Embeddings of α-Resolvable Steiner Triple Systems*

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Abstract

Let α-resolvable STS(v) denote a Steiner triple system of order v whose blocks are partitioned into classes such that each point of the design occurs in precisely α blocks in each class. We show that for $v \equiv u \equiv 1 \pmod 6$ and $v \geq 3u + 4$ there exists an α-resolvable STS(v) containing an α-resolvable sub-STS(u) for all suitable α.

1 Introduction

A Steiner triple system of order v, denoted STS(v), is an ordered pair $(X, \mathcal{A})$ where X is a v-set and $\mathcal{A}$ is a collection of 3-subsets (called *blocks* or *triples*) of X such that each pair of distinct elements of X is contained in exactly one triple.

An STS(v) is said to be α-resolvable if its blocks can be partitioned into classes (called *α-resolution classes*) such that each point of the design occurs in precisely α blocks in each class. It is easy to show that the existence of an STS(v) implies that $3|\alpha v$ and $\alpha|\frac{v-1}{2}$.

For $\alpha = 1$, a 1-resolvable STS(v) is also known as a Kirkman triple system (or KTS(v)). It is well known ([8]) that a KTS(v) exists if and only if $v \equiv 3 \pmod 6$.

Jungnickel, Mullin and Vanstone [7] have shown that an α-resolvable STS(v) exists if and only if $(v-1) \equiv 0 \pmod 2$, $v(v-1) \equiv 0 \pmod 6$, $3|\alpha v$ and $\alpha|\frac{v-1}{2}$. We then have

Theorem 1.1 ([7]) There exists a 3-resolvable STS(v) for $v \equiv 1 \pmod 6$.

We are interested in α-resolvable STS(v)s which contain α-resolvable STS(u)s as subsystems. Let $(X, \mathcal{A})$ be an α-resolvable STS(v) and $(Y, \mathcal{B})$ be an α-resolvable STS(u). If $Y \subset X$ and $\mathcal{B}$ is a subcollection of $\mathcal{A}$, and each α-resolution class of $\mathcal{B}$ is a part of some α-resolution class of $\mathcal{A}$, then

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$(Y, \mathcal{B})$ is called a *subsystem* of $(X, \mathcal{A})$, or $(Y, \mathcal{B})$ is said to be *embedded* in $(X, \mathcal{A})$. We shall describe the subsystem as an α-resolvable sub-STS(u).

For $\alpha = 1$, the existence of a KTS(v) containing a KTS(u) has been solved by Rees and Stinson in [11].

Theorem 1.2 ([11]) There is a KTS(v) containing a sub-KTS(u) if and only if $v \equiv u \equiv 3 \pmod 6$ and $v \geq 3u$.

Thus, for $v \equiv u \equiv 3 \pmod 6$, subsystems can be constructed by collecting α different 1-resolution classes to obtain the desired α-resolution classes whenever $\alpha | \frac{v-1}{2}$ and $\alpha | \frac{u-1}{2}$.

For $v \equiv u \equiv 1 \pmod 6$, it is obvious that if an α-resolvable STS(u) can be embedded in an α-resolvable STS(v), then v and u satisfy $v \geq 3u + 4$. The necessary conditions $3|\alpha v$ and $3|\alpha u$ yield the condition $3|\alpha$. A solution for the case $\alpha = 3$ will then provide solutions for all α where $3|\alpha$. One can combine $\frac{\alpha}{3}$ 3-resolution classes into one α-resolution class. Hence, we only need to consider the existence of a 3-resolvable STS(v) containing a 3-resolvable sub-STS(u).

The purpose of this paper is to give a complete solution to the problem of embedding problem a 3-resolvable STS(u) in a 3-resolvable sub-STS(v) for $v \equiv u \equiv 1 \pmod 6$ and $v \geq 3u + 4$.

Theorem 1.3 Let $v \equiv u \equiv 1 \pmod 6$ and $v \geq 3u + 4$. Then there exists a 3-resolvable STS(v) containing a 3-resolvable sub-STS(u).

2 Some constructions

We need to define several types of designs. A *group divisible design* is a triple $(X, \mathcal{G}, \mathcal{A})$ where

1. $\mathcal{G}$ is a partition of X into subsets called *groups*,

2. $\mathcal{A}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point, and

3. every pair of points from distinct groups occurs in a unique block.

