Factor-conformability and total chromatic number

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1 Introduction

The graphs considered here are simple (that is, they have no loops or multiple edges). A *component* of a graph is a maximal connected subgraph. The *total chromatic number*, $\chi_T(G)$, of G is the least number of colours required to colour the edges and vertices of G so that no two adjacent or incident elements receive the same colour.

In 1965 and 1968 Behzad [1] and Vizing [6] independently conjectured that, for any graph G,

$$\Delta(G) + 1 \leq \chi_T(G) \leq \Delta(G) + 2$$

where $\Delta(G)$ is the maximum degree of any vertex of G. A graph G is said to be of *type 1* if $\chi_T(G) = \Delta(G) + 1$, and of *type 2* otherwise (it is clear that $\chi_T(G) > \Delta(G)$).

In this paper, a *type 1 total colouring* of a graph G is a total colouring that uses just $\Delta + 1$ colours; thus G is of type 1 if and only if it possesses such a colouring.

In 1986 Chetwynd and Hilton [2] made a conjecture, later modified by Hamilton, Hilton and Hind [5], based on the following concept: a (proper) vertex colouring of G with $\Delta(G) + 1$ colours is *conformable* if the number of colour classes of parity different from that of $|V(G)|$ is at most $\text{def}(G) = \sum_{v \in V(G)} (\Delta(G) - d(v))$ where $d(v)$ is the degree of a vertex v.

The modified Conformability Conjecture is:

Conjecture 1.1 *Let G be a graph satisfying* $\Delta(G) \geq \frac{1}{2}(|V(G)| + 1)$. *Then G is type 2 if and only if G contains a subgraph H with* $\Delta(G) = \Delta(H)$ *which is either non-conformable or, when* $\Delta(G)$ *is even, consists of* $K_{\Delta(G)+1}$ *with one edge subdivided.*

We note that for graphs of low degree compared with $|V(G)|$, conformability of G (rather than of all its subgraphs with the same maximum degree, as would be required by the conjecture) does not provide a particularly good predictor that G is type 1. Of the first fifty total-colouring critical graphs in the catalogue provided by Hamilton, Hilton and Hind ([4], also in [5]), thirty-one are conformable (see Appendix for further details). The purpose of this paper is to discuss a stronger variant of the conformability concept, according to which only one of the above fifty is conformable.

2 Spined graphs

A *spined graph* is a graph in which some vertices may possess *spines* (sometimes known as *dangling edges* or *semi-edges* in the literature), that are incident with only one vertex rather than two. (Spines are not loops, since a loop contributes 2 to the degree of a vertex.) For any vertex v of an un-spined graph G, we define $\text{def}(v) = \Delta(G) - d(v)$. It is convenient in this paper to replace G by the spined graph (still denoted by G) where each vertex v has $\text{def}(v)$ spines; thus every vertex of G now has exactly $\Delta(G)$ incident edges or spines, and the total number of spines is $\text{def}(G)$. We make this assumption throughout the paper. The *edge degree* of a vertex v of the spined graph G is the number of incident *edges* at v (that is, the degree of the original graph); the spines do not contribute, but the new terminology aids clarity. We continue, however, to refer to the *maximum degree* of a graph G, since the above construction does not alter this.

The vertices and spines of G together are the *vesps* of G; the vesps and edges together are the *parts* of G. The subgraph of G *induced* by a set of vertices of G includes any incident spines.

A *vesp colouring* of a graph G of maximal degree Δ is a colouring of the vesps, using up to $\Delta + 1$ colours, such that the vertex colours form a proper vertex colouring of G and, moreover, the spine colours at any vertex v are distinct from each other and from that of v.

The concept of type 1 total colouring now extends as follows. A *type 1 total colouring* of a spined graph G is a colouring of the parts of G using $\Delta + 1$ colours, whose restriction to the vertices and edges is a total colouring and whose restriction to the vesps is a vesp colouring.

Clearly, a type 1 total colouring of a graph G (not spined) extends to a type 1 total colouring of the corresponding spined graph, since at each vertex there are just enough spare colours to deal with the spines. Thus adding spines as described above does not change $\chi_T(G)$.

