USING SIGNED PERMUTATIONS TO REPRESENT WEYL GROUP ELEMENTS OF A SYMMETRIC SPACE

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ABSTRACT. Most computer algebra packages for Weyl groups use generators and relations and the Weyl group elements are expressed as reduced words in the generators. This representation is not unique and leads to computational problems. In [HHR06], the authors introduce the representation of Weyl group elements uniquely as signed permutations. This representation is useful for the study of symmetric spaces and their representations.

A computer algebra package enabling one to do computations related to symmetric spaces would be an important tool for researchers in many areas of mathematics, including representation theory, Harish Chandra modules, singularity theory, differential and algebraic geometry, mathematical physics, character sheaves, Lie theory, etc. In this paper, we use the representation of Weyl group elements as signed permutations to improve the algorithms of [DH05]. These algorithms compute the fine structure of symmetric spaces and nice bases for local symmetric spaces.

1. INTRODUCTION

Until recently very few algorithms existed for computations in symmetric spaces. The first algorithms for computations related to symmetric spaces were developed a few years ago (see [H96, H00, DH05, DH07, GH06]). The representation of Weyl group elements as signed permutations can be used to speed of the implementation of these algorithms.

A Weyl group is a reflection group of a root system Φ. Given Φ in a Euclidean vector space, for each vector α in Φ, define s_α as the reflection through α. The Weyl group, $W(\Phi)$, is the group generated by all the reflections s_α with $\alpha \in \Phi$. Weyl group elements are usually given in terms of generators and relations. This representation is not unique and therefore leads to computational problems. Given two Weyl group elements, $w_1, w_2 \in W(\Phi)$, one usually has to compute the product $w_1 w_2^{-1}$ to determine if w_1 and w_2 represent the same element. In [HHR06], the authors introduce the representation of Weyl group elements as *signed permutations*.

The classical Weyl groups are those that correspond to Dynkin diagrams of type A, B, C, and D. The roots of a classical Weyl group are sums or differences of at most two standard basis vectors, e_i. An element w in $W(\Phi)$ can be described completely by its action on the e_i. In this paper, we use signed permutations to represent Weyl group elements related to symmetric spaces.

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2. Symmetric Spaces, Restricted Weyl Groups, and θ-diagrams

In [DH05] it was shown that the fine structure of the real Riemannian symmetric spaces is the same as that of the symmetric varieties over algebraically closed fields.

2.1. **Symmetric spaces.** Let G denote a reductive algebraic group over an algebraically closed field, $\theta \in \mathrm{Aut}(G)$ an involution, K the fixed point group of θ, and $P = \{A\theta(A)^{-1} \mid A \in G\}$. The variety P is called a *symmetric variety* or also a *reductive symmetric space.* If G is semisimple, P is also called a *semisimple symmetric space.* Let $\mathfrak{g}$ denote the Lie algebra of G and denote the involution of $\mathfrak{g}$ induced by θ also by θ. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k} = \{x \in \mathfrak{g} \mid \theta(x) = x\}$ is the Lie algebra of K and $\mathfrak{p} = \{x \in \mathfrak{g} \mid \theta(x) = -x\}$ is the tangent space in the identity of P. $\mathfrak{p}$ is called a *local symmetric space.*

2.2. **Root space decomposition.** Let $\mathfrak{t}$ be a maximal toral subalgebra of the Lie algebra $\mathfrak{g}$. For $\alpha \in \mathfrak{t}^*$, let $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [t,x] = \alpha(t)x \text{ for all } t \in \mathfrak{t}\}$ and let $\Phi(\mathfrak{t}) = \{\alpha \in \mathfrak{t}^* \mid \mathfrak{g}_\alpha \neq 0\}$. The elements of $\Phi(\mathfrak{t})$ are called roots and the subspaces $\mathfrak{g}_\alpha$ are called root-subspaces. Then

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha\in\Phi(\mathfrak{t})} \mathfrak{g}_\alpha.$$

2.3. **Root space decomposition for a local Symmetric space.** Let $\mathfrak{a}$ be a maximal toral subalgebra in $\mathfrak{p}$ and consider the root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\lambda\in\Phi(\mathfrak{a})} \mathfrak{g}_\lambda$$

Here $\mathfrak{g}_\lambda = \{x \in \mathfrak{g} \mid [t,x] = \lambda(t)x \text{ for all } t \in \mathfrak{a}\}$ and $\Phi(\mathfrak{a}) = \{\lambda \in \mathfrak{a}^* \mid \lambda \neq 0 \text{ and } \mathfrak{g}_\lambda \neq 0\}$.