The *group-type* (or *type*) of the GDD $(X, \mathcal{G}, \mathcal{A})$ is the multiset $\{|G| : G \in \mathcal{G}\}$ and we usually use the "exponential" notation for its description: group-type $1^i 2^j 3^k \cdots$ denotes i occurrences of groups of size 1, j occurrences of size 2, and so on. We shall sometimes refer to a GDD $(X, \mathcal{G}, \mathcal{B})$ as a K-GDD if $|A| \in K$ for every block $A \in \mathcal{A}$. A $\{k\}$-GDD of type m^k is called a transversal design TD(k, m). It is well known that the existence of a

TD(k, m) is equivalent to the existence of $k - 2$ mutually orthogonal Latin squares of order m.

Now we define the idea of a GDD with a hole. Informally, an incomplete GDD, or IGDD, is a GDD from which a sub-GDD is missing (This is the "hole"). We give a formal definition. An IGDD is a quadruple $(X, Y, \mathcal{G}, \mathcal{A})$ which satisfies the following properties:

1. X is a set of points, and $Y \subseteq X$,
2. $\mathcal{G}$ is a partition of X into groups,
3. $\mathcal{A}$ is a set of blocks, each of which intersect each group in at most one point,
4. no block contains two members of Y, and
5. every pair of points $\{x, y\}$ from distinct groups, where at least one of x, y is in $X \backslash Y$, occurs in a unique block of $\mathcal{A}$.

We say that an IGDD $(X, Y, \mathcal{G}, \mathcal{A})$ is K-IGDD if $|A| \in K$ for every block $A \in \mathcal{A}$. The type of the IGDD is defined to be the multiset of ordered pairs $\{(|G|, |G \cap Y|) : G \in \mathcal{G}\}$. As with GDDs, we shall use an exponential notation to describe types. Note that if $Y = \emptyset$, then the IGDD is a GDD.

For group divisible designs, we have the following existence results.

Lemma 2.1

(1) ([2]) There exists a $TD(6, m)$ for all positive integers $m \geq 5$ and $m \neq 6$, 10, 14, 18, 22.

(2) ([12]) There exists a 4-GDD of type $1^h m^1$ if and only if $h \geq 2m + 1$ and either m, $h + m \equiv 1$ or 4 (mod 12) or m, $h + m \equiv 7$ or 10 (mod 12).

(3) ([5]) There exists a 4-GDD of type $g^4 m^1$ if and only if $g \equiv m \equiv 0 \pmod 3$ and $0 \leq m \leq 3g/2$ except for $(g, m) = (6, 0)$.

(4) ([6]) There exists a 4-GDD of type $6^h m^1$ for each $h \geq 4$ and $m \equiv 0 \pmod 3$ with $0 \leq m \leq 3(h - 1)$ except for $(h, m) = (4, 0)$ and except possibly for $(h, m) \in E$, where

$E = \{(7,15), (11,21), (11,24), (11,27), (13,27), (13,33), (17,39), (17,42), (19,45), (19,48), (19,51), (23,60), (23,63)\}$.

We also need an auxiliary design, specifically (K, α)-frames. Let $(X, \mathcal{G}, \mathcal{A})$ be a K-GDD. A subset $\mathcal{P}$ of $\mathcal{A}$ is called a *partial* α-resolution class with *hole* G if $\mathcal{P}$ is a collection of blocks in which each point of X occurs in precisely α blocks and no block contains any point of G. $(X, \mathcal{G}, \mathcal{A})$ is called a (K, α)-frame if $\mathcal{A}$ can be partitioned into partial α-resolution classes. The *type* of a (K, α)-frame is the same as that of a GDD.

For $(3, 1)$-frames, we have the following existence result.

Lemma 2.2 ([1]) There exists a $(3,1)$-frame of type g^u if and only if $u \geq 4$, g is even and $g(u-1) \equiv 0 \pmod 3$.

We have already defined an α-resolvable STS(v) containing an α-resolvable sub-STS(u). If we allow the subsystem to be missing, we have an incomplete α-resolvable design which we will denote by α-resolvable (v,u)-ISTS. Next, we employ a more general type of incomplete STS. Suppose we have an α-resolvable STS(v) containing an α-resolvable sub-STS(u_1) and an α-resolvable sub-STS(u_2) which intersect in an α-resolvable sub-STS(u_3). If we remove these subsystems, we obtain an incomplete system which we will denote by α-resolvable $(v; u_1, u_2; u_3)$-$\diamond$-ISTS.

