmod{4}$, we choose $y_i = \begin{cases} 3 & \text{if} \quad i = 1; \\ 1 & \text{if} \quad i \geq 2. \end{cases}$

Suppose m, n are both odd. If $m, n \equiv 3 \pmod{4}$, we choose $x = 1$ and $y_i = 2$. If $m, n \equiv 1 \pmod{4}$, we choose $x = 1$ and $y_i = \begin{cases} 3 & \text{if} \quad i = 1, 2; \\ 2 & \text{if} \quad i \geq 3. \end{cases}$

These labelings will satisfy the equations (4.3) and (4.4).

If $m \equiv 1 \pmod{4}$ and $n \equiv 3 \pmod{4}$, then we will be forced to choose $x = 2$ and will end up with the equation $y_1 + y_2 + \cdots + y_n \equiv 0 \pmod{4}$, which does not have solution in $\{1, 3\}$. □

Theorem 4.8. *Using the above notation, let $f(a) = f(d) = 1$, and $f(b) = f(c) = m$. Then $IM(G_f) = \mathbb{N} - \{2\}$.*

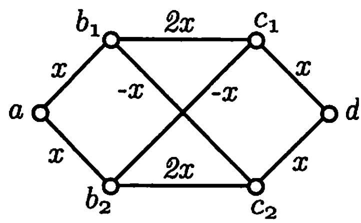

Figure 8: A typical magic labeling of G_f.

Proof. First observe that $\deg(a) = m$ and $\deg(b_1) = m+1$, which have different parity. Therefore, the graph is not 2-magic. Next, given $h \geq 3$, we choose $x \in \mathbb{Z}_h$ such that $2x \not\equiv 0 \pmod h$ and label all the edges of G_f by x except

$$l(b_i c_j) = \begin{cases} 2x & \text{if } j = i; \\ -x & \text{if } j = i+1 \ 1 \leq i \leq n-1; \\ -x & \text{if } j = 1, \ i = n; \\ x & \text{otherwise.} \end{cases}$$

The labeling of the edges $b_i c_j$ is illustrated in the following table.

	c_1	c_2	c_3	c_4	$\dots$	c_{m-1}	c_m
b_1	$2x$	$-x$	x	x	$\dots$	x	x
b_2	x	$2x$	$-x$	x	$\dots$	x	x
b_3	x	x	$2x$	$-x$	$\dots$	x	x
b_4	x	x	x	$2x$	$\dots$	x	x
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
b_{m-1}	x	x	x	x	$\dots$	$2x$	$-x$
b_m	$-x$	x	x	x	$\dots$	x	$2x$

(4.6)

□

Problem 4.9. *Find the integer-magic spectrum of* $Ext(P_4, f)$*, where* $f(a) = f(d) = 1$*,* $f(b) = m$*, and* $f(c) = n$ $(m \neq n)$*.*

Theorem 4.10. *Using the notation of Figure 5, let* $f(a) = f(b) = 1$*, and* $f(c) = f(d) = n$*. Then*

$$IM(G_f) \subset \mathbb{N} - \{h \mid h = 2 \text{ or } h \text{ is a divisor of one of } n \pm 1\}.$$

Proof. First observe that $\deg(b) = n+1$ and $\deg(c_1) = n$, which have different parity. Therefore, the graph is not 2-magic. Next, this graph is isomorphic to $K(n, n+1)$ having a pendant attached to one of the vertices of the partite set that has $n+1$ elements.

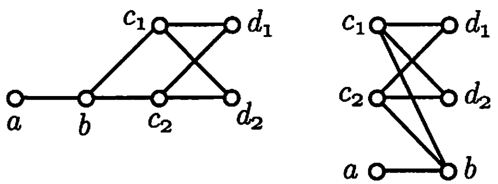

Figure 9: G_f and $K(n, n+1)$ with a pendant attached to one of its vertices.