3 Factor-conformability

Let σ be a vesp colouring of a spined graph G using colours $c_1, \ldots, c_{\Delta+1}$ (σ could of course be a restriction of a type 1 colouring). For each $v \in V(G)$ we denote by $C_\sigma(v)$ the set of colours *at* v; that is, the colours on v and its attached spines. Then, for $i = 1, \ldots, \Delta(G) + 1$, we denote by $S_i(\sigma)$ the set $\{v \in V(G)\colon c_i \in C_\sigma(v)\}$, and by $G_i(\sigma)$ the spined subgraph of G induced by $V(G) - S_i(\sigma)$. When there is no risk of confusion, we drop the reference to σ.

In similar vein, for all pairs i, j ($i, j = 1, \ldots, \Delta(G) + 1$), we denote by G_{ij} the spined subgraph of G induced by $V(G) - (S_i \cup S_j)$; we shall use this notation in Theorem 4.2 and thereafter.

Let σ be a vesp colouring of G. Then σ is said to be *factor-conformable* if G_i has a 1-factor for each $i = 1, \ldots, \Delta + 1$; and the graph G is said to be *factor-conformable* if it has a factor-conformable vesp colouring.

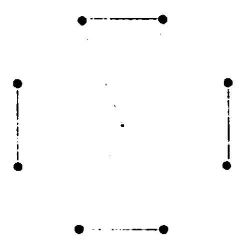
Lemma 3.1 *If G is type 1, then G is factor-conformable.*

Proof. Let μ be a type 1 total colouring of G. Then for each $i = 1, \ldots, \Delta + 1$, the set $\{e \in E(G)\colon \mu(e) = c_i\}$ is a 1-factor of G_i. ■

Lemma 3.2 *If G is factor-conformable, then G is conformable.*

Proof. Let σ be a factor-conformable vesp colouring of G. For $i = 1, \ldots, \Delta(G) + 1$, the number of vesps coloured c_i has the parity of $|V(G)|$. Thus for every vertex colour class whose cardinality differs in parity from that of $|V(G)|$, there must be at least one spine of that colour. Therefore def(G) is at least the number of such classes. ■

We observe that the graph numbered 12 of the list in [4], [5] is the 8-cycle with edges added so that opposite vertices are adjacent. (Note that, being regular, this graph has no spines.)



Up to isomorphism, there are just three factor-conformable vertex 4-colourings, which are (taking the vertices in cyclic order):

$c_1, c_2, c_1, c_3, c_4, c_3, c_4, c_2$;

$c_1, c_2, c_3, c_1, c_4, c_3, c_4, c_2$;

$c_1, c_2, c_3, c_1, c_4, c_3, c_2, c_4$.

However, it may be checked that none of the other forty-nine critical graphs listed in [4], [5] has any factor-conformable $(\Delta + 1)$-colouring. (See Appendix for further observations.) Thus, factor-conformability strengthens the concept of conformability in a manner that seems to improve considerably its ability to predict whether a graph is of type 1 or 2.

We now consider in detail graphs of maximum degree at most 3, and propose some conjectures concerning conditions under which such graphs are type 1. There is no loss of generality in considering only connected graphs.

4 Graphs of low degree

We dispose first of graphs of maximum degree at most 2.

Theorem 4.1 *Every factor-conformable, connected graph of maximum degree at most 2 is type 1.*

Proof. Let G be a connected graph of maximum degree $\Delta \leq 2$. If $\Delta = 0$ then G is a singleton vertex and is type 1; if $\Delta = 1$ then $G = K_2$ and is not factor-conformable. Thus we may assume $\Delta = 2$. Paths, and cycles of order divisible by 3, are type 1, so we need only consider cycles of order $3q + 1$ or $3q + 2$ (where $q > 0$). But any vertex 3-colouring of such a cycle must have three successive vertices coloured c_i, c_j, c_i for some pair i, j of $\{1, 2, 3\}$. This colouring is clearly not factor-conformable. ■

Now let G be a connected graph of maximum degree 3. The *truncate* of G is the graph obtained by successively deleting vertices of edge-degree 0 and 1 until there are no such vertices. Then G is type 1 if and only if its truncate is type 1, and it is thus reasonable to restrict our attention to connected graphs with $\Delta = 3$ and minimum edge-degree at least 2. In this case each vertex of edge-degree 2 lies on a unique path $vw_1w_2...w_qx$ such that v and x have edge-degree 3, all intermediate vertices being of edge-degree 2. Then, provided that $q \geq 3$, we may *prune* G, to a graph H, by deleting the w_i and all the edges of the path (so that the edge-degrees of v and x are now 2, and v and x each acquire a spine). A straightforward check of the possible colours at v and x shows that G is type 1 if and only if H is type 1. Thus it is reasonable to restrict our attention still further, to connected graphs all of whose edge-degrees are 2 or 3 and such that there are no sequences of three or more successive vertices of edge-degree 2. Such a graph is said to be *cubic* if it is regular of degree 3; we shall say that it is *semi-cubic* otherwise.

