OC-irredundance, CO-irredundance and maximum degree in trees

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Abstract

A vertex subset X of a simple graph is called OC-irredundant (respectively CO-irredundant) if for each $v \in X$, $N(v) - N[X - \{v\}] \neq \emptyset$ (respectively $N[v] - N(X - \{v\}) \neq \emptyset$). Sharp bounds involving order and maximum degree for the minimum cardinality of a maximal OC-irredundant set and a maximal CO-irredundant set of a tree are obtained and extremal trees are exhibited.

Keywords: OC-irredundance; CO-irredundance, OO-irredundant
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1 Introduction

We first define four types of vertex subsets of a simple graph $G = (V, E)$. The set $X \subseteq V$ is

$$\left.\begin{array}{l}\text{CC-}\textit{irredundant}\\ \text{OC-}\textit{irredundant}\\ \text{CO-}\textit{irredundant}\\ \text{OO-}\textit{irredundant}\end{array}\right\} \text{ iff for each } v \in X, \left\{\begin{array}{l} N[v] - N[X - \{v\}] \\ N(v) - N[X - \{v\}] \\ N[v] - N(X - \{v\} \\ N(v) - N(X - \{v\}) \end{array}\right. \neq \emptyset.$$

For example the prefix CO is used in the name CO-irredundant set because the first neighbourhood used in its definition is <u>C</u>losed and the second is <u>O</u>pen.

These sets may be characterised in terms of existence of private neighbours which we now define. For $v \in X$, the vertex t is an X-*external private neighbour* (X-*epn*) of v if

$$t \in V - X \quad \text{and} \quad N(t) \cap X = \{v\};$$

an X-*internal private neighbour* (X-*ipn*) of v if

$$t \in X \quad \text{and} \quad N(t) \cap X = \{v\};$$

an X-*self private neighbour* (X-*spn*) of v if

$$t = v \quad \text{and} \quad v \text{ is isolated in } G[X].$$

The following four characterisations are easily proved:

$$X \text{ is } \left.\begin{matrix} \text{CC} \\ \text{OC} \\ \text{CO} \\ \text{OO} \end{matrix}\right\} \text{-irredundant iff each } v \in X \text{ has an } \left\{\begin{matrix} X\text{-spn or } X\text{-epn.} \\ X\text{-epn.} \\ X\text{-epn, } X\text{-ipn or } X\text{-spn.} \\ X\text{-epn or } X\text{-ipn.} \end{matrix}\right.$$

The property of CC-irredundance (usually the CC prefix is omitted) has been well studied due partially to its intimate connection with domination. For a survey of this theory, the reader is directed to [8].

The other three types of sets defined above were studied in [5]. We note that OC-irredundance was originally introduced in [4] usingr the term "open irredundance." The reader is referred to [1, 6] for results concerning these sets, further generalisations and extensive bibliography.

The present work is concerned with lower bounds involving order n and maximum degree Δ for the smallest cardinality of maximal sets with these properties. Let

$$\left.\begin{matrix} \text{ir}(G) \\ \text{oir}(G) \\ \text{coir}(G) \\ \text{ooir}(G) \end{matrix}\right\} \text{ denote the smallest cardinality of a maximal } \left\{\begin{matrix} \text{CC-} \\ \text{OC-} \\ \text{CO-} \\ \text{OO-} \end{matrix}\right.$$

irredundant set. The following sharp bounds are known for general graphs.

Theorem 1 [3] *For* $\Delta \geq 2$, $\text{ir}(G) \geq \dfrac{2n}{3\Delta}$.

A more elegant proof of Theorem 1 is given in [7].

Theorem 2 [6] *If G has no isolated vertex, then*

$$\text{oir}(G) \geq \begin{cases} \frac{n}{3} & \Delta = 2 \\ \frac{2n}{11} & \Delta = 3 \\ \frac{n}{8} & \Delta = 4 \\ \dfrac{(3\Delta - 1)n}{2\Delta^3 - 5\Delta^2 + 8\Delta - 1} & \Delta \geq 5. \end{cases}$$

Theorem 3 [6]

$$\operatorname{coir}(G) \geq \begin{cases} \frac{n}{2} & \Delta = 2 \\ \frac{4n}{13} & \Delta = 3 \\ \dfrac{2n}{3\Delta - 3} & \Delta \geq 4. \end{cases}$$

The bound of Theorem 1 has been improved for trees with $\Delta \geq 3$.