2.3.1. *Root systems and Weyl groups.* $\theta|\mathfrak{t}$ induces an involution on $\mathfrak{t}^*$ and hence on $\Phi(\mathfrak{t})$. By abuse of notation, we will denote the restricted involution also by θ. The Weyl group of a root system $\Phi(\mathfrak{t})$ will be denoted by $W(\Phi(\mathfrak{t}))$.

Notation 1. *Let $\Delta(\mathfrak{t})$ denote a basis for $\Phi(\mathfrak{t})$.*

Definition 1. *For a subset $S \subset \Phi(\mathfrak{t})$, we let $W(S)$ denote the subgroup of $W(\Phi(\mathfrak{t}))$ generated by the reflections s_α with $\alpha \in S$.*

Let $X_0(\theta) = \{\chi \in \text{ the root lattice of } \Phi(\mathfrak{t}) \mid \theta(\chi) = \chi\}$, $\Phi_0(\theta) = \Phi(\mathfrak{t}) \cap X_0(\theta)$, and $\Delta_0(\theta) = \Delta(\mathfrak{t}) \cap X_0(\theta)$. Identify $W_0(\theta)$ with the subgroup $W(\Phi_0(\theta))$ of $W(\Phi(\mathfrak{t}))$. Let

$$W_1(\theta) = \{w \in W(\Phi(\mathfrak{t})) \mid w(X_0(\theta)) = X_0(\theta)\}.$$

Let $\overline{\Phi} = \pi(\Phi(\mathfrak{t}) - \Phi_0(\theta))$ denote the set of *restricted roots* of $\Phi(\mathfrak{t})$ relative to θ. Define the projection π by $\pi(\alpha) = \frac{1}{2}(\alpha - \theta(\alpha))$. All $w \in W_1(\theta)$ induce a mapping $\pi(w)$ such that $\pi(w(\chi)) = \pi(w)(\pi(\chi))$. Define $\overline{W} = \{\pi(w) \mid w \in W_1(\theta)\}$.

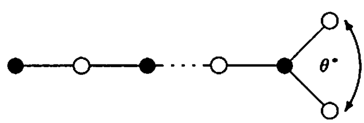
Notation 2. *Let $W(\mathfrak{a})$ denote the Weyl group of the restricted root system $\Phi(\mathfrak{a})$.*

Theorem 1 ([H88]). *Let $\overline{\Phi}$, $\overline{W}$, etc. be as above. Then*

(1) $\overline{\Phi} = \Phi(\mathfrak{a})$.
(2) $W(\mathfrak{a}) = \overline{W} \cong W_1(\theta)/W_0(\theta)$.

Definition 2. $W(\mathfrak{a})$ *is called the* restricted Weyl group *with respect to the action of* θ *on* $R(\mathfrak{t})$.

Let $w_0(\theta) \in W_0(\theta)$ be the involution $w_0(\theta)(\Delta_0(\theta)) = -\Delta_0(\theta)$ and $\theta^* = -\,\mathrm{id} \circ \theta \circ w_0(\theta)$. As in [H88] we make a diagrammatic representation of the action of θ on a basis for $\Phi(\mathfrak{t})$. Color black those vertices of the ordinary Dynkin diagram, which represent roots in $\Delta_0(\theta)$ and indicate the action of θ^* on $\Delta - \Delta_0(\theta)$ by arrows. This diagram is called the θ-diagram. An example in type D_n is:



A classification of involutions, reductive algebraic groups, and their associated Lie algebras using these θ-diagrams is given in [H88]. For the simple Lie algebras of classical type we list in Table 1 below the type of the Lie algebra and involution and its θ-diagram.

