Some 2-coloured 5-cycle decompositions

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Abstract

Let C be the set of distinct ways in which the vertices of a 5-cycle may be coloured with at most two colours, called *colouring types*, and let $S \subseteq C$. Suppose we colour the vertices of K_v with at most two colours. If $\mathcal{D}$ is a 5-cycle decomposition of K_v such that the colouring type of each 5-cycle is in S, and every colouring type in S is represented in $\mathcal{D}$, then $\mathcal{D}$ is said to have a *proper colouring type* S. For all S, $|S| \leq 2$, we determine some necessary conditions for existence of a 5-cycle decomposition of K_v with proper colouring type S. In many cases, we show that these conditions are also sufficient.

1 Introduction

Let G and H be graphs. A G-decomposition of H is a set $\mathcal{G} = \{G_1, G_2, \ldots, G_p\}$ such that G_i is isomorphic to G, for $1 \leq i \leq p$, and $\mathcal{G}$ partitions the edge set of H. Most commonly, $H = K_v$, the complete graph on v vertices. The problem of determining all values of v for which there exists a G-decomposition of K_v is called the *spectrum problem* for G.

An m-cycle, denoted by $(x_1, x_2, \ldots, x_m)$, is the graph with vertex set $\{x_1, x_2, \ldots, x_m\}$ and edge set $\{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_m, x_1\}\}$. The spectrum problem for m-cycles has recently been solved; see [1] and [3].

A variant of the spectrum problem for m-cycles arises when the vertices of K_v have been coloured and there are demands on how each m-cycle in the decomposition must be coloured. An m-cycle is said to be *monochromatic* if all m vertices are the same colour, and a *weak colouring* of an m-cycle decomposition results in no monochromatic m-cycles While most work has considered weak colourings, new colouring systems are emerging.

JCMCC **64** (2008), pp. 187-202

Suppose that the vertices of K_v have been coloured with at most two colours, say black and white (denoted by B and W respectively), and suppose that there exists an m-cycle decomposition of K_v, denoted $\mathcal{D}$.

Let $C_1C_2\ldots C_m$ denote the colouring of an m-cycle which assigns the colour C_i to the vertex x_i, where $C_i \in \{B, W\}$, for $i = 1, 2, \ldots, m$. Then $C_1C_2\ldots C_m$ is said to be a *colouring type*.

We let C be the set of all distinct colouring types, and we let S be a non-empty subset of C. Then the decomposition $\mathcal{D}$ is said to be of *proper colouring Type S* if both the colouring type of every m-cycle in $\mathcal{D}$ is in S, and every colouring type in S is represented in $\mathcal{D}$. (Some colouring types can be obtained from other types by interchanging the colours of all vertices. If S_1 and S_2 are sets of such colouring types, then we write $S_1 \equiv S_2$.)

The existence problem for 4-cycle decompositions of K_v with proper colouring Type S has been completely settled for all possible S; see [4], [7] and [8]. In this paper, we extend this work by considering 5-cycle decompositions of K_v. As such, we often use the following well-known theorem.

Theorem 1.1 [5] *A 5-cycle decomposition of K_v exists if and only if* $v \equiv 1, 5 \pmod{10}$, $v \geq 5$.

Definition 1.2 *Let the colouring* BBBBB *be denoted* Type A1, WWWWW *be denoted* Type A2, BBBBW *be denoted* Type B1, WWWWB *be denoted* Type B2, BBBWW *be denoted* Type C1, WWWBB *be denoted* Type C2, BBWBW *be denoted* Type D1 *and* WWBWB *be denoted* Type D2.

Let $S \subseteq \{$A1, A2, B1, B2, C1, C2, D1, D2$\}$. (For the sake of brevity, we omit the word *Type*.) Our main results are Theorems 1.3 and 1.4 .

Theorem 1.3 *If there exists a 5-cycle decomposition of K_v with proper colouring Type S, where* $S \in \{\{$B1, B2$\}, \{$B1, C1$\}, \{$B2, C2$\}, \{$B1, C2$\}, \{$B2, C1$\}, \{$C1, D2$\}, \{$C2, D1$\}\}$, *then the conditions in Table 1 are satisfied, where w denotes the number of white vertices in K_v. Also, for* $\{$B1, C1$\} \equiv \{$B2, C2$\}$, $\{$B1, C2$\} \equiv \{$B2, C1$\}$ *and* $\{$C1, D2$\} \equiv \{$C2, D1$\}$, *we have found a suitable decomposition for the smallest admissible value of v.*

S	Admissible v
$\{$B1, B2$\}$	$v \equiv 1 \pmod{30}$, $v \geq 39601$, $\sqrt{5v(v+4)} \in \mathbb{Z}$, $w \equiv 0, 1 \pmod{6}$
$\{$B1, C1$\} \equiv \{$B2, C2$\}$	$v \equiv 1, 5 \pmod{10}$, $v \geq 45$, $\sqrt{5v(v+4)} \in \mathbb{Z}$
$\{$B1, C2$\} \equiv \{$B2, C1$\}$	$v \equiv 1, 5 \pmod{10}$, $v \geq 45$, $\sqrt{5v(v+4)} \in \mathbb{Z}$
$\{$C1, D2$\} \equiv \{$C2, D1$\}$	$v \equiv 1, 5 \pmod{10}$, $v \geq 15$, $\sqrt{25 + 20v(v-1)} \in \mathbb{Z}$

Table 1: Some necessary conditions for existence of a 5-cycle decomposition of K_v with proper colouring type S.

