AN INEQUALITY FOR FINITE PLANAR SPACES WITH NO DISJOINT PLANES

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ABSTRACT. We prove that for a finite planar space $S = (\mathcal{P}, \mathcal{L}, \mathcal{H})$ with no disjoint planes and with a constant number of planes on a line, the number b of lines is greater than or equal to the number c of planes, and the equality holds to be true if and only if S is either the finite desarguesian 4-dimensional projective space $PG(4, q)$, or the complete graph K_5.

1. INTRODUCTION

Let $S = (\mathcal{P}, \mathcal{L})$ be a linear space, a subset X of points is a *subspace* if any line connecting two points of X is wholly contained in X.

Clearly the set theoretical intersection of subspaces in S is a subspace, and so the notion of spanned subspace makes sense.

A *plane* is a subspace spanned by three non-collinear points, that is the intersection of all subspaces of S containing the three given points.

A *planar space* is a linear space endowed with a family of planes. Examples of finite planar spaces are the affine and projective spaces of dimension at least 3 with respect to their lines and their planes.

Let $\mathcal{H}$ denote the family of planes of a planar space.

Assume S is a finite planar space, that is $|\mathcal{P}| < \infty$, and put $b = |\mathcal{L}|$, $c = |\mathcal{H}|$, and for any plane π let v_π and b_π be the numbers of its points and lines, respectively. If ℓ is a line, let c_ℓ denote the number of planes passing through ℓ. Moreover, put $\mu = \min_{\pi \in \mathcal{H}} b_\pi$ and $h = \max_{\ell \in \mathcal{L}} c_\ell$.

Let $PG(r, q)$ be the finite desarguesian projective r-dimensional space of order q, with $r \in \{3, 4\}$, then any two planes have non-empty intersection (cf e.g. [5]), and

$$b = (q^2 + 1)(q^2 + q + 1),\ c = (q + 1)(q^2 + q + 1),\ \text{if } r = 3,$$

$$b = c = q^4 + q^3 + q^2 + q + 1,\ \text{if } r = 4.$$

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In the literature on finite geometries, one can find a number of results finding inequalities among their parameters, and the extremal cases of these inequalities are studied in order to describe the structure of the geometry or to get characterizations of classical examples (for finite linear spaces see e.g. [1, 2, 4, 6]). Recently, continuing this tradition, in [3] the authors consider a class of finite planar spaces with no disjoint planes, which they call *regular planar spaces of type* (k,n). They investigate some properties of such planar spaces, and in particular they prove the following result.

Theorem 1.1 (Durante, Lo Re, Olanda (2002)). *Let* $(\mathcal{P},\mathcal{L},\mathcal{H})$ *be a finite planar space with* b *lines,* c *planes, with constant line size* $k+1$*, and such that for every point* p *in every plane each pencil of lines has size* $n+1$*. Then* $b \geq c$*, and the equality holds if and only if* $(\mathcal{P},\mathcal{L},\mathcal{H})$ *is* $PG(4,n)$.

In this paper, we prove a similar result (Theorem 2.2) for finite planar spaces with no disjoint planes and with a constant number of planes on a line.

Since, in a regular planar space of type (k,n) the planes have all the same size, then counting v via the planes through a line ℓ it follows that through any line there is a constant number of planes, and so our assumptions are weaker than those of Theorem 1.1.

2. THE RESULT

Lemma 2.1. *Let* S *be a finite planar space with no disjoint planes. Assume that there exists a line-plane pair* (ℓ,π) *with* ℓ *disjoint with* π*. Then* $c_\ell \leq v_\pi$.

PROOF. Clearly, each point of π gives a plane through ℓ. Hence, the assertion follows since every plane through ℓ meets π. □

Theorem 2.2. *Let* S *be a finite planar space with no disjoint planes, and with a constant number* h *of planes on the lines. Then* $b \geq c$*, and the equality holds if and only if* S *is either the finite desarguesian 4–dimensional projective space* $PG(4,q)$ *or the complete graph* K_5 *on five vertices.*

PROOF. Assume on the contrary that $b \leq c$. We will prove that $b = c$ and that either S is $PG(4,q)$ or K_5.

Double counting gives

$$\sum_{\pi\in\mathcal{H}} b_\pi = \sum_{\ell\in\mathcal{L}} c_\ell, \tag{2.1}$$

so being $b \leq c$ it follows that

$$c\mu \leq \sum_{\pi\in\mathcal{H}} b_\pi = \sum_{\ell\in\mathcal{L}} c_\ell = bh \leq ch. \tag{2.2}$$

Thus, $h \geq \mu$.

CASE 1. $h = \mu$.

In such a case all the inequalities in Equation (2.2) are equalities, and so $h = \mu$, $b = c$ and $c_\ell = b_\pi = \mu$ for every $(\ell, \pi) \in \mathcal{L} \times \mathcal{H}$.

Let π be a plane, and let ℓ be a line meeting π in exactly one point p. Since $b_\pi = \mu = c_\ell$ and since π contains lines not passing through p, there is at least one plane through ℓ meeting π exactly in the point p.

It follows that for any plane π there is a line ℓ' disjoint with π, and so by Lemma 2.1 it follows that $b_\pi = \mu = h = c'_\ell \leq v_\pi$.

Thus, from the Fundamental Theorem on finite linear spaces [2] it follows that $v_\pi = b_\pi$, and therefore the planes of $(\mathcal{P}, \mathcal{L})$ are either projective planes or near–pencils.

If every plane is a projective one, then $(\mathcal{P}, \mathcal{L})$ is a projective space $PG(r, q)$, and by $b = c$ it follows that $(\mathcal{P}, \mathcal{L})$ is $PG(4, q)$.

Assume now that there is a plane π which is a near–pencil. Let $\pi = \ell \cup \{p\}$. Each plane meeting π in a line through p is a near–pencil since it has a line of length 2. Also, each plane meeting π in ℓ is a near–pencil since it has $|\ell| + 1 = b_\pi$ lines. So, the lines outside π and meeting ℓ have length 2. It follows that all the planes are near–pencils. Since all the planes have the same size it follows that all the lines have length 2, hence $v_\pi = b_\pi = 3$, and so $\mu = 3$. It follows that $v = 5$ and so $(\mathcal{P}, \mathcal{L})$ is the complete graph K_5.

CASE 2. $h \geq \mu + 1$. In such a case, there is no line disjoint with a plane π such that $b_\pi = \mu$. Indeed, if ℓ is a line disjoint with π, and $b_\pi = \mu$, then by the Fundamental Theorem on finite linear spaces $v_\pi \leq b_\pi = \mu < c_\ell = h$, which contradicts Lemma 2.1.

Let π be a plane with $b_\pi = \mu$, and ℓ be a line not contained in π. Since $b_\pi < c_\ell = h$, not all the planes through ℓ meet π in a line, that is there is a plane α meeting π into exactly one point. Clearly, α contains a line disjoint with π, a contradiction.

Thus, $h = \mu$, $b = c$ and S is either the desarguesian 4–dimensional projective space $PG(4, q)$ or the complete graph K_5 on five vertices. □

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