Thus, a non-cubic graph of maximum degree 3 is type 1 if and only if the graph obtained by truncating, then successively pruning until the resulting graph is semi-cubic, is type 1. (Note that the result of truncating and pruning a non-cubic graph of maximum degree 3 is never cubic.)

In this context, it is useful to introduce a special definition of connectivity and to import some ideas from vertex colouring theory.

Let S be a set of parts of a spined graph. We say that S is *chain-connected* if it cannot be partitioned into two non-empty sets S_1, S_2 such that no element of S_1 is incident with any element of S_2 and no edge or spine of S_1 is adjacent to any edge or spine of S_2. For example, consider vertices v, w, x, edges $e = vw$, $f = wx$, and a spine s at v. Then $\{e, f\}$ is chain-connected, as are $\{s, e, f\}$, $\{v, e, f\}$, $\{v, e, w\}$ and $\{s, v\}$, but $\{v, w\}$ is not chain-connected.

Now let μ be a type 1 total colouring of a spined graph G using the colours $c_1, ..., c_{\Delta(G)+1}$. If c_i and c_j are two colours, then an (i, j) *Kempe chain* is a maximal chain-connected set of parts coloured c_i or c_j. Thus, any Kempe chain is either:

(i) the edge set of a cycle, or

(ii) the edge set of a path, together with either a vertex or a spine at each end of the path.

We describe these chains as *Kempe cycles* and *Kempe paths*, respectively. The *length* of such a chain is the number of edges that it contains. The *internal vertices* of a Kempe path are the vertices (if any) incident with two edges of the path. If X is such a chain, we denote by $\hat{X}$ the union of X with the incident vesps.

A Kempe path is *short* if it contains no internal vertices; that is, if it has length 0 or 1. There are four kinds of short Kempe paths: a single vertex with an attached spine; an edge with an incident vertex at each end; an edge with an incident vertex at one end and an incident spine at the other end; and edge with an incident spine at each end. (A Kempe path is defined by two colours, so a single vertex cannot be such a path.)

We now show that any type 1 graph G that is cubic or semi-cubic must obey three simple counting conditions, and conjecture that these conditions, together with factor-conformability, are sufficient to ensure that G is type 1. It is convenient to begin with cubic graphs. (In the statement of this and succeeding theorems and conjectures, *component* means 'connected component with respect to the usual definition of connectivity', and *odd component* means 'component with an odd number of vertices'.)

Theorem 4.2 *Let G be a cubic graph. Then a necessary condition for G to be type 1 is that G should have a factor-conformable vertex colouring such that, for every pair i, j from* $\{1, \ldots, 4\}$, *the subgraph* G_{ij} *has:*

(i) at least $(|S_i| + |S_j|)/2$ *components;*

(ii) at most $(|S_i| + |S_j|)/2$ *components that have fewer than four vertices;*

(iii) an even number $2q$ *of odd components, where*
$2q \leq \min\{|S_i|, |S_j|\}$.

Proof. Note that G has an even number of vertices. Suppose that G is type 1 and let μ be a total 4-colouring of G; then the restriction of μ to $V(G)$ is a factor-conformable vertex colouring.

Let $\{i, j\}$ be a pair from $\{1, \ldots, 4\}$, and let $\{k, l\} = \{1, \ldots, 4\} \setminus \{i, j\}$. The (i, j) Kempe chains partition the set of all edges coloured c_i or c_j; if X is such a chain, then:

if X is a cycle, all the vertices of $\hat{X}$ have colour c_k or c_l;

if X is a path, all the internal vertices of $\hat{X}$ have colour c_k or c_l and at each end there is a vertex of colour c_i or c_j. (Since G is cubic, there are no spines or short paths.)

Since each vertex coloured c_k or c_l is incident with an edge of each colour c_i, c_j, then each such vertex belongs to a unique such $\hat{X}$. Thus, for every component Z of G_{ij}, $V(Z)$ is the set of vertices of an (i, j) Kempe cycle or the set of *internal* vertices of an (i, j) Kempe path. (The vertices at the end of such a path do not belong to G_{ij} at all.) Moreover, each vertex of colour c_i or c_j is at the *end* of a unique (i, j) Kempe path (and does *not* belong to $V(Z)$ where Z is the corresponding component of G_{ij}). This proves part (i); indeed, the number of components that are paths is exactly $(|S_i| + |S_j|)/2$.