The following "Filling in Holes" construction, which is a variant of Stinson's "Filling in Holes" construction, provides a powerful tool for the embeddings of α-resolvable Steiner triple systems.

Construction 2.3 (Filling in Holes) Let $a \geq 0$. Suppose that the following designs exist:

1. a $(3,\alpha)$-frame of type $t_1 t_2 \cdots t_n$,
2. an α-resolvable $(t_i + a, a)$-ISTS, for $1 \leq i \leq n-1$, and
3. an α-resolvable STS$(t_n + a)$.

then there exists an α-resolvable $(t+a, t_n+a)$-ISTS, where $t = \sum_{1 \leq i \leq n} t_i$.

We also use incomplete (K,α)-frames, which bear the same relationship to frames as IGDDs do to GDDs. We also construct STS containing sub-STS by filling in the holes of incomplete frames with $\diamond$-ISTS.

Construction 2.4 (Generalized Filling In Holes) Let $b \geq a \geq 0$. Suppose that the following designs exist:

1. an incomplete $(3,\alpha)$-frame of type $\{(t_1,u_1),(t_2,u_2),\cdots,(t_n,u_n)\}$,
2. an α-resolvable $(t_i + b; u_i + a, b; a)$-$\diamond$-ISTS, for $1 \leq i \leq n-1$, and
3. an α-resolvable $(t_n + b, u_n + a)$-ISTS.

then there exists an α-resolvable $(t+b, u+a)$-ISTS, where $t = \sum_{1 \leq i \leq n} t_i$ and $u = \sum_{1 \leq i \leq n} u_i$.

In applying the "Filling in Holes" construction, we will require more incomplete group divisible designs and incomplete frames. To get these, we use the following recursive constructions.

Construction 2.5 (Fundamental IGDD Construction) ([13]) Suppose $(X, Y, \mathcal{G}, \mathcal{A})$ is an IGDD, and let $t,\, s:\ X \to Z^+ \cup \{0\}$ be functions such that $t(x) \leq s(x)$ for every $x \in X$. For every block $A \in \mathcal{A}$, suppose that we have a K-IGDD of type $\{(s(x), t(x)) : x \in A\}$. Suppose also that we have a K-IGDD of type $\{(\sum_{x \in G \cap Y} s(x), \sum_{x \in G \cap Y} t(x)) : G \in G\}$. Then there exists a K-IGDD of type $\{(\sum_{x \in G} s(x), \sum_{x \in G} t(x)) : G \in G\}$.

Construction 2.6 (Fundamental Frame Construction) ([11]) Suppose $(X, Y, \mathcal{G}, \mathcal{A})$ is an IGDD, and let $s : X \rightarrow Z^+ \cup \{0\}$ be a function. For every block $A \in \mathcal{A}$, suppose that we have a (K, α)-frame of type $\{s(x) : x \in A\}$. Then there exists an incomplete (K, α)-frame of type $\{(\sum_{x \in G} s(x), \sum_{x \in G \cap Y} s(x)) : G \in \mathcal{G}\}$.

Suppose that $K = \{3\}$ and $\alpha = 1$. For each hole of the $(3,1)$-frame in Construction 2.6, combine three partial 1-resolution classes into one partial 3-resolution class. We thus obtain a $(3,3)$-frame of type $\{(\sum_{x \in G} s(x), \sum_{x \in G \cap Y} s(x)) : G \in \mathcal{G}\}$ if $\sum_{x \in G} s(x) \equiv \sum_{x \in G \cap Y} s(x) \equiv 0 \pmod 6$ for every $G \in \mathcal{G}$.

3 The cases $u \geq 121$

In this section, we will prove the existence of 3-resolvable STS(v)s containing 3-resolvable sub-STS(u)s when $u \geq 121$. First we give several designs with small holes.

Lemma 3.1 There exists a 3-resolvable $(3u + 4, u)$-ISTS for $u \equiv 1 \pmod 6$.