Theorem 1.4 *For each $S \subseteq$ {A1, A2, B1, B2, C1, C2, D1, D2}, where $|S| \in \{1,2\}$, and excluding the cases $S \in$ {{B1, B2}, {B1, C1}, {B2, C2}, {B1, C2}, {B2, C1}, {C1, D2}, {C2, D1}}, there exists a 5-cycle decomposition of K_v with proper colouring Type S if and only if the conditions in Table 2 are satisfied.*

S	Admissible v
{A1} ≡ {A2}	$v \equiv 1,5 \pmod{10}$, $v \geq 5$
{B1} ≡ {B2}	$v = 5$
{C1} ≡ {C2}	none
{D1} ≡ {D2}	none
{A1, A2}	none
{A1, B1} ≡ {A2, B2}	$v \equiv 1,5 \pmod{10}$, $v \geq 11$
{A1, B2} ≡ {A2, B1}	$v \equiv 5,21 \pmod{40}$, $v \geq 21$
{A1, C1} ≡ {A2, C2}	none
{A1, C2} ≡ {A2, C1}	none
{A1, D1} ≡ {A2, D2}	none
{A1, D2} ≡ {A2, D1}	none
{B1, D1} ≡ {B2, D2}	none
{B1, D2} ≡ {B2, D1}	none
{C1, C2}	none
{C1, D1} ≡ {C2, D2}	$v \equiv 5 \pmod{10}$, $v \geq 5$
{D1, D2}	none

Table 2: Necessary and sufficient conditions for existence of a 5-cycle decomposition of K_v with proper colouring type S.

For $|S| \geq 3$, it remains an open problem to determine the spectrum of 5-cycle decomposition of K_v with proper colouring type S.

We now introduce some terminology and notation. We say that an edge is *pure-coloured* if it connects two vertices of the same colour, and that an edge is *mixed-coloured* if it connects two vertices of different colours. We let the lower case letters b and w denote the number of black and white vertices in K_v respectively. We let $G - H$ denote the graph G with the edges of the graph H removed.

We define a *pairwise balanced design*, $PBD(v, K, \lambda)$, to be a pair $(V, \mathcal{B})$ such that V is a v-set of elements and $\mathcal{B}$ is a collection of subsets of V, such that $|B| \in K$, for each $B \in \mathcal{B}$, and every unordered pair of elements in V occurs together in precisely λ subsets in $\mathcal{B}$.

Lemma 1.5 [2], [9] *For all odd integers $v > 1$ there exists a $PBD(v, 3, 1)$ or a $PBD(v, \{3, 5^*\}, 1)$.*

We define a *group divisible design*, $GDD[K, \lambda, M; v]$, to be a triple $(V, \Gamma, \mathcal{B})$ such that V is a v-set of elements, $\Gamma = \{G_1, G_2, \ldots\}$ is a partition of V, and $\mathcal{B}$ is a collection of subsets of V, such that $|G_i| \in M$ for each $G_i \in \Gamma$, $|B| \in K$ for each $B \in \mathcal{B}$, and for all $x, y \in V$, x and y occur together in precisely λ subsets of $\mathcal{B}$ if they do not appear together in G_i, for all i, and x and y occur together in no subsets of $\mathcal{B}$ otherwise.

Corollary 1.6 *For all even integers v there exists a $GDD[3, 1, 2; v]$ or a $GDD[3, 1, \{2, 4^*\}; v]$.*

Finally, a path of length m, denoted $[v_0, v_1, \ldots, v_m]$, is the graph with vertex set $\{v_0, v_1, \ldots, v_m\}$ and edge set $\{\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{m-1}, v_m\}\}$. We write P_m to denote the path on m vertices; that is, P_m denotes the path of length $m - 1$. Furthermore, we say that a P_m-design of K_v, denoted $P(v, m, 1)$, is a decomposition of K_v into paths of length $m - 1$.

Theorem 1.7 [6] *A P_m-design $P(v, m, 1)$ exists if and only if $v \geq m$ (if $v > 1$) and $v(v - 1) \equiv 0 \pmod{2(m - 1)}$.*

2 Preliminary designs

Lemma 2.1 [8] *There exist 5-cycle decompositions of: K_5 with proper colouring types {B1} ≡ {B2} and {C1, D1} ≡ {C2, D2}; K_{15} with proper colouring type {C1, D2} ≡ {C2, D1}; K_{21} with proper colouring types {A1} ≡ {A2} and {A1, B2} ≡ {A2, B1}; $K_{45} - K_5$ with proper colouring types {A1, B2} ≡ {A2, B1}, {B1, C1} ≡ {B2, C2} and {B1, C2} ≡ {B2, C1}; $K_{85} - K_5$ with proper colouring type {A1, B2} ≡ {A2, B1}; $K_{3(5)}$ with proper colouring types {A1} ≡ {A2} and {C1, D1} ≡ {C2, D2}; $C_{5(5)}$ with proper colouring types {A1} ≡ {A2} and {B1} ≡ {B2}; $K_{5(4)}$ with proper colouring type {A1, B2} ≡ {A2, B1}; and $K_{5(5)}$ with proper colouring type {C1, D1} ≡ {C2, D2}.*

Lemma 2.2 [8] *There exists a decomposition of $K_{3(4)}$ into 5-cycles with colouring type {B2} and one 3-cycle with three black vertices.*

Lemma 2.3 *There exists a 5-cycle decomposition of $K_{3(20)}$ with proper colouring type {A1, B2} ≡ {A2, B1}.*

Proof. Take a copy of $K_{3(4)}$ and, within each part, colour one vertex black and three vertices white. By Lemma 2.2, there exists a decomposition of this graph into 5-cycles with colouring type {B2} and one 3-cycle with three black vertices. Replace each vertex by five new vertices, colouring each one the same colour as the vertex it replaced. By Lemma 2.1, we can place a copy of a 5-cycle decomposition of $K_{3(5)}$ with colouring type {A1} on the set of vertices arising from the 3-cycle and, by Lemma 2.1, we can place a copy of a 5-cycle decomposition of $C_{5(5)}$ with colouring type {B2} on each set of vertices arising from a 5-cycle. □

Lemma 2.4 *There exists a* 5*-cycle decomposition of* $K_{5(20)}$ *with proper colouring type* {A1, B2} ≡ {A2, B1}.

Proof. Take a copy of $K_{5(4)}$ and, within each part, colour one vertex black and three vertices white. By Lemma 2.1, there exists a 5-cycle decomposition of this graph with colouring type {A1, B2}. Replace each vertex by five new vertices, colouring each one the same colour as the vertex it replaced. By Lemma 2.1, we can place a copy of a 5-cycle decomposition of $C_{5(5)}$ with colouring type {A1} or {B2} respectively on each set of vertices arising from a 5-cycle. □

Lemma 2.5 *There exists a* 5*-cycle decomposition of* K_{45} *with proper colouring type* {B1, C1} ≡ {B2, C2}.