Table 1: θ-diagram

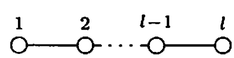
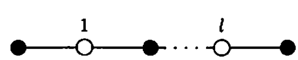
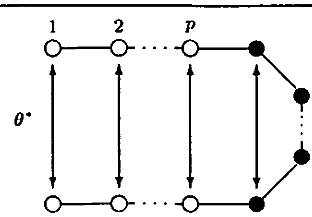
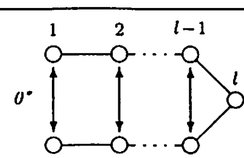
Type θ	θ-diagram
AI	
AII	
$AIII_a$ (AIV ($p = 1$)) ($1 \leq 2p \leq l$)	
$AIII_b$ ($l \geq 2$)	

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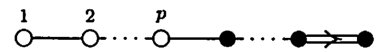
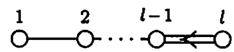
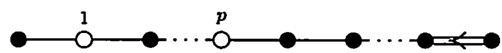
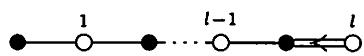
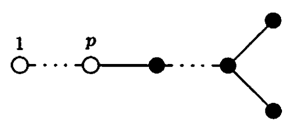
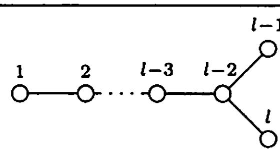
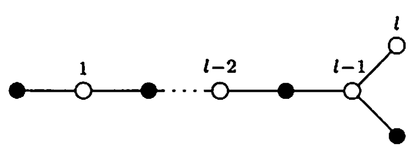
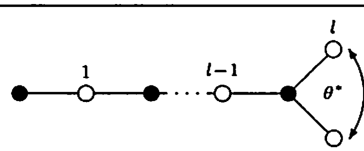
Table 1: *continued*

Type θ	θ-diagram
BI (BII ($p=1$)) ($l \geq 2,\ 1 \leq p \leq l$)	
CI	
CII_a ($l \geq 3$) ($1 \leq p \leq \frac{1}{2}(l-1)$)	
CII_b ($l \geq 2$)	
DI_a (DII ($p=1$)) ($l \geq 4,\ 1 \leq p \leq l-1$)	
DI_b ($l \geq 4$)	
$DIII_a$ ($l \geq 2$)	
$DIII_b$ ($l \geq 2$)	

In [DH05], the authors give an algorithm to compute the fine structure of each of these local symmetric spaces, which is roughly as follows:

(i) For each case, recover the action of the involution on the original root system from the θ-diagram.
(ii) Determine all the positive roots that project down to each root in the base for the restricted root system.

(iii) Find representatives in the original Weyl group for each element in the restricted Weyl group.
(iv) Use the Weyl group representatives to find a complete list of positive roots in the restricted root system.
(v) Determine all the positive roots of $\mathfrak{g}$ that project down to each root in the base for the restricted root system using the restricted Weyl group and the representatives of its elements in the original Weyl group as in (iii).

Using signed permutations in steps (i) and (iii) will speed up the computation of steps (iv) and (v). In (i), θ can be recovered from the diagram by the computation $\theta = -\operatorname{id} \circ \theta^* \circ w_0(\theta)$ where $-\operatorname{id}$, θ^*, and $w_0(\theta)$ commute and $w_0(\theta)$ is represented as a signed permutation. In (iii), each Weyl group representative will be a signed permutation.

3. Weyl Groups and Signed Permutations

From [B81], the classical Lie algebras over a vector space V consist of roots that can be written as a sum or difference of at most two standard basis vectors e_i.

Table 2: Basis for the Classical Root Systems

Lie algebra type	V	Basis
A_n	$\mathbb{R}^{n+1}$	$\alpha_i = e_i - e_{i+1}$
B_n	$\mathbb{R}^n$	$\alpha_i = e_i - e_{i+1}$ for $i = 1, \cdots, n-1$ and $\alpha_n = e_n$.
C_n	$\mathbb{R}^n$	$\alpha_i = e_i - e_{i+1}$ for $i = 1, \cdots, n-1$ and $\alpha_n = 2e_n$.
D_n	$\mathbb{R}^n$	$\alpha_i = e_i - e_{i+1}$ for $i = 1, \cdots, n-1$ and $\alpha_n = e_{n-1} + e_n$.