Theorem 4 [2] *If a tree T has $\Delta \geq 3$, then*

$$\operatorname{ir}(T) \geq \frac{2(n+1)}{2\Delta + 3}.$$

In Section 2 (respectively Section 3) we prove sharp bounds for $\operatorname{oir}(T)$ (respectively $\operatorname{coir}(T)$) thus improving Theorem 2 (respectively Theorem 3) for the class of trees.

Each of the proofs will involve the basic partition (X, B, C, R) of V induced by the vertex subset X, where

$$\begin{aligned} B &= \{w \in V - X : |N(w) \cap X| = 1\} \\ C &= \{w \in V - X : |N(w) \cap X| \geq 2\} \\ R &= \{w \in V - X : |N(w) \cap X| = 0\}. \end{aligned}$$

In each proof cardinalities of sets denoted by upper case letters will be denoted by the corresponding lowercase letter. The only exception to this notation will be that (as usual) $|V|$ is denoted by n.

2 OC-irredundance and maximum degree

Recall that X is an OC-irredundant set of an arbitrary graph if every $v \in X$ has an X-epn and observe that the set of X-epns of $v \in X$ is precisely $N(v) \cap B$. The following condition for maximality of an OC-irredundant set X (involving the basic partition) was proved in [6].

Proposition 5 [6] *The OC-irredundant set X is maximal if and only if for each $w \in N(R)$*

$$\textit{there exists } v \in X \textit{ such that } N(v) \cap B \subseteq N[w]. \tag{1}$$

We now confine our attention to trees and state our first principal result.

Theorem 6 *For an n-vertex tree T with maximum degree* Δ,

$$\operatorname{oir}(T) \geq \begin{cases} \frac{n+1}{5} & \Delta = 3 \\ \dfrac{n+\Delta-3}{2\Delta-2} & \Delta \geq 4. \end{cases}$$

Proof. Suppose that a tree T with n vertices and maximum degree Δ has a maximal OC-irredundant set X. We refine the basic partition of V induced by X as follows. Let

$$\begin{aligned} X_1 &= \{u \in X : u \text{ has precisely one } X\text{-epn}\} \\ X_2 &= X - X_1 \\ B_i &= B \cap N(X_i) \quad (i = 1, 2) \\ C_1 &= C \cap N(R) \\ Y &= B_2 \cap N(R). \end{aligned}$$

Observe that

$$\begin{aligned} b &= b_1 + b_2 \\ &\leq x_1 + \Delta x_2 \\ &= x + (\Delta - 1)\, x_2. \end{aligned} \tag{2}$$

Since T has no cycles, if v and w satisfy (1), then $v \in X_1$. Note that $C_1 \cup Y \subseteq N(R)$. Hence by Proposition 5 each vertex of $C_1 \cup Y$ is adjacent to a vertex of B_1. The presence of these edges enables us to estimate r.

Each vertex of Y sends an edge to both X and B_1, hence

$$|N(Y) \cap R| \leq (\Delta - 2)\, y. \tag{3}$$

Each vertex of C_1 sends an edge to B_1 and at least two edges to X, hence

$$|N(C_1) \cap R| \leq (\Delta - 3)\, c_1. \tag{4}$$

The vertices of B_1 send $b_1 = x_1$ edges to X_1 and at least $c_1 + y$ edges to $C_1 \cup Y$, hence

$$\begin{aligned} |N(B_1) \cap R| &\leq \Delta x_1 - x_1 - c_1 - y \\ &= (\Delta - 1)(x - x_2) - c_1 - y. \end{aligned} \tag{5}$$