Now consider the complete bipartite graph $K(n, n+1)$ with the partite sets $A = \{c_1, c_2, \cdots c_n\}$ and $B = \{d_1, d_2, \cdots d_n, b\}$. We will label the edges of $K(n, n+1)$ in such a way that $l^+(c_i) = l^+(d_j) = n+1$ and $l^+(b) = 0$. Then, we will label the edge ab by $n+1$. This labeling is illustrated in the following table:

	d_1	d_2	d_3	$\dots$	d_n	b	$l^+(c_i)$
c_1	1	1	1	$\dots$	1	1	$n+1$
c_2	1	1	1	$\dots$	1	1	$n+1$
c_3	1	1	1	$\dots$	1	1	$n+1$
c_4	1	1	1	$\dots$	1	1	$n+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
c_{n-1}	1	1	1	$\dots$	1	1	$n+1$
c_n	2	2	2	$\dots$	2	$1-n$	$n+1$
$l^+(d_i)$	$n+1$	$n+1$	$n+1$	$\dots$	$n+1$	0	

To guarantee that this is a valid labeling, we need to make sure that $n \pm 1 \not\equiv 0 \pmod{h}$, and that is why we are excluding the divisors of $n \pm 1$. □

Problem 4.11. *Using the notation of the Theorem 4.10, show that*

$$IM(G_f) = I\!N - \{h \mid h = 2 \text{ or } h \text{ is a divisor of one of } n \pm 1\}.$$

Problem 4.12. *Using the notation of Figure 5, find the integer-magic spectrum of Ext(P_4, f), where $f(a) = f(b) = 1$, $f(c) = m$, and $f(d) = n$ ($m \neq n$).*

Problem 4.13. *Using the notation of Figure 5, find the integer-magic spectrum of Ext(P_4, f), where $f(a) = 1$ and $f(b) = f(c) = f(d) = n$. Solve this problem when $f(b)$, $f(c)$, and $f(d)$ are distinct integers bigger than 1.*

Problem 4.14. *Using the notation of Figure 5, find the integer-magic spectrum of Ext(P_4, f), where $f(b) = 1$ and $f(a) = f(c) = f(d) = n$. Solve this problem when $f(a)$, $f(c)$, and $f(d)$ are distinct integers bigger than 1.*

We conclude this paper with the following theorem, that can easily be generalized to P_n ($n > 4$).

Theorem 4.15. *Using the notation of Figure 5, let* $f(a) = m$, $f(b) = n$, $f(c) = p$, *and* $f(d) = q$, *where* m, n, p, *and* q *are distinct integers bigger than* 1. *Then*

$$IM(G_f) = \begin{cases} \mathbb{N} & \text{if } n,\ m+p,\ n+q,\ p \text{ have the same parity };\\ \mathbb{N} - \{2\} & \text{otherwise.} \end{cases}$$

Proof. Let $A = \{a_1, a_2, \cdots, a_m\}$, $B = \{b_1, b_2, \cdots, b_n\}$, $C = \{c_1, c_2, \cdots, c_p\}$, and $D = \{d_1, d_2, \cdots, d_q\}$ be the vertices of G_f. Consider the complete bipartite graphs $K(m, n)$ with partite sets A, B; $K(n, p)$ with partite sets B, C; and $K(p, q)$ with partite sets C, D. The graph G_f can be viewed as these three complete bipartite graphs combined in such a way that their corresponding vertices, same partite sets, are identified. By the theorem 2.3, one can label the edges of these complete bipartite graphs so that the sum be 0. Therefore, when the corresponding vertices are identified, the sum still would be 0. To determine whether G_f is 2-magic or not, we observe that $\deg(a_i) = n$, $\deg(b_i) = m + p$, $\deg(c_i) = n + q$, and $\deg(d_i) = p$. And G_f is 2-magic if and only if the numbers n, $m + p$, $n + q$, and p have the same parity. □

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