Proof. Let the vertex set of K_{45} be $U \cup V$, where $|U| = 5$ and $|V| = 40$. Colour four vertices in U and twenty-nine vertices in V black. Colour the remaining vertices white. By Lemma 2.1, we can place a copy of a 5-cycle decomposition of K_5 with colouring type {B1} on U. By Lemma 2.1, we can place a copy of a 5-cycle decomposition of $K_{45} - K_5$ with colouring type {B1, C1} on $U \cup V$, where the hole is on the vertices in U. □

Lemma 2.6 *There exists a* 5*-cycle decomposition of* K_{45} *with proper colouring type* {B1, C2} ≡ {B2, C1}.

Proof. The proof mirrors that given for Lemma 2.5, with the vertices in U and V coloured in the same way. In this case, we use a 5-cycle decomposition of K_5 with colouring type {B1} and a 5-cycle decomposition of $K_{45} - K_5$ with colouring type {B1, C2}. These decompositions exist by Lemma 2.1. □

3 The Constructions

Theorem 3.1 *There exists a* 5*-cycle decomposition of* K_v *with proper colouring type* {A1} (≡ {A2}) *if and only if* $v \equiv 1, 5 \pmod{10}$, $v \geq 5$.

Proof. This follows from Theorem 1.1 (colour every vertex black). □

Theorem 3.2 *There exists a* 5*-cycle decomposition of* K_v *with proper colouring type* {B1} (≡ {B2}) *if and only if* $v = 5$.

Proof. By Lemma 2.1, the decomposition exists when $v = 5$. Suppose that the decomposition exists, for some $v > 5$. Since each 5-cycle in the decomposition contains one white vertex, no pure-coloured white edges and two mixed-coloured edges, then $w = 1$ and $bw = v - 1 = v(v-1)/5$. Hence $v = 5$, a contradiction. □

Theorem 3.3 *There exist no* 5*-cycle decompositions of* K_v *with proper colouring type* {C1} (≡ {C2}).

Proof. Suppose that the decomposition exists. Then, by Theorem 1.1, $v \equiv 1, 5 \pmod{10}$, $v \geq 5$. Note that a 5-cycle of Type C1 contains two pure-coloured black edges, one pure-coloured white edge and two mixed-coloured edges. Therefore $bw = w(w-1) = b(b-1)/2$. Solving this system for b and w, we find that $b = -3$ and $w = -2$. Hence $v = -5$, a contradiction. □

Theorem 3.4 *There exist no* 5*-cycle decompositions of* K_v *with proper colouring type* {D1} (≡ {D2}).

Proof. Suppose that the decomposition exists. A 5-cycle of Type D1 contains no pure-coloured white edges. Therefore $w \leq 1$. However, there are two white vertices in such a 5-cycle and so $w \geq 2$, a contradiction. □

Theorem 3.5 *There exist no* 5*-cycle decompositions of* K_v *with proper colouring type* {A1, A2}.

Proof. Suppose that the decomposition exists. Then $b \geq 5$ and $w \geq 5$, and so there are at least twenty-five mixed-coloured edges in K_v. However, neither a 5-cycle of Type A1 nor A2 contains mixed-coloured edges. □

Theorem 3.6 *There exists a* 5*-cycle decomposition of* K_v *with proper colouring type* {A1, B1} (≡ {A2, B2}) *if and only if* $v \equiv 1, 5 \pmod{10}$, $v \geq 11$.

Proof. The necessary conditions follow from Theorem 1.1 and the fact that such a decomposition of K_5 is clearly impossible.

To prove sufficiency, let $v \geq 11$ and take an uncoloured 5-cycle decomposition of K_v. Choose an arbitrary vertex x. Then x does not appear in $(v-1)(v-5)/10$ of the 5-cycles; a quantity which is greater than zero. Now colour x white and colour all other vertices black to obtain the necessary decomposition. □

Theorem 3.7 *There exists a* 5*-cycle decomposition of* K_v *with proper colouring type* {A1, B2} (≡ {A2, B1}) *if and only if* $v \equiv 5, 21 \pmod{40}$, $v \geq 21$.

Proof. First note that the decomposition is impossible if $v = 5$. Now suppose that the decomposition exists for some $v > 5$, and that it contains n 5-cycles of Type B2. Then $v \equiv 1, 5 \pmod{10}$ and $n = w(w-1)/6 = bw/2$. Solving this system for b and w we find that $b = (v-1)/4$ and $w = (3v+1)/4$.

Since each black vertex in the decomposition is adjacent to either two black or two white vertices in each cycle, then b is odd. Thus $b = (v-1)/4 \equiv 1 \pmod 2$ and so $v \equiv 5 \pmod 8$. Hence the decomposition is possible only if $v \equiv 5, 21 \pmod{40}$, $v \geq 21$. We consider these two cases separately.

Case 1: $v \equiv 5 \pmod{40}$.

Let $v = 40x + 5 = 20(2x) + 5$, for $x \geq 1$. By Corollary 1.6, we can take either a GDD$[3, 1, 2; 2x]$ or a GDD$[3, 1, \{2, 4^*\}; 2x]$ and replace each element of the design by twenty new vertices, colouring five new vertices black and fifteen new vertices white. Now adjoin five additional vertices $\infty_1, \infty_2, \ldots, \infty_5$, colouring ∞_1 black and ∞_2, ∞_3, ∞_4 and ∞_5 white. By Lemma 2.1, we can place a copy of a 5-cycle decomposition of K_5 with colouring type {B2} on $\{\infty_1, \infty_2, \ldots, \infty_5\}$. By Lemma 2.3, we can place a copy of a 5-cycle decomposition of $K_{3(20)}$ with colouring type {A1, B2} on each set of vertices arising from a block of the design. Finally, by Lemma 2.1, we can place a copy of a 5-cycle decomposition of $K_{45} - K_5$ or $K_{85} - K_5$, both with colouring type {A1, B2}, on $g \cup \{\infty_1, \infty_2, \ldots, \infty_5\}$ for each set of g vertices arising from a group of the design of size 2 or 4 respectively.