Since every component of G_{ij} that is a cycle is an even cycle (and hence of order at least 4), part (ii) is also proved.

Where a component Z of G_{ij} is a path with both end vertices the same colour, c_i or c_j, it is of even order. Thus each odd component has exactly one vertex of each colour c_i, c_j; so there must be at most $\min\{|S_i|, |S_j|\}$ of these. Now, since $|V(G)|$ is even and μ is factor-conforming, $|S_i|$ and $|S_j|$ must be even, and hence the number of odd components of G_{ij} must be even. This proves part (iii). ■

Remark This result gives insight into why the graph HHH12 is type 2. Up to isomorphism this graph has just the three distinct factor-conformable vertex colourings given above. Although in each colouring, for all choices of c_i and c_j we have an even number of odd components, it is also the case that, for each colouring, there are some choices of c_i and c_j such that G_{ij} has either (i) fewer than $(|S_i| + |S_j|)/2$ components, or (ii) more than $(|S_i| + |S_j|)/2$ components that have fewer than four vertices.

Conjecture 4.3 *Let G be a cubic graph. Then the necessary condition for G to be type* 1, *stated in Theorem* 4.2, *is also sufficient.*

A proper vertex k-colouring of a graph G is *acyclic* if for each pair of colours, the subgraph induced by that pair of colour classes has no cycles [3]. The following conjecture (which is implied by Conjecture 4.3) may perhaps be easier to prove.

Conjecture 4.4 *Let G be a cubic graph. If G has an acyclic factor-conformable vertex* 4*-colouring such that, for every pair i, j from* 1, …, 4, *the subgraph* G_{ij} *has:*

(*i*) *exactly* $(|S_i| + |S_j|)/2$ *components;*

(*ii*) *an even number* $2q$, *of odd components, where*
$2q \leq \min\{|S_i|, |S_j|\}$;

then G is a type 1 *graph.*

In order to extend the above results to semi-cubic graphs, we require to account for short paths.

Let G be a type 1, semi-cubic graph with a vesp colouring σ using the colours $c_1, \ldots, c_4$. For each pair $\{i, j\}$ of distinct elements of $\{1, \ldots, 4\}$ we denote by $\zeta(\{i, j\})$ the number of short (i, j) paths with respect to σ. These may be enumerated as follows. For each spine s of G, let v_s be the incident vertex and define a function ζ_s taking values on the pairs $\{i, j\}$ of distinct elements of $\{1, \ldots, 4\}$, as follows:

$\zeta_s(\{i, j\}) = 1$ if there is a short (i, j) path involving s and no other spine (that is, with one end at s and the other end at v_s or a vertex adjacent to v_s);

$\zeta_s(\{i, j\}) = ½$ if there is a short (i, j) path involving the spine s and another spine (which will be incident with one of the vertices adjacent to v_s);

$\zeta_s(\{i, j\}) = 0$ otherwise.

Clearly, $\zeta(\{i, j\}) = \sum_{s \text{ a spine of } G} \zeta_s(\{i, j\})$ for each pair $\{i, j\}$, since each short path is summed by a 1 or two ½s.

Theorem 4.5 *Let G be a semi-cubic graph. Then a necessary condition for G to be type 1 is that G should have a factor-conformable vesp colouring such that, for every pair i, j from* 1, ..., 4, *the subgraph* G_{ij} *has:*

(i) at least $(|S_i + |S_j|)/2 - \zeta(\{i, j\})$ *components;*

(ii) at most $(|S_i| + |S_j|)/2 - \zeta(\{i, j\})$ *components that have fewer than four vertices;*

(iii) at most $\min\{|S_i|, |S_j|\} - \zeta(\{i, j\})$ *odd components, the parity being that of* $\zeta(\{i, j\})$.

Proof. Suppose that G is type 1 and let μ be a total 4-colouring of G; then the restriction of μ to the vesps is factor-conformable.