Proof Write $u = 6t + 1$ and $t \geq 1$. Start with a KTS(6t+3) and adjoin infinite points to $3t$ 1-resolution classes of the KTS. We get a 4-GDD of type $3^{2t+1}(3t)^1$. Give every point of the resulting GDD weight 2 and apply Construction 2.6, using a $(3,1)$-frame of type 2^4 in Lemma 2.2, to form a $(3,3)$-frame of type $6^{2t+1}(6t)^1$. We then apply Construction 2.3 with such a frame to obtain the desired design.

Lemma 3.2 There exists a 3-resolvable $(3u + 10, u)$-ISTS for $u \equiv 1 \pmod 6$.

Proof Write $u = 6h - 5$ and $h \geq 1$. Start with a 4-GDD of type $6^u(3u - 3)^1$ in Lemma 2.1. Give every point of the GDD weight 2 and apply Construction 2.6. We get a $(3,3)$-frame of type $12^h(6h - 6)^1$. We then apply Construction 2.3 with such a frame to obtain the desired design.

Lemma 3.3 There exists a 3-resolvable $(v, 7)$-ISTS for $v \equiv 1 \pmod 6$ and $v \geq 25$.

Proof Write $u = 6h + 1$ and $h \geq 4$. We apply Construction 2.3 with a $(3,3)$-frame of type 6^h in Lemma 2.2 to obtain the desired design.

Lemma 3.4 There exists a 3-resolvable $(v, 13)$-ISTS for $v \equiv 1 \pmod 6$ and $v \geq 43$.

Proof For $v = 43$, the required design is in Lemma 3.1. For $v = 12h + 7$ and $h \geq 4$, start with a 4-GDD of type $6^h 3^1$ in Lemma 2.1. Give every point of the GDD weight 2 and apply Construction 2.6. We get a $(3,3)$-frame of type $12^h 6^1$. We then apply Construction 2.3 with such a frame to obtain the desired design. For $v = 12h + 1$ and $h \geq 4$, we can apply Construction 2.3 with a $(3,3)$-frame of type 12^h in Lemma 2.2 to obtain the desired design.

Now we present some particular classes of 3-resolvable $\diamond$-ISTSs which play an important role in our constructions.

Lemma 3.5 There exists a 3-resolvable $(18m + 7; 6m + 1, 7; 1)$-$\diamond$-ISTS.

Proof Removing two holes of size 7 and $6t + 1$ from the proof of Lemma 3.1 actually gives a 3-resolvable $(18m + 7; 6m + 1, 7; 1)$-$\diamond$-ISTS.

Lemma 3.6 There exists a 3-resolvable $(18m + 25; 6m + 7, 25; 7)$-$\diamond$-ISTS for $m \geq 3$.

Proof There exists a KTS$(6m + 9)$ containing a sub-KTS(9). Adjoin infinite points to $3m+3$ 1-resolution classes ensuring that all of the blocks of the sub-KTS(9) are included. This produces a 4-GDD of type $3^{2m+3}(3m+3)^1$ containing a subsystem of type 3^4 in which the last group is contained in the group of size $3m + 3$ in the master GDD. Now give every point of this GDD weight 2. By applying Construction 2.6, we get a $(3,3)$-frame of type $6^{2m+3}(6m + 6)^1$ containing a $(3,3)$-frame of type 6^4 as a subsystem in which the last group is contained in the group of size $6m + 6$ in the master incomplete frame. Then add an infinite point, and fill in 3-resolvable STS(7)s and a 3-resolvable $(6m + 7, 7)$-ISTS in which the hole of size 7 is exactly the last group in the sub-frame aligning with the infinite point. This gives a 3-resolvable $(18m+25, 7)$-ISTS containing a 3-resolvable $(25, 7)$-ISTS and a 3-resolvable $(6m + 7, 7)$-ISTS, and these ISTSs have a common hole of size 7. Finally we remove the two sub-ISTSs and obtain the required design.

Similarly, we have the following design.

Lemma 3.7 There exists a 3-resolvable $(18m+43; 6m+13, 43; 13)$-$\diamond$-ISTS for $m \geq 5$.

Lemma 3.8 Suppose there exists a $TD(6,m)$, a 3-resolvable $(18t+3a+4; 6t+a, 3a+4; a)$-$\diamond$-ISTS and a 3-resolvable $(12s+a, a)$-ISTS where $0 \leq t \leq m$, $0 \leq s \leq m$ and $a = 1$, 7 or 13. Then there exists a 3-resolvable (v,u)-ISTS where $v = 72m + 18t + 12s + 3a + 4$, $u = 24m + 6t + a$.