Case 2: $v \equiv 21 \pmod{40}$.

Let $v = 40x + 21 = 20(2x + 1) + 1$, for $x \geq 0$. By Theorem 1.5, we can take either a PBD$(2x + 1, 3, 1)$ or a PBD$(2x + 1, \{3, 5^*\}, 1)$, and replace the i^{th} element of the design with twenty new vertices, five of which are coloured black and fifteen of which are coloured white, and which are contained in the set V_i, for $1 \leq i \leq 2x + 1$. Adjoin one new vertex ∞, which is coloured white. By Lemma 2.1, we can place a copy of a 5-cycle decomposition of K_{21} with colouring type {A1, B2} on $\infty \cup V_i$, for $1 \leq i \leq 2x + 1$. By Lemmas 2.3 and 2.4, we can place a copy of a 5-cycle decomposition of $K_{3(20)}$ or $K_{5(20)}$, both with colouring type {A1, B2}, on each set of vertices arising from a block of the design of size 3 or 5 respectively. □

Theorem 3.8 *There exist no 5-cycle decompositions of K_v with proper colouring type* {A1, C1} (≡ {A2, C2}).

Proof. Suppose that the decomposition exists. Then $v \equiv 1, 5 \pmod{10}$, $v \geq 5$. Since a 5-cycle of Type C1 contains one pure-coloured white edge and two mixed-coloured edges, while a 5-cycle of Type A1 contains neither of these types of edges, then $w(w - 1) = bw$. Solving for b and w we find that $b = (v - 1)/2$ and $w = (v + 1)/2$. Also, since the number of mixed-coloured edges in K_v must be less than twice the number of 5-cycles in the decomposition, then $(v + 1)(v - 1)/4 < v(v - 1)/5$. Hence $v < -5$, a contradiction. □

Theorem 3.9 *There exist no 5-cycle decompositions of K_v with proper colouring type* {A1, C2} (≡ {A2, C1}).

Proof. The proof mirrors that given for Theorem 3.8. □

Theorem 3.10 *There exist no 5-cycle decompositions of K_v with proper colouring type* {A1, D1} (≡ {A2, D2}).

Proof. The proof mirrors that given for Theorem 3.4. □

Theorem 3.11 *There exist no* 5*-cycle decompositions of* K_v *with proper colouring type* {A1, D2} (≡ {A2, D1}).

Proof. Suppose that the decomposition exists. Then $v \equiv 1, 5 \pmod{10}$, $v \geq 5$, and $bw = 2w(w-1)$. Solving for w we find that $w = (v+2)/3$. Also, since every black vertex in the decomposition occurs between either two black or two white vertices, then w is even. Hence v is also even, a contradiction. □

We now present some general observations relating to decompositions in which every 5-cycle contains two mixed-coloured edges.

Theorem 3.12 *Suppose that a* 5*-cycle decomposition of* K_v *exists such that each* 5*-cycle in the decomposition has two mixed-coloured edges. Then* $v \equiv 1, 5 \pmod{10}$, $v > 5$ *and* $\sqrt{5v(v+4)} \in \mathbb{Z}$.

Proof. No such decompositions of K_5 is possible. Now suppose that the required decomposition exists. Then $v \equiv 1, 5 \pmod{10}$, $v > 5$. Furthermore, since each 5-cycle contributes two mixed-coloured edges, $bw = v(v-1)/5$, from which we obtain a quadratic given by $5w^2 - 5vw + v(v-1) = 0$, the solution to which is $w = (5v \pm \sqrt{5v(v+4)})/10$. Without loss of generality let $w = (5v + \sqrt{5v(v+4)})/10$. Since $w \in \mathbb{Z}$, then $\sqrt{5v(v+4)} \in \mathbb{Z}$. □

Theorem 3.13 *If there exists a* 5*-cycle decomposition of* K_v *with proper colouring type* {B1, B2}, *then* $v \equiv 1, 25 \pmod{30}$, $v \geq 25$, $\sqrt{5v(v+4)} \in \mathbb{Z}$ *and* $w \equiv 0, 1 \pmod 6$.

Proof. Suppose that the decomposition exists. Then $v \equiv 1, 5 \pmod{10}$, $v > 5$, and there is a $P(b, 4, 1)$ and a $P(w, 4, 1)$ embedded in the decomposition. That is, we have a P_4-design of order b (on the set of black vertices) and another of order w (on the set of white vertices), and each of these is embedded in the 5-cycle system of order $b + w$.

By Theorem 1.7, a P_4-design of order b (w) exists if and only if $b \equiv 0, 1 \pmod 6$ ($w \equiv 0, 1 \pmod 6$). Since $v = b + w$ must be odd, if $b \equiv 0 \pmod 6$, then $w \equiv 1 \pmod 6$, and vice versa. Hence $v \equiv 1 \pmod 6$. Combining this with Theorem 3.12, we find that $v \equiv 1, 25 \pmod{30}$, $v \geq 25$, $\sqrt{5v(v+4)} \in \mathbb{Z}$ and $w \equiv 0, 1 \pmod 6$. □

In the next lemma we formulate an iterative set of equations which calculates families of values of v for which $\sqrt{5v(v+4)} \in \mathbb{Z}$.

Lemma 3.14 *Given an initial solution* (x_1, y_1) *to the equation* $x_1^2 = 5y_1^2 + 4$, *let* $x_{n+1} = 9x_n + 20y_n$ *and* $y_{n+1} = 4x_n + 9y_n$. *Then* $v_n = x_n - 2$, $n \geq 1$, *is a solution to* $5v_n(v_n+4) = s_n^2$, *where* $s_n = 5y_n$, *and all positive solutions are generated using three initial solutions, namely* $(x_1, y_1) = (3, 1)$, $(7, 3)$ *or* $(18, 8)$.

Proof. We want to determine integer values of v for which $5v(v+4) = s^2$, $s \in \mathbb{Z}$. This is equivalent to finding all integer solutions of $x^2 = 5y^2 + 4$, where $v = x - 2$ and $s = 5y$.