Now r is bounded above by the sum of the right hand sides of the inequalities (3), (4) and (5). We use these and (2) to obtain

$$\begin{aligned} n &= x + b + r + c \\ &\leq x + (x + (\Delta - 1)\, x_2) \\ &\qquad + ((\Delta - 3)\, y + (\Delta - 1)(x - x_2) + (\Delta - 4)\, c_1) + c \\ &= (\Delta + 1)\, x + (\Delta - 3)\, y + (\Delta - 4)\, c_1 + c. \end{aligned} \tag{6}$$

We now estimate y by using the fact that $H = T[X \cup B \cup C]$ is a forest. In H there are at least $c_1 + y$ edges from $C_1 \cup Y$ to B_1, at least $2c$ edges from X to C and precisely $x_1 + b_2$ edges from X to B. Hence

$$2c + c_1 + y + x_1 + b_2 \leq x + (x_1 + b_2) + c - 1$$

i.e.

$$y \leq x - c_1 - c - 1. \tag{7}$$

There are now two cases to consider.

Case 1 $\Delta \geq 4$:
Elimination of y from (6) and (7) gives:

$$\begin{aligned} n &\leq (\Delta + 1)\,x + (\Delta - 3)\,(x - c_1 - c - 1) + (\Delta - 4)\,c_1 + c \\ &= (2\Delta - 2)\,x - (\Delta - 4)\,c - c_1 - (\Delta - 3) \qquad (8) \\ &\leq (2\Delta - 2)\,x - (\Delta - 3)\,. \end{aligned}$$

Hence $x \geq \dfrac{n + \Delta - 3}{2\Delta - 2}$ as required.

Case 2 $\Delta = 3$.
In this case $c_1 = 0$ (since a vertex of C_1 sends two edges to X and one edge to each of R and B_1). Therefore (6) and (7) become

$$n \leq 4x + c$$

and

$$y \leq x - c - 1.$$

Elimination of c from these two inequalities gives

$$n \leq 5x - y - 1 \leq 5x - 1. \tag{9}$$

Therefore $x \geq \dfrac{n+1}{5}$ as required. ■

We show that the bounds of Theorem 6 are sharp. If T is an extremal tree and X is a maximal OC-irredundant set of cardinality oir (T), then we have equality in all of the inequalities used in the derivation of the bounds.

Equality in (2) shows that each $v \in X_2$ has Δ neighbours in B and so there are no edges from X_2 to C. Hence the forest $T[X_1 \cup C]$ has at least $2c$ edges and we deduce $2c \leq c + x_1 - 1$ i.e.

$$c \leq x_1 - 1. \tag{10}$$

First assume that $\Delta = 3$. Then $c_1 = 0$ and equality in (9) shows that $y = 0$. From (10) and equality in (7) it follows that $x_1 - 1 \geq x_1 + x_2 - 1$, so that $x_2 = 0$. Equality in (5) shows that each $w \in B_1$ joins two vertices in R.

The maximum value of $c = x_1 - 1$ (from (10)) is attained when $T[C \cup X_1]$ is a path. Hence there is precisely one extremal tree T for each value of $\operatorname{oir}(T)$. The extremal tree with $\operatorname{oir}(T) = 4$ is depicted in Figure 1.

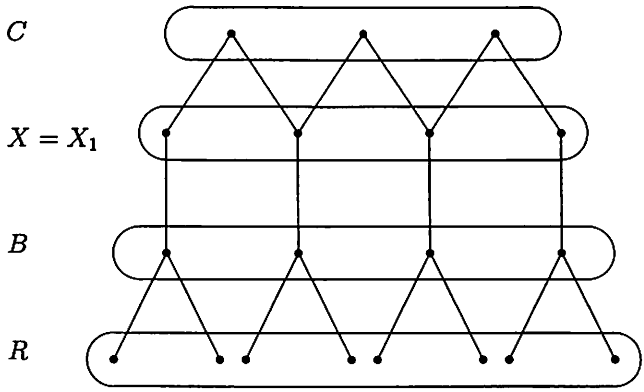

Figure 1 Extremal Tree of Theorem 6 with $\Delta = 3$.

Next, let $\Delta \geq 4$. Figure 2 depicts an extremal tree for $\Delta = 4$ where $c \neq 0$.