Proof Start with a $TD(6,m)$, delete $m-t$ points from the fifth group and $m-s$ points from the last group. We get a $\{4,5,6\}$-GDD of type $m^4t^1s^1$. Give the points in the first five groups weight $(9,3)$ and the points in the last group weight $(6,0)$, and apply the Fundamental IGDD Construction, using 4-IGDDs of type $(9,3)^4$, $(9,3)^5$, $(9,3)^46^1$ and $(9,3)^56^1$ (whose existence is shown in [11]). We obtain a 4-IGDD of type $(9m,3m)^4(9t,3t)^1(6s)^1$. Next, assign every point of the resulting IGDD weight 2 and apply Construction 2.6. We thus form an incomplete $(3,3)$-frame of type $(18m,6m)^4(18t,6t)^1$ $(12s)^1$. We will fill 3-resolvable $\diamond$-ISTS into the holes of the frame, using Construction 2.4. We adjoin a total of $3a+4$ points, a of which are incorporated into the sub-ISTS. Then we will fill in 3-resolvable $(18m+3a+4; 6m+a, 3a+4; a)$-$\diamond$-ISTSs, a 3-resolvable $(18t+3a+4; 6t+a, 3a+4; a)$-$\diamond$-ISTS and a 3-resolvable $(12s+a, a)$-ISTS, which come from Lemma 3.3-Lemma 3.7. Hence we obtain the required design.

Lemma 3.9 There exists a 3-resolvable (v,u)-ISTS for $v \equiv u \equiv 1 \pmod 6$, $3u+4 \leq v \leq 3u+64$ and $v \geq 121$.

Proof Write $v = 72m + 18t + 12s + 7$ and $u = 24m + 6t + a$. For $v \geq 121$, there exists a $TD(6,m)$ for $m \geq 5$ and $m \neq 6, 10, 14, 18, 22$. Apply Lemma 3.8 with all t and s such that $0 \leq t \leq m$ and $0 \leq s \leq m$. This gives a 3-resolvable (v,u)-ISTS for $3u+4 \leq v \leq 3u+64$ and $v \geq 121$, except that $u = 157$ and 163. For $u = 157$ and 163, take $m = t = 5$, $a = 7$ and 13 respectively in Lemma 3.8. We then obtain the required designs.

Lemma 3.10 Suppose there exists a $TD(6,m)$, a 3-resolvable $(6m+a, a)$-ISTS and a 3-resolvable $(6(m+n_i)+a, a)$-ISTS where $i = 1, 2$ and $0 \leq n_i \leq m$. Then there exists a 3-resolvable (v,u)-ISTS where $v = 36m + 6(n_1+n_2) + a$, $u = 6(m+n_1) + a$.

Proof Start with a $TD(6,m)$. Give each of n_1 points of the fifth group and each of n_2 points of the last group weight 12, give each of the remaining $6m - n_1 - n_2$ points weight 6. In order to apply Fundamental frame Construction, we need $(3,3)$-frames of type 6^6, 6^512^1 and 6^412^2, which are obtained as follow. The first is from Lemma 2.2. We can obtain the second by applying the Fundamental frame Construction to a 4-GDD of type 6^13^5, giving every point weight 2 (this GDD is obtained by adjoining infinite points to 6 1-resolution classes of a KTS(15)). Similarly, we can obtain the last one by applying the Fundamental frame Construction to a 4-GDD of type $6^2 3^4$ (whose existence is shown in [13]) and giving every point weight 2. We then apply Construction 2.3 to the resulting $(3,3)$-frame of type $(6m)^4(6m+6n_1)^1(6m+6n_2)^1$ to obtain the required design.

Lemma 3.11 There exists a 3-resolvable (v,u)-ISTS for $v \equiv u \equiv 1 \pmod 6$, $3u+58 \le v \le 6u+1$ and $u \ge 79$.