Let (x_n, y_n) be an integer solution, and define

$$x_{n+1} = 9x_n + 20y_n, \quad (1)$$

$$y_{n+1} = 4x_n + 9y_n. \quad (2)$$

Then note that (x_{n+1}, y_{n+1}) is also an integer solution. Thus any solution inductively generates an infinite family of solutions.

Making x_n and y_n the subject of Equations 1 and 2, we see that if (x_{n+1}, y_{n+1}) is an integer solution, then (x_n, y_n) is also an integer solution.

We now find the set of smallest, positive initial solutions which generate all solutions. This is done by finding a lower bound on x_1 and by finding a positive upper bound on y_n, given that y_{n-1} is negative.

We begin by using $x_{n+1}^2 = 5y_{n+1}^2 + 4$, together with Equations 1 and 2 to generate expressions for x_n and y_n in terms of x_{n+1} and y_{n+1} respectively. These are given by

$$x_n = 9x_{n+1} - 4\sqrt{5}\left(x_{n+1}^2 - 4\right)^{1/2} \text{ and} \quad (3)$$

$$y_n = 9y_{n+1} - 4\left(5y_{n+1}^2 + 4\right)^{1/2}. \quad (4)$$

Temporarily regarding Equation 3 as a real-valued function and differentiating with respect to x_{n+1}, we get $\frac{dx_n}{dx_{n+1}} = 9 - 4\sqrt{5}x_{n+1}\left(x_{n+1}^2 - 4\right)^{-1/2}$, which equals zero when $x_{n+1} = 18$. Consequently, $x_n \geq 2$, and so $x_1 \geq 2$.

We now show that y_n decreases as n gets smaller, provided that $y_{n+1} \geq 0$. Using Equation 4, we find that $y_n - y_{n+1} = 8y_{n+1} - 4\left(5y_{n+1}^2 + 4\right)^{1/2} < 8y_{n+1} - 4\sqrt{5}y_{n+1} < 0$.

We now wish to find the maximum value of y_{n+1}, given $y_n \leq 0$. Substituting for x_n into Equation 2, we obtain $y_{n+1} = 4\left(5y_n^2 + 4\right)^{1/2} + 9y_n$, the derivative of which (with respect to y_n), $\frac{dy_{n+1}}{dy_n}$, is strictly positive. Thus, to find the maximum value of y_{n+1} under the condition $y_n \leq 0$, we simply substitute the boundary value $y_n = 0$ into the equation for y_{n+1}. This leads us to conclude that if $y_n \leq 0$, then $y_{n+1} \in [1, 8]$. Consequently, we have established that the set of initial solutions which generate all possible answers to our system must satisfy $x_1 \geq 2$ and $1 \leq y_1 \leq 8$.

The only values of y_1 in this range for which x_1 is an integer are $y_1 = 1$, 3 and 8. Consequently, the only initial solutions we require to generate all solutions to our system are $(x_1, y_1) = (3, 1)$, $(7, 3)$ or $(18, 8)$. □

We will call the system of equations given in Lemma 3.14 "System 1". Given Theorem 3.12, without loss of generality, we may express the number of white vertices in K_{v_n} as $w_n = (5v_n + s_n)/10$.

In Table 3, we give modified equations for x_{n+1} and y_{n+1} which can be used to return every second or every fourth solution returned by System 1. We call the system that returns every second solution "System 2", and we call the system that returns every fourth solution "System 4". Henceforth, we differentiate between the systems through the use of the superscripts (1), (2) and (4), which are associated with the variables of each system. For convenience, we also include the equations of System 1 in Table 3. Note also that for each system $v_n^{(*)} = x_n^{(*)} - 2$, $s_n^{(*)} = 5y_n^{(*)}$ and $w_n^{(*)} = (5v_n^{(*)} + s_n^{(*)})/10$, where $(*) = (1), (2)$ or (4).

System	$x_{n+1}^{(*)}$	$y_{n+1}^{(*)}$
1	$x_{n+1}^{(1)} = 9x_n^{(1)} + 20y_n^{(1)}$	$y_{n+1}^{(1)} = 4x_n^{(1)} + 9y_n^{(1)}$
2	$x_{n+1}^{(2)} = 161x_n^{(2)} + 360y_n^{(2)}$	$y_{n+1}^{(2)} = 72x_n^{(2)} + 161y_n^{(2)}$
4	$x_{n+1}^{(4)} = 51841x_n^{(4)} + 115920y_n^{(4)}$	$y_{n+1}^{(4)} = 23184x_n^{(4)} + 51841y_n^{(4)}$

Table 3: Equations for $x_{n+1}^{(*)}$ and $y_{n+1}^{(*)}$ for Systems 1, 2 and 4.

Lemma 3.15 $x_{n+1}^{(4)} \equiv x_n^{(4)} \pmod{30}$.

Proof. $x_{n+1}^{(4)} = x_n^{(4)} + 30(1728x_n^{(4)} + 3864y_n^{(4)}) \equiv x_n^{(4)} \pmod{30}$. □

Lemma 3.16 $v_{n+1}^{(4)} \equiv v_n^{(4)} \pmod{30}$.

Proof. Since $v_n^{(4)} = x_n^{(4)} - 2$, this follows from Lemma 3.15. □

Lemma 3.17 $w_{n+1}^{(4)} \equiv w_n^{(4)} \pmod{6}$.

Proof. $w_{n+1}^{(4)} = (x_{(n+1)}^{(4)} + y_{(n+1)}^{(4)} - 2)/2 = (x_n^{(4)} + y_n^{(4)} - 2)/2 + 6(6262x_n^{(4)} + 13980y_n^{(4)}) \equiv w_n^{(4)} \pmod{6}$. □

For each of the starting solutions to System 4 (based on the initial solutions to System 1, as given by Lemma 3.14), we give the congruence class for $v_n^{(4)}$ and $w_n^{(4)}$ in Table 4.