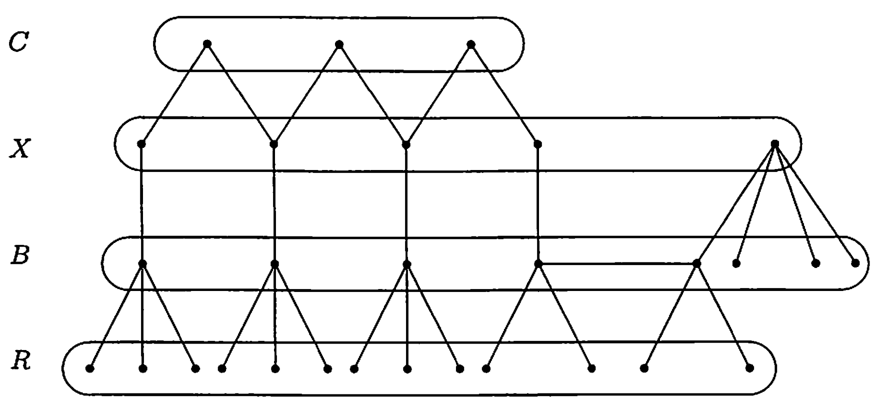

Figure 2 An extremal tree for Theorem 6 with $\Delta = 4$ and $c \neq 0$.

Equality in (8) shows that $c = 0$ for $\Delta \geq 5$ and it will be seen that there are also extremal trees for $\Delta = 4$ with $c = 0$. Using this and (7) we deduce that such a tree has $y = x - 1$. For given Δ there is flexibility in the choice of x_1 and x_2. We now present a family of extremal trees, one for each Δ.

Fix $\Delta \geq 4$, let $x = \Delta + 1$ and $x_2 = 1$. Then $b_1 = b_2 = y = \Delta$. Let $T[B]$ be a matching from B_1 to B_2. Form R by joining $\Delta - 2$ leaves to each vertex of B so that $r = 2\Delta(\Delta - 2)$. It is easily seen that X is maximal OC-irredundant in T and that T is an extremal tree for Theorem 6.

In addition we note that the stars $K_{1,\Delta}$ $(\Delta \geq 4)$ are also extremal trees for Theorem 6 with $\text{oir}(T) = 1$.

3 CO-irredundance and maximum degree

Recall that $X \subseteq V$ is a CO-irredundant set of an arbitrary graph if and only if for each $v \in X$,

$$PN(v, X) = \{t : t \text{ is an } X\text{-spn, } X\text{-epn or } X\text{-ipn of } v\} \neq \emptyset.$$

Define $Z = \{v \in X | v \text{ is isolated in } G[X]\}$. Note that Z is precisely the set of X-spns.

The following maximality condition was proved in [6].

Proposition 7 [6] *In any graph the CO-irredundant set X is maximal if and only if for each $w \in N(Z) \cup N[R]$*

$$\textit{there exists } v \in X \textit{ such that } PN(v, X) \subseteq N(w). \tag{11}$$

When (11) is satisfied we say that w *annihilates* v. We now state and prove the second principal result.

Theorem 8 *For an n-vertex tree T with maximum degree $\Delta \geq 3$,*

$$\text{coir}(T) \geq \begin{cases} \frac{n}{\Delta} & (\text{coir}(T) \ \textit{even}) \\ \frac{n+1}{\Delta} & (\text{coir}(T) \ \textit{odd}). \end{cases}$$

Proof. Suppose that X is a maximal CO-irredundant set of the tree T. We define a variety of sets from the basic partition induced by X. First let

$$X_1 = \{v \in X : v \text{ has precisely one } X\text{-epn, no } X\text{-ipn and no } X\text{-spn}\}.$$

Denote by v' the unique X-epn of $v \in X_1$, define $S = N(X_1) \cap B$ and partition X_1 into P_1, P_2, P_3 where

$$\begin{aligned} P_1 &= \{v \in X_1 : v' \in N(R) \text{ and } v' \text{ is isolated in } T[S]\}, \\ P_2 &= \{v \in X_1 : v' \in N(R) \text{ and } v' \text{ has degree } \geq 1 \text{ in } T[S]\} \\ \text{and } \ P_3 &= \{v \in X_1 : v' \notin N(R)\}. \end{aligned}$$