Proof Let $T_6 = \{m > 1 : \text{there exists a } TD(6,m)\}$. It follows that for each $v \equiv u \equiv 1 \pmod 6$, $u \ge 79$, there exists a $m_0 \in T_6$ such that $4 \le 30m_0 - 2u \le 52$. Write $u = 6m_0 + 6n_1 + a$, $0 \le n_1 \le m_0$, $a \equiv 1 \pmod 6$ and $a \le 3m_0 - 2$. Then by Lemma 3.10, there exists a 3-resolvable (v,u)-ISTS for all $u \equiv 1 \pmod 6$, $30m_0 + u \le v \le 36m_0 + u$. Further, it is easy to see that there exist positive integers s and $m_i \in T_6$, $0 \le i \le s$, $m_0 < m_1 < \cdots < m_s$, such that $36m_{i-1} + 6 \ge 30m_i$, $0 \le i \le s$ and $6m_s + 1 \le u \le 6m_s + 13$. Let $v_0 = 30m_0 + u$ and $v_1 = 36m_s + u$. Then $3u + 4 \le v_0 \le 3u + 52$ and $v_1 \ge 6u + 1$ and there exists a 3-resolvable (v,u)-ISTS for each $v \equiv 1 \pmod 6$, $v_0 \le v \le v_1$. This completes the proof.

Lemma 3.12 Suppose there exists a $TD(6,m)$, a 3-resolvable $(6m+a,a)$-ISTS and a 3-resolvable $(6n_i + a, a)$-ISTS where $i = 1,2$ and $0 \le n_i \le m$. Then there exists a 3-resolvable (v,u)-ISTS where $v = 24m + 6(n_1+n_2) + a$, $u = 6m + a$ or $u = 6n_i + a$.

Proof Start with a $TD(6,m)$ and delete $m - n_1$ points from the fifth group and $m - n_2$ points from the last group, we get a $\{4,5,6\}$-GDD of type $m^4 {n_1}^1 {n_2}^1$. Give every point of the resulting GDD weight 6 and apply Construction 2.6, using $(3,3)$-frames of type 6^4, 6^5 and 6^6 in Lemma 2.2, to form a $(3,3)$-frame of type $(6m)^4(6n_1)^1(6n_2)^1$. We then apply Construction 2.3 with such a frame to obtain the desired design.

Lemma 3.13 There exists a 3-resolvable (v,u)-ISTS for $v \equiv u \equiv 1 \pmod 6$, $v \ge 5u - 4$ and $u \ge 31$.

Proof Write $m = n_1 + s$ and $u = 6n_1 + 1$ in Lemma 3.12. Then $v = 5u - 4 + 24s + 6n_2$, where $0 \le n_2 \le m$, $s \ge 0$. By Lemma 3.12, there exists a 3-resolvable (v,u)-ISTS for all v such that $v \equiv 1 \pmod 6$ and $v \ge 5u-4$, except that $u = 37, 61, 85, 109$ or 133 and $v = 5u-4, 5u+2, 5u+8$ or $5u+14$.

For $u = 37$, we can first remove $v = 5u - 4$ by applying Construction 2.3 with $a = 1$ to a $(3,3)$-frame of type 36^5 in Lemma 2.2. Next we employ Lemma 3.10 with $a = 7$ to handle each cases corresponding to u. When $u = 37$, take $m = 5$, $n_1 = 0$ and $n_2 = 0, 1, 2$ respectively; when $u = 61$, take

$m = 8$, $n_1 = 2$ and $n_2 = 0, 1, 2, 3$ respectively; when $u = 85$, take $m = 11$, $n_1 = 3$ and $n_2 = 1, 2, 3, 4$ respectively; when $u = 109$, take $m = 13$, $n_1 = 5$ and $n_2 = 7, 8, 9, 10$ respectively; when $u = 133$, take $m = 17$, $n_1 = 5$ and $n_2 = 3, 4, 5, 6$ respectively.

This completes the proof.

Combine Lemma 3.9, Lemma 3.11 and Lemma 3.13, we have established the following result.

Theorem 3.14 Let $v \equiv u \equiv 1 \pmod 6$ and $v \geq 3u + 4$, then there exists a 3-resolvable STS(v) containing a 3-resolvable sub-STS(u) for $u \geq 121$.

4 The cases $u < 121$

In this section, we will prove the existence of 3-resolvable STS(v)s containing 3-resolvable sub-STS(u)s when $u < 121$. First we give several infinite classes.