Lemma 3.18 *Let* $(x_1^{(1)}, y_1^{(1)}) = (39603, 17711)$ *or* $(271443, 121393)$. *Then every fourth value of* $v_n^{(1)}$ *generated by System* 1, *beginning either with* $v_1^{(1)} =$ 39601 *or* 271441, *will satisfy the following conditions:* $v_n^{(1)} \equiv 1 \pmod{30}$, $v_n^{(1)} \geq 39601$, $\sqrt{5v_n^{(1)}(v_n^{(1)} + 4)} = (s_n^{(1)})^2$, *where* $s_n^{(1)} = 5y_n^{(1)}$, *and* $w_n^{(1)} \equiv 0, 1 \pmod{6}$. *Furthermore, these are the only values of* $v_n^{(1)}$ *for which all these conditions hold.*

$(x_1^{(4)}, y_1^{(4)})$	$v_n^{(4)}\ (\text{mod}\ 30)$	$w_n^{(4)}\ (\text{mod}\ 6)$
$(3,1)$	1	1
$(47,21)$	15	3
$(843,377)$	1	3
$(15127,6765)$	5	1
$(7,3)$	5	4
$(123,55)$	1	4
$(2207,987)$	15	0
$(39603,17711)$	1	0
$(18,8)$	16	0
$(322,144)$	20	4
$(5778,2584)$	16	4
$(103682,46368)$	0	0

Table 4: Congruence of $v_n^{(4)}$ and $w_n^{(4)}$ for all initial solutions to System 4.

Proof. This follows from Lemmas 3.16 and 3.17, and Table 4. □

Theorem 3.19 *If there exists a 5-cycle decomposition of K_v with proper colouring type* {B1, B2}, *then* $v \equiv 1\ (\text{mod}\ 30)$, $v \geq 39601$, $\sqrt{5v(v+4)} \in \mathbb{Z}$ *and* $w \equiv 0, 1\ (\text{mod}\ 6)$.

Proof. From Theorem 3.13, $v \equiv 1, 25\ (\text{mod}\ 30)$, $v \geq 25$, $\sqrt{5v(v+4)} \in \mathbb{Z}$ and $w \equiv 0, 1\ (\text{mod}\ 6)$. By Lemma 3.14, every v satisfying $\sqrt{5v(v+4)} \in \mathbb{Z}$ is generated using the initial solutions $(3,1)$, $(7,3)$ and $(18,8)$ in System 1. By Lemma 3.18, the only values of v which also satisfy the additional conditions of Theorem 3.13 are obtained by taking every fourth value generated by System 1 starting from either the solution $(39603, 17711)$ or $(271443, 121393)$. In either case, $v \equiv 1\ (\text{mod}\ 30)$ and $v \geq 39601$. □

Open Problem Let $(x_1, y_1) = (39603, 17711)$ or $(271443, 121393)$. Let $x_{n+1} = 51841x_n + 115920y_n$, $y_{n+1} = 23184x_n + 51841y_n$, $v_n = x_n - 2$, $s_n = 5y_n$ and $w_n = (5v_n + s_n)/10$. Does there exist a 5-cycle decomposition of K_{v_n} with proper colouring type {B1, B2}, for all $n \geq 1$?

Lemma 3.20 *Let* $(x_1^{(1)}, y_1^{(1)}) = (47, 21)$ *or* $(123, 55)$. *Then System* 1 *produces values of* $v_n^{(1)}$ *such that* $v_n^{(1)} \equiv 1, 5\ (\text{mod}\ 10)$, $\sqrt{5v_n^{(1)}(v_n^{(1)}+4)} = (s_n^{(1)})^2$, *where* $s_n^{(1)} = 5y_n^{(1)}$, *and* $v_n^{(1)} \geq 45$.

Proof. This follows from Theorem 3.12, Lemmas 3.14 and Table 4. □

Theorem 3.21 *If there exists a 5-cycle decomposition of K_v with proper colouring type* {B1, C1} ($\equiv$ {B2, C2}) *or* {B1, C2} ($\equiv$ {B2, C1}), *then* $v \equiv 1, 5 \pmod{10}$, $v \geq 45$ *and* $\sqrt{5v(v+4)} \in \mathbb{Z}$.

Proof. This follows from Theorem 3.12 and Lemma 3.20. □

Open Problem Let $(x_1, y_1) = (47, 21)$ or $(123, 55)$. Let $x_{n+1} = 9x_n + 20y_n$, $y_{n+1} = 4x_n + 9y_n$, $v_n = x_n - 2$, $s_n = 5y_n$ and $w_n = (5v_n + s_n)/10$. Does there exist a 5-cycle decomposition of K_{v_n} with colouring type {B1, C1} ($\equiv$ {B2, C2}) or {B1, C2} ($\equiv$ {B2, C1}), for all $n \geq 1$?

Note that by Lemmas 2.5 and 2.6, there exist 5-cycle decompositions of K_{45} with proper colourings types {B1, C1} and {B1, C2} respectively.

Theorem 3.22 *There exist no 5-cycle decompositions of K_v with proper colouring type* {B1, D1} ($\equiv$ {B2, D2}).

Proof. The proof mirrors that given for Theorem 3.4. □

Theorem 3.23 *There exist no 5-cycle decompositions of K_v with proper colouring type* {B1, D2} ($\equiv$ {B2, D1}).

Proof. Trivially, the decomposition cannot exist for $v = 5$. Suppose that the decomposition exists and that it contains n_1 5-cycles of Type B1 and n_2 5-cycles of Type D2. Then $v \equiv 1, 5 \pmod{10}$, $v \geq 11$. Note that a 5-cycle of Type B1 contains three pure-coloured black edges and no pure-coloured white edges, while a 5-cycle of Type D2 contains no pure-coloured black edges and one pure-coloured white edge. Hence, $n_1 = b(b-1)/6$, $n_2 = w(w-1)/2$, and $2n_1 + 4n_2 = bw$. Substituting for n_1 and n_2, and performing some simple manipulations, we obtain a quadratic in b given by $10b^2 + 5(1-3v)b + 6v(v-1) = 0$, the solution to which is $b = (5(3v-1) \pm \sqrt{-15v^2 + 90v + 25})/20$.