The standard generators for the Weyl groups correspond to reflections. For type A_n, we have that $\alpha_i = e_i - e_{i+1}$ so the Weyl group element s_{α_i} corresponds to the transposition $(i, i+1)$. Likewise s_{α_i} corresponds to the transposition $(i, i+1)$ for $i = 1, \cdots, n-1$ when the root system is of type B_n or C_n, and s_{α_n} corresponds to the transposition $(n, -n)$. For type D_n we have that s_{α_i} corresponds to the transposition $(i, i+1)$ for $i = 1, \cdots, n-1$ and $\widehat{s}_{\alpha_n}$ corresponds to the product of transpositions $(n, -(n-1))(n-1, -n)$.

An element $w \in W(\Phi)$ can be described entirely by its action on the e_i, i.e., $w(e_i) = \pm e_j$ for all i. As in [HHR06], for $a \in \mathbb{R}^+$, define $\operatorname{sgn}(a) = \begin{cases} + & \text{if } a > 0, \\ - & \text{if } a < 0. \end{cases}$ Represent $w \in W(\Phi)$ by the vector $(a_1, a_2, \cdots, a_n)$ where $w(e_i) = \operatorname{sgn}(a_i) e_{|a_i|}$. This signed permutation corresponds to the first n places in the bottom row of the standard matrix representation of the permutation.

Example 1. *For the Lie algebra D_8, consider $(8, -7)(7, -8)$, a representative for s_{α_8}. Notice that this representation is not unique. The matrix representation for*

s_{α_8} *in* S_8 *is*

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & -8 & -7 \end{pmatrix}.$$

The bottom row of this matrix, $(1, 2, 3, 4, 5, 6, -8, -7)$*, is the* unique *representation of* s_{α_8} *in signed permutation notation.*

The next proposition from [HH05] describes multiplication using signed permutation notation.

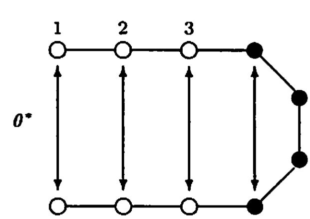
Proposition 1. *For* $1 \leq i < n$*,*

(i) $(a_1, a_2, \cdots, a_n)s_{\alpha_i} = (a_1, a_2, \cdots, a_{i+1}, a_i, \cdots, a_n)$
(ii) *If* $a_k = i$*, and* $a_l = i + 1$ *then* $s_{\alpha_i}(a_1, a_2, \cdots, a_n) =$
$(a_1, a_2, \cdots, a_{k-1}, \mathrm{sgn}(a_k)|a_l|, a_{k+1}, \cdots, a_{l-1}, \mathrm{sgn}(a_l)|a_k|, a_{l+1}, \cdots, a_n)$
(iii) $(a_1, a_2, \cdots, a_n)s_{\alpha_n} = (a_1, a_2, \cdots, a_{n-1}, -a_n)$
(iv) $(a_1, a_2, \cdots, a_n)\widehat{s}_{\alpha_n} = (a_1, a_2, \cdots, -a_n, -a_{n-1})$
(v) *If* $a_l = n$ *then* $s_{\alpha_n}(a_1, a_2, \cdots, a_n) = (a_1, a_2, \cdots, -a_l, \cdots, a_n)$
(vi) *If* $a_k = n - 1$ *and* $a_l = n$ *then* $\widehat{s}_{\alpha_n}(a_1, a_2, \cdots, a_k, \cdots, a_l, \cdots, a_n) =$
$(a_1, a_2, \cdots, -\mathrm{sgn}(a_k)|a_l|, \cdots, -\mathrm{sgn}(a_l)|a_k|, \cdots, a_n)$.

4. RESULTS

For each of the θ-diagrams in Table 1, the Weyl group element $w_0(\theta) \in W_0(\theta)$ must be computed. Recall that $w_0(\theta)$ is the involution that satisfies $w_0(\theta)(\Delta_0(\theta)) = -\Delta_0(\theta)$. Let $m = |\Delta_0(\theta)|$.

Example 2. *Consider the* θ*-diagram of type* $A_{10}^3(III_a)$*. Recall from* [H88] *that the notation* $A_{10}^3(III_a)$ *means that the original root system,* $\Phi(\mathfrak{t})$*, is of type* A_{10}*,* θ *is a type* III_a *involution in* [H78]*, and the resulting restricted root system satisfies* $|\Delta(\mathfrak{a})| = 3$.