Let

$$B_i = N(P_i) \cap B, i = 1, 2, 3$$
$$\text{and } B^* = N(Z) \cap B.$$

Next we deduce some properties from the maximality of X and Proposition 7. Since there are no cycles,

$$\text{if (11) is satisfied, then } |PN(v, X)| = 1. \tag{12}$$

Let $w \in R$. By Proposition 7, w annihilates some $v \in X$. No X-ipn or X-spn is in $N(w)$. Hence by (12),

$$v \in P_1 \cup P_2 \text{ and } w \text{ is adjacent to } v' \in B_1 \cup B_2. \tag{13}$$

Let $w \in B^*$. By Proposition 7, w annihilates some $v \in X$. If $v \in Z$, then the X-spn v is in $N(w)$, therefore w is an X-epn of v. But $w \notin N(w)$ and hence (11) is not satisfied. We conclude that $v \notin Z$. Moreover no X-ipn is in $N(w)$. We deduce (by (12)) that

$$v \in X_1 \text{ and } w \text{ is adjacent to } v' \in B_1 \cup B_2 \cup B_3. \tag{14}$$

Next consider $v' \in B_1$ where v' is the unique X-epn of $v \in P_1$. By Proposition 7, v' annihilates some $u \in X$. Now v' does not annihilate v because $v' \notin N(v')$. Further v' is not adjacent to the X-epn of any other vertex of X_1. Hence $u \notin X_1$. Further v' is not adjacent to X-spns and so $u \notin Z$. Using (12) we conclude that u has the unique X-ipn v and no X-epn. If u had degree one in $T[X]$, then u would be an X-ipn of v, a contradiction with $v \in X_1$. Hence u has degree at least two in $T[X]$. Further no two vertices of B_1 can annihilate the same vertex u (since u has only one X-ipn). Define

$$X_2 = \{u \in X : u \text{ is annihilated by some } v' \in B_1\}.$$

Then by the above

$$\left.\begin{array}{l}\text{each } u \in X_2 \text{ has a unique } X\text{-ipn in } P_1, \text{ has no } X\text{-epn} \\ \text{and has degree at least two in } G[X]. \text{ Also } x_2 = p_1.\end{array}\right\} \tag{15}$$

We now define additional sets. Let

$$X_3 = \{v \in X - (X_1 \cup X_2) : v \text{ has degree at least two in } T[X]\},$$
$$X_4 = \{v \in X - (X_1 \cup X_2) : v \text{ has degree one in } T[X]\} \cap N(C)$$
$$\text{and } X_5 = \{v \in X - (X_1 \cup X_2) : v \text{ has degree one in } T[X]\} - N(C).$$

Observe that Z, X_1, X_2, X_3, X_4, X_5 is a partition of X.

We now estimate r. Each $w \in B_1 \cup B_2$ joins a vertex of X and each $w \in B_2$ joins a vertex of S, hence by (13),

$$\begin{aligned} r &\leq |N(B_1) \cap R| + |N(B_2) \cap R| \\ &\leq (\Delta - 1) b_1 + (\Delta - 2) b_2 \\ &= (\Delta - 1) p_1 + (\Delta - 2) p_2. \end{aligned} \tag{16}$$

To estimate b, observe that vertices of X_1, X_2 have one X-epn, no X-epn respectively. Vertices of X_3, X_4, X_5 have at most $\Delta - 2$, $\Delta - 2$ and $\Delta - 1$ X-epns respectively. Therefore

$$b \leq x_1 + (\Delta - 2) x_3 + (\Delta - 2) x_4 + (\Delta - 1) x_5 + b^*. \tag{17}$$