In order for b to exist, we require $-15v^2 + 90v + 25 \geq 0$. It is not difficult to check that the only integer values of v for which this holds are $v \in \{0, 1, 2, 3, 4, 5, 6\}$. Thus we have a contradiction. □

Theorem 3.24 *There exist no 5-cycle decompositions of K_v with proper colouring type* {C1, C2}.

Proof. Suppose that the decomposition exists and that it contains n_1 5-cycles of Type C1 and n_2 5-cycles of Type C2. Then $v \equiv 1, 5 \pmod{10}$, $v \geq 5$, $n_1 + n_2 = v(v-1)/10$ and $n_1 + 2n_2 = w(w-1)/2$. Solving for n_2, we find that $n_2 = (5w(w-1) - v(v-1))/10$, which must be strictly greater than zero. That is, $5w(w-1) - v(v-1) > 0$. By Theorem 3.12, $w = (5v \pm \sqrt{5v(v+4)})/10$. Without loss of generality let $w = (5v - s)/10$, where $s = \sqrt{5v(v+4)}$. Substituting for w, and performing some simple manipulations, we find that $v^2 - (1+s)v + s > 0$, an inequality which is satisfied only for $v < 1$ or $v > s$. Since $v \geq 5$, we ignore the former inequality. Therefore $v > s$ and we deduce that $v^2 > 5v(v+4)$, since v is positive. Thus $v^2 < -5v$, a contradiction. □

Theorem 3.25 *There exists a 5-cycle decomposition of K_v with proper colouring type* {C1, D1} ($\equiv$ {C2, D2}) *if and only if* $v \equiv 5 \pmod{10}$, $v \geq 5$.

Proof. Suppose that the decomposition exists and that it contains n_1 5-cycles of Type C1 and n_2 5-cycles of Type D1. Then $n_1 + n_2 = v(v-1)/10$, $2n_1 + n_2 = b(b-1)/2$, $n_1 = w(w-1)/2$ and $2n_1 + 4n_2 = bw$, where $v \equiv 1, 5 \pmod{10}$, $v \geq 5$. Solving this system of equations for b and w we find that $b = 3v/5$ and $w = 2v/5$. Since $b, w \in \mathbb{Z}$, $v \equiv 5 \pmod{10}$, $v \geq 5$.

We now prove sufficiency. Let $v = 10x + 5$, for $x \geq 0$. By Theorem 1.5, we can take either a PBD$(2x+1, 3, 1)$ or a PBD$(2x+1, \{3, 5^*\}, 1)$ and replace each element of the design with five vertices, colouring three vertices black and two vertices white. Let the five vertices replacing the i^{th} element be in the set V_i, for $1 \leq i \leq 2x+1$.

By Lemma 2.1, we can place a copy of a 5-cycle decomposition of K_5 with colouring type {C1, D1} on V_i, for $1 \leq i \leq 2x+1$. Furthermore, by Lemma 2.1, we can place a copy of a 5-cycle decomposition of either $K_{3(5)}$ or $K_{5(5)}$, both with colouring type {C1, D1}, on each set of vertices arising from a block of the design of size 3 or 5 respectively. □

Theorem 3.26 *If there exists a 5-cycle decomposition of K_v with proper colouring type* {C1, D2} ($\equiv$ {C2, D1}), *then* $v \equiv 1, 5 \pmod{10}$, $v > 5$ *and* $\sqrt{20v^2 - 20v + 25} \in \mathbb{Z}$.

Proof. The first condition follows from Theorem 1.1, while the second arises because it is clearly impossible to find such a decomposition of K_5. Furthermore, if the decomposition exists, then the number of pure-coloured white edges must equal the number of 5-cycles in the decomposition. That is, $v(v-1)/10 = w(w-1)/2$. From this, we construct a quadratic in w, $5w^2 - 5w - v(v-1) = 0$, the solution to which is given by $w = (5 \pm \sqrt{20v^2 - 20v + 25})/10$. Since $w \in \mathbb{Z}$, then $\sqrt{20v^2 - 20v + 25} \in \mathbb{Z}$. □

In the next lemma, we formulate an iterative set of equations which calculates families of values of v for which $\sqrt{20v^2 - 20v + 25} \in \mathbb{Z}$.

Lemma 3.27 *Given an initial solution (x_1, y_1) to the equation $x_1^2 = 5y_1^2 - 4$, let $x_{n+1} = 9x_n + 20y_n$ and $y_{n+1} = 4x_n + 9y_n$. Then $v_n = (x_n + 1)/2$, $n \geq 1$, is a solution to $20v_n^2 - 20v_n + 25 = s_n^2$, where $s_n = 5y_n$, and all positive, integer solutions are generated using two initial solutions, namely $(x_1, y_1) = (1, 1)$ and $(11, 5)$.*

Proof. We want to determine integer values of v for which $20v^2 - 20v + 25 = s^2$, $s \in \mathbb{Z}$. This is equivalent to finding all integer solutions of $x^2 = 5y^2 - 4$, where $v = (x+1)/2$ and $s = 5y$.

Let (x_n, y_n) be an integer solution. Then (x_{n+1}, y_{n+1}), where x_{n+1} and y_{n+1} are defined in Equations 1 and 2, is also an integer solution.

We now find the set of smallest, positive initial solutions which generate all solutions to our system. Again, we proceed in much the same way as we did in the proof of Lemma 3.14, and some simple manipulation gives us the following expressions:

$$x_n = 9x_{n+1} - 4\sqrt{5}\left(x_{n+1}^2 + 4\right)^{1/2}, \text{ and} \quad (5)$$

$$y_n = 9y_{n+1} - 4\left(5y_{n+1}^2 - 4\right)^{1/2}. \quad (6)$$

We now show that x_n decreases as n gets smaller, provided that $x_{n+1} \geq 0$. Using Equation 6, we find that $x_n - x_{n+1} = 8x_{n+1} - 4\sqrt{5}\left(x_{n+1}^2 + 4\right)^{1/2}$ $< 8x_{n+1} - 4\sqrt{5}x_{n+1} < 0$. We now wish to find the maximum value of x_{n+1}, given $x_n \leq 0$ and $x_{n+1} = 9x_n + 4\sqrt{5}\left(x_n^2 + 4\right)^{1/2}$. Differentiating with respect to x_n, we find that $\frac{dx_{n+1}}{dx_n}$ is strictly positive. Thus, to find the maximum value of x_{n+1} under the condition $x_n \leq 0$, we substitute the boundary value $x_n = 0$ into the equation for x_{n+1}. Thus, we conclude that, if $x_n \leq 0$, then $x_{n+1} \in [1, 17]$, and so $x_1 \in [1, 17]$.