Lemma 4.1 There exists a 3-resolvable (v, u)-ISTS for $v \geq 3u + 4$ and either v, $u \equiv 1$ or $7 \pmod{24}$ or v, $u \equiv 13$ or $19 \pmod{24}$.

Proof Start with a 4-GDD of type $1^h m^1$ in Lemma 2.1, where $h \geq 2m + 1$ and either m, $h + m \equiv 1$ or $4 \pmod{12}$ or m, $h + m \equiv 7$ or $10 \pmod{12}$. Delete one point from the group of size m. We get a 4-GDD of type $3^{h/3}(m-1)^1$. Give every point of the resulting GDD weight 2 and apply Construction 2.6. We form a (3,3)-frame of type $6^{h/3}(2m-2)^1$. We then apply Construction 2.3 with such a frame to obtain the desired 3-resolvable $(2h + 2m - 1, 2m - 1)$-ISTS.

Lemma 4.2 There exists a 3-resolvable (v, u)-ISTS for $v \geq 3u + 10$ and either v, $u \equiv 1 \pmod{12}$ or v, $u \equiv 7 \pmod{12}$, except possibly for $(v, u) \in F_1$, where

$$F_1 = \{(12h + 2m + 1, 2m + 1) : (h, m) \in E\}.$$

Proof Start with a 4-GDD of type $6^h m^1$ in Lemma 2.1, where $h \geq 4$ and $m \equiv 0 \pmod 3$ with $0 \leq m \leq 3(h-1)$ except for $(h, m) \in E \cup \{(4, 0)\}$. Give every point of the GDD weight 2 and apply Construction 2.6. We form a (3,3)-frame of type $12^h(2m)^1$. We then apply Construction 2.3 with such a frame to obtain the desired 3-resolvable $(12h + 2m + 1, 2m + 1)$-ISTS.

Combine Lemma 4.1 with Lemma 4.2, we have established the following result.

Lemma 4.3 There exists a 3-resolvable (v, u)-ISTS for $v \equiv u \equiv 1 \pmod 6$

and $v = 3u + k,\ k \equiv 4,\ 10,\ 22 \pmod{24}$ except possibly for $(v, u) \in F_1$.

For our purpose, we only need to consider the values in F_1 and the cases $v = 3u + k,\ k \equiv 16 \pmod{24}$.

Lemma 4.4 There exists a 3-resolvable (v, u)-ISTS for $v \equiv u \equiv 1 \pmod{6}$ and $4u - 3 \leq v \leq (11u - 9)/2$.

Proof Start with a 4-GDD of type g^4m^1 in Lemma 2.1, where $g \equiv m \equiv 0 \pmod{3}$ and $0 \leq m \leq 3g/2$ except for $(g, m) = (6, 0)$. Give every point of the GDD weight 2 and apply Construction 2.6. We form a (3,3)-frame of type $(2g)^4(2m)^1$. We then apply Construction 2.3 with such a frame to obtain the desired 3-resolvable $(8g + 2m + 1, 2g + 1)$-ISTS.

Combine Lemma 4.4, Lemma 3.9, Lemma 3.11 and Lemma 3.13. For the cases $v = 3u+k,\ k \equiv 16 \pmod{24}$, we only need to consider the values in F_2, where

$$F_2 = \{(3u + k, u) : k = 16, 40,\ u \equiv 1 \pmod{6} \text{ and } 49 \leq u \leq 115\} \cup \{(3u + 16, u) : u = 31, 37, 43\} \cup \{(283, 73)\}$$

Lemma 4.5 There exists a 3-resolvable $(18h + 6s + 1, 6h + 1)$-ISTS for $h \equiv 0 \pmod{4}$, $h \geq 8$ and $0 \leq s \leq h - 1$.

Proof Start with a 4-RGDD of type 3^h (whose existence is shown in [3]), where $h \equiv 0 \pmod{4}$ and $h \geq 8$. Adjoin infinite points to s of the 1-resolution classes of the GDD. We get a $\{4, 5\}$-GDD of type 3^hs^1 in which every block of size 5 hits the group of size s, where $0 \leq s \leq h - 1$. Assign weight (3,1) to every point of the original GDD, and assign weight (3,0) to the s infinite points. Apply Construction 2.5, using 4-IGDD of types $(3, 1)^4(3, 0)^1$ and $(3, 1)^4$