In order to estimate c, we use the tree property. Let $Y_1, \ldots, Y_k$ be the vertex sets of the components of $T[X_1]$. If $Y_i = \{v\}$, then v is adjacent to $u \in X - X_1$ and since v has no X-ipn, we conclude that $u \in X_2 \cup X_3$. Otherwise suppose that v is an end vertex of $T[Y_i]$ and $w \in Y_i$ is adjacent to v. Since w has no X-ipn, v is also adjacent to $u \in X_2 \cup X_3$. We conclude

$$\text{there is an edge from each } Y_i \text{ to } X_2 \cup X_3. \tag{18}$$

For $i = 1, \ldots, k$ let T_i be the subtree $T[Y_i \cup (N(Y_i) \cap B)]$. Consider the family $\mathcal{F}$ of subtrees which contains $T_1, \ldots, T_k$ and the single vertices of $C \cup Z \cup B^* \cup X_2 \cup X_3 \cup X_4$. Joining these subtrees we have k edges by (18), $2c$ edges by definition of C, b^* edges from B^* to Z, b^* edges from B^* to the T_i's by (14). These edges number less than the number of subtrees of $\mathcal{F}$ (except when this number is zero), hence

$$\begin{aligned} &2c + 2b^* + k \leq c + z + b^* + x_2 + x_3 + x_4 + k - 1 \\ \text{i.e.} \quad &b^* + c \leq z + x_2 + x_3 + x_4 - 1. \end{aligned} \tag{19}$$

From (17) and (19)

$$b + c \leq x_1 + x_2 + (\Delta - 1)(x_3 + x_4 + x_5) + z - 1. \tag{20}$$

Now $n = (x + r) + (b + c)$. Using $x = \sum_{i=1}^{5} x_i + z$, (16), (20) and a little arithmetic we obtain

$$n \leq [2x_1 + (\Delta - 2)(p_1 + p_2)] + [p_1 + 2x_2] + \Delta(x_3 + x_4 + x_5) + 2z - 1. \tag{21}$$

Now $p_1 + p_2 \leq x_1$, so that the first square bracket in (21) is at most Δx_1 and by (15) the second square bracket is equal to $3x_2$. Hence

$$n \leq \Delta(x_1 + x_3 + x_4 + x_5) + 3x_2 + 2z - 1.$$

Since $\Delta \geq 3$, we deduce that provided $\mathcal{F}$ has at least one subtree,

$$n \leq \Delta x - 1. \tag{22}$$

If $\mathcal{F}$ is empty, then (by (19)) $c = z = b^* = x_1 = x_2 = x_3 = x_4 = 0$. Therefore $x = x_5$. Each $v \in X_5$ has at most $\Delta - 1$ X-epns and hence

$$n \leq \Delta x. \tag{23}$$

Note that $X = X_5$ is impossible with x odd. This completes the proof. ■

If the bound (23) is attained (see above discussion), then $x = x_5$ is even, $T[X] = \frac{x}{2}K_2$ and each $v \in X$ has precisely $\Delta - 1$ neighbours in B. Hence T contains $\frac{x}{2}$ disjoint subtrees T', where T' is obtained by joining the central vertices of two copies of the star $K_{1,\Delta-1}$. Then T is completed by adding $\frac{x}{2} - 1$ edges incident with vertices of B, in any manner which avoids cycles and maintains maximum degree Δ. An example with $\Delta = 4$ is shown in Figure 3.

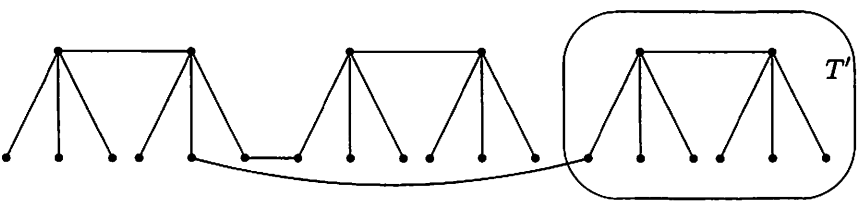

Figure 3 An extremal tree of Theorem 8 with even coir (T).

We finally show that the bound (22) may be attained for any odd x (≥ 3) and any Δ (≥ 3). Let T'' be formed from P_3 by adding $\Delta - 1$ leaves to the end vertices and $\Delta - 2$ leaves to the central vertex. Form T from $T'' \cup \left(\frac{x-3}{2}\right) T'$ by adding $\frac{x-3}{2}$ edges incident with the leaves in any manner which avoids cycles and maintains maximum degree Δ. Then T attains the bound (22).

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