Here $m = 4$*, the number of roots fixed by* θ*, and* $\Delta_0(\theta) = \{\alpha_4, \alpha_5, \alpha_6, \alpha_7\}$*. From* [DH05]*, a representative in* $W_0(\theta)$ *for* $w_0(\theta)$ *is*

$$s_{\alpha_4}s_{\alpha_5}s_{\alpha_4}s_{\alpha_6}s_{\alpha_5}s_{\alpha_4}s_{\alpha_7}s_{\alpha_6}s_{\alpha_5}s_{\alpha_4}.$$

Notice that this representation is not unique. Writing as a product of transpositions and multiplying, we get

$$(4,5)(5,6)(4,5)(6,7)(5,6)(4,5)(7,8)(6,7)(5,6)(4,5) = (4,8)(5,7).$$

The unique *representation of* $w_0(\theta)$ *is* $(1, 2, 3, 8, 7, 6, 5, 4, 9, 10, 11)$.

Table 3 summarizes the results for each θ-diagram.

Table 3: $w_0(\theta)$

Type θ	$w_0(\theta)$
$A_l^l(\mathrm{I})$	$(1, 2, \cdots, l, l+1)$
$A_{2l+1}^l(\mathrm{II})$	$(2, 1, 4, 3, \cdots, i+1, i, \cdots, 2l+2, 2l+1)$
$A_l^p(\mathrm{III}_a)$ $l = 2p + m$	$(1, \cdots, p, p+m, \cdots, p+2, p+1, p+m+1, \cdots, l+1)$
$A_{2l-1}^l(\mathrm{III}_b)$ $(l \geq 2)$	$(1, 2, \cdots, 2l-1, 2l)$
$B_l^p(\mathrm{I})$ $l = p + m$	$(1, 2, \cdots, p, -(p+1), \cdots, -(p+m))$
$C_l^l(\mathrm{I})$	$(1, 2, \cdots, l-1, l)$
$C_l^p(\mathrm{II}_a)$ $l = 2p + m$	$(2, 1, 4, 3, \cdots, 2p, 2p-1, -(2p+1), \cdots, -(2p+m))$
$C_{2l}^l(\mathrm{II}_b)$	$(2, 1, 4, 3, \cdots, i+1, i, \cdots, 2l, 2l-1)$
$D_l^p(\mathrm{I}_a)$ $l = p + m$	$(1, 2, \cdots, p-1, p, -(p+1), \cdots, -(p+m))$
$D_l^l(\mathrm{I}_b)$ $(l \geq 4)$	$(1, 2, \cdots, l-1, l)$

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Table 3: *continued*

Type θ	$w_0(\theta)$
$D^l_{2l}(\mathrm{III}_a)$ $(l \geq 2)$	$(2,1,4,3,\cdots,i+1,i,\cdots,2l,2l-1)$
$D^l_{2l+1}(\mathrm{III}_b)$ $(l \geq 2)$	$(2,1,4,3,\cdots,i+1,i,\cdots,2l-2,2l,2l-1,2l+1)$

From Theorem 1, we have that $W(\mathfrak{a}) \simeq W_1(\theta)/W_0(\theta)$. For each λ_i in $\Phi(\mathfrak{a})$, we find a representative $w_i \in W_1(\theta)$ for s_{λ_i} and represent it in signed permutation notation.

Example 3. *Recall from* [H88] *that the notation $A_7^3(II)$ means that the original root system, $\Phi(\mathfrak{t})$ is of type A_7, the involution θ is $\theta(A) = J(-A)^T J^{-1}$ where*

$$J = \begin{pmatrix} 0 & I_4 \\ -I_4 & 0 \end{pmatrix}$$

which is a type II involution in [H78], *and that the resulting restricted root system satisfies $|\Delta(\mathfrak{a})| = 3$. The restricted root system has basis roots λ_1, λ_2 and λ_3, given by*

$$\frac{1}{2}(\alpha_1 + 2\alpha_2 + \alpha_3) = \lambda_1,$$

$$\frac{1}{2}(\alpha_3 + 2\alpha_4 + \alpha_5) = \lambda_2, \text{ and}$$

$$\frac{1}{2}(\alpha_5 + 2\alpha_6 + \alpha_7) = \lambda_3.$$

A representative in $W_1(\theta)$ for s_{λ_1} is $w_1 = s_{\alpha_2}s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}$. Writing w_1 as a product of transpositions and multiplying, we get