We now consider the smallest, positive value that y_1 can take. Temporarily regarding Equation 6 as a real-valued function and differentiating with respect to y_{n+1}, we find that $\frac{dy_n}{dy_{n+1}} = 9 - 20y_{n+1}\left(5y_{n+1}^2 - 4\right)^{-1/2}$, which equals zero when $y_{n+1} = \sqrt{64.8}$. Since we are interested only in integer solutions, we find that $y_1 \geq 1$.

Consequently, we have established that the set of initial solutions which generate all possible answers to our system must satisfy $1 \leq x_1 \leq 17$ and $y_1 \geq 1$. The only values of x_1 in this range for which y_1 is also an integer are $x_1 = 1$, 4 and 11. However, we discard $x_1 = 4$, because $x_{n+1} = 9x_n + 20y_n$ will always be even if $x_1 = 4$ and, therefore, $v_n \notin \mathbb{Z}$. Hence, the only initial solutions we require are $(x_1, y_1) = (1, 1)$, and $(11, 5)$. □

We will call the system of equations given in Lemma 3.27 "System A". Given Theorem 3.26, we may express the number of white vertices in K_{v_n} as $w_n = (5 + s_n)/10$.

In Table 5, we give modified equations for x_{n+1} and y_{n+1} which can be used to return every second solution returned by System A. We call the system that returns every second solution "System B". Henceforth, we differentiate between the systems through the use of the superscripts (A) and (B). For convenience, we also include the equations of System A in

Table 5. Note also that for each system $v_n^{(*)} = (x_n^{(*)} + 1)/2$, $s_n^{(*)} = 5y_n^{(*)}$ and $w_n^{(*)} = (5 + s_n^{(*)})/10$, where $(*) = $ (A) or (B).

System	$x_{n+1}^{(*)}$	$y_{n+1}^{(*)}$
A	$x_{n+1}^{(A)} = 9x_n^{(A)} + 20y_n^{(A)}$	$y_{n+1}^{(A)} = 4x_n^{(A)} + 9y_n^{(A)}$
B	$x_{n+1}^{(B)} = 161x_n^{(B)} + 360y_n^{(B)}$	$y_{n+1}^{(B)} = 72x_n^{(B)} + 161y_n^{(B)}$

Table 5: Equations for $x_{n+1}^{(*)}$ and $y_{n+1}^{(*)}$ for Systems A and B.

Lemma 3.28 $x_{n+1}^{(B)} \equiv x_n^{(B)} \pmod{20}$.

Proof. $x_{n+1}^{(B)} = x_n^{(B)} + 20(8x_n^{(B)} + 18y_n^{(B)}) \equiv x_n^{(B)} \pmod{20}$. □

Lemma 3.29 $v_{n+1}^{(B)} \equiv v_n^{(B)} \pmod{10}$.

Proof. Since $x_n^{(B)} = (x_n^{(B)} + 1)/2$, this follows from Lemma 3.28. □

For each of the starting solutions to System B (based on the initial solutions to System A, as given by Lemma 3.27), we give the congruence class for $v_n^{(B)}$ in Table 6.

$(x_1^{(B)}, y_1^{(B)})$	$v_n^{(B)} \pmod{10}$	$(x_1^{(B)}, y_1^{(B)})$	$v_n^{(B)} \pmod{10}$
(1, 1)	1	(11, 5)	6
(29, 13)	5	(199, 89)	0

Table 6: Congruence of $v_n^{(B)}$ given all initial solutions to System B.

Lemma 3.30 *Let* $(x_1^{(A)}, y_1^{(A)}) = (29, 13)$. *Then System A produces values of* $v_n^{(A)}$ *such that* $v_n^{(A)} \equiv 1, 5 \pmod{10}$, $\sqrt{20(v_n^{(A)})^2 - 20v_n^{(A)} + 25} \in \mathbb{Z}$, *and* $v_n^{(A)} \geq 15$.

Proof. This follows from Lemma 3.27 and Table 6. □

Theorem 3.31 *If there exists a* 5*-cycle decomposition of* K_v *with proper colouring type* {C1, D2} ($\equiv$ {C2, D1}), *then* $v \equiv 1, 5 \pmod{10}$, $v \geq 15$ *and* $\sqrt{20v^2 - 20v + 25} \in \mathbb{Z}$.

Proof. This follows from Theorem 3.26 and Lemma 3.30. □

Open Problem Let $(x_1, y_1) = (29, 13)$. Let $x_{n+1} = 9x_n + 20y_n$, $y_{n+1} = 4x_n + 9y_n$, $v_n = (x_n + 1)/2$ and $s_n = 5y_n$. Let $w_n = (5 + s_n)/10$ be the number of white vertices in K_{v_n}. Does there exist a 5-cycle decomposition of K_{v_n} with proper colouring type {C1, D2} ($\equiv$ {C2, D1}), for all $n \geq 1$?

Note that in Lemma 2.1, we give a 5-cycle decomposition of K_{15} with proper colouring type {C1, D2}.

Theorem 3.32 *There exist no 5-cycle decompositions of K_v with colouring type* {D1, D2}.

Proof. Suppose that the decomposition exists and that it contains n_1 5-cycles of Type D1 and n_2 5-cycles of Type D2. Then $v \equiv 1, 5 \pmod{10}$, $v \geq 5$, and $n_1 + n_2 = v(v-1)/10 = bw/4$. Hence $5b^2 - 5bv + 2v(v-1) = 0$ and so $b = 5v \pm \sqrt{5v(8-3v)})/10$. For b to be a positive integer, we require $5v(8-3v) \geq 0$ which implies that $v \leq 8/3$, a contradiction. □

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