Let $\{i, j\}$ be a pair from $\{1, ..., 4\}$. As in the proof of Theorem 4.2, the (i, j) Kempe chains partition the edges coloured c_i or c_j and there are exactly $(|S_i| + |S_j|)/2$ such chains that are paths; moreover, any components of G_{ij} arise from such paths. However, $\zeta(\{i, j\})$ of these paths are short and do not therefore contribute components of G_{ij}. Thus $\zeta(\{i, j\})$ must be subtracted from each of the expressions given in the statement of Theorem 4.2. The result follows. ■

We now make the equivalent conjectures as in the case of regular graphs.

Conjecture 4.6 *Let G be a semi-cubic graph. Then the necessary condition for G to be type 1, stated in Theorem* 4.5, *is also sufficient.*

Conjecture 4.7 *Let G be a semi-cubic graph. If G has an acyclic factor-conformable vesp colouring such that, for each pair* S_i, S_j *of colour sets, the subgraph* G_{ij}:

(*i*) *has exactly* $(n_i + n_j)/2 - \zeta(\{i, j\})$ *components;*

(*ii*) *has at most* $\min\{|S_i|, |S_j|\} - \zeta(\{i, j\})$ *odd components, the parity being that of* $\zeta(\{i, j\})$;

then G is a type 1 *graph.*

Note

In [5], some consideration was given to a somewhat stronger condition than conformability, namely the existence of a proper vertex-colouring of G such that

$$\operatorname{def}(G) \geq \sum_{i=1}^{\Delta+1} \xi_i^+$$

where, for $i = 1, \ldots, \Delta + 1$, ξ_i is the number of vertices of G whose neighbourhoods consist entirely of vertices with colour c_i, and ξ_i^+ is defined as follows. Let V_i be the set of vertices of G coloured c_i; then ξ_i^+ is equal to ξ_i or $\xi_i + 1$ according to the parity of $\xi_i + |V_i| - |V(G)|$.

(The significance of this condition is that, if G is conformable, the minimum degree of G is at least 2 and $\Delta(G) \geq |V(G)|/2 + 4$, then G has such a vertex colouring if and only if G is not a complete graph of odd order with one edge subdivided; see [5], Theorem 2.5.)

Clearly this condition implies conformability. We now show that factor-conformability in turn implies the condition.

Suppose σ is a factor-conformable vesp colouring of G. Then for each $i = 1, \ldots, \Delta + 1$, let $S_i(\sigma) = T_i \cup V_i$ where T_i is the set of vertices having c_i as a spine colour and V_i is as above. Clearly $|T_i| \geq \xi_i$ and $|T_i| + |V_i| \equiv |V(G)| \pmod 2$. Thus there are at least ξ_i^+ spines coloured c_i $(i = 1, \ldots, \Delta + 1)$, and so $\operatorname{def}(G) \geq \sum_{i=1}^{\Delta+1} \xi_i^+$.

The authors are grateful to the referee for drawing their attention to this feature of factor-conformability, and also for comments that have very considerably improved the presentation of this paper.

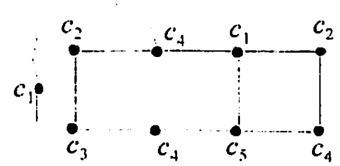
Appendix: fifty critical graphs

In [4] and [5] there is given a Catalogue of Critical Graphs, containing the fifty graphs on at most ten vertices that are critical with respect to total colouring. Both of these references remark that the seventeen graphs numbered 1, 3, 7, 10, 14, 16, 27, 28, 29, 30, 31, 44, 45, 46, 48, 49 and 50 are non-conformable. In fact a further two, namely those numbered 4 and 25, are also non-conformable.

The graph numbered 4 is the 5-cycle graph, which has deficiency 0 and is easily seen to be non-conformable as no colour class can contain more than two vertices.

The graph numbered 25 also has deficiency 0; it has nine vertices, is regular of degree 4 and contains two disjoint copies of K_4. It follows easily that any proper vertex colouring using at most five colours has at least two colour classes containing two vertices, and so this graph is not conformable.

The remaining thirty-one graphs are conformable. The first-named author's PhD thesis [7] gives conformable colourings of all of these except the graph numbered 21; a conformable colouring of this graph (which has deficiency 2) is shown below.



By Lemma 3.2, the nineteen non-conformable graphs are not factor-conformable. Of the thirty conformable graphs other than that numbered 12, it is straightforward to check that the vertex colouring above, and those given in [4] (augmented by spine colours where necessary) are not factor-conformable. In order to verify that these graphs are not factor-conformable, it necessary to carry out this check for every isomorphism class of proper vertex colourings; this was done by hand by the first-named author as unpublished work towards her doctoral thesis.

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