$$w_1 = (2,3)(1,2)(3,4)(23) = (1,3)(2,4).$$

The matrix representation for s_{λ_1} in S_8 is $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 1 & 2 & 5 & 6 & 7 & 8 \end{pmatrix}$. The bottom row of this matrix $(3,4,1,2,5,6,7,8)$ is the unique *representation of s_{λ_1} in signed permutation notation.*

Table 4 summarizes the results for each θ-diagram.

Table 4: $W(\mathfrak{a})$

Type θ	s_{λ_i}-representative
$A_l^l(\mathrm{I})$	$w_i = (1, 2, \cdots, i-1, i+1, i, i+2, \cdots, l, l+1)$
$A_{2l+1}^l(\mathrm{II})$	$w_i = (1, 2, \cdots, 2i-2, 2i+1, 2i+2, 2i-1, 2i, 2i+3, \cdots, 2l+2)$
$A_l^p(\mathrm{III}_a)$ $l = 2p + m$	$w_i = (1, \cdots, i-1, i+1, i, i+2, \cdots, l-i, l+2-i, l+1-i, l+3-i, \cdots, l+1)$ $w_p = (1, \cdots, p-1, p+m+2, p+1, \cdots, p+m+1, p, p+m+3, \cdots, l+1)$
$A_{2l-1}^l(\mathrm{III}_b)$ $(l \geq 2)$	$w_i = (1, \cdots, i-1, i+1, i, i+2, \cdots, 2l-i-1, 2l-i+1, 2l-i, 2l-i+2, \cdots, 2l)$ $w_l = (1, \cdots, 2l-2, 2l, 2l-1)$
$B_l^p(\mathrm{I})$ $l = p + m$	$w_i = (1, 2, \cdots, i-1, i+1, i, i+2, \cdots, l)$ $w_p = (1, \cdots, p-1, -p, p+1, \cdots, p+m)$
$C_l^l(\mathrm{I})$	$w_i = (1, 2, \cdots, i-1, i+1, i, i+2, \cdots, l)$ $w_l = (1, \cdots, l-1, -l)$
$C_l^p(\mathrm{II}_a)$ $l = 2p + m$	$w_i = (1, 2, \cdots, 2i-2, 2i+1, 2i+2, 2i-1, 2i, 2i+3, \cdots, l)$

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Table 4: *continued*

Type θ	s_{λ_i}-representative
	$w_p = (1, \cdots, 2p-2, -(2p), -(2p-1), 2p+1, \cdots, l)$
$C^l_{2l}(\mathrm{II}_b)$	$w_i = (1, 2, \cdots, 2i-2, 2i+1, 2i+2, 2i-1, 2i, 2i+3, \cdots, 2l)$
	$w_l = (1, \cdots, 2l-2, -(2l), -(2l-1))$
$D^p_l(\mathrm{I}_a)$ $l = p + m$	$w_i = (1, 2, \cdots, i-1, i+1, i, i+2, \cdots, l)$
	$w_p = (1, \cdots, p-1, -p, p+1, \cdots, l-1, -l)$
$D^l_l(\mathrm{I}_b)$ $(l \geq 4)$	$w_i = (1, 2, \cdots, i-1, i+1, i, i+2, \cdots, l)$
	$w_l = (1, \cdots, l-2, -(l), -(l-1))$
$D^l_{2l}(\mathrm{III}_a)$ $(l \geq 2)$	$w_i = (1, 2, \cdots, 2i-2, 2i+1, 2i+2, 2i-1, 2i, 2i+3, \cdots, 2l)$
	$w_l = (1, \cdots, 2l-2, -(2l), -(2l-1))$
$D^l_{2l+1}(\mathrm{III}_b)$ $(l \geq 2)$	$w_i = (1, 2, \cdots, 2i-2, 2i+1, 2i+2, 2i-1, 2i, 2i+3, \cdots, 2l+1)$
	$w_l = (1, \cdots, 2l-2, -(2l), -(2l-1), 2l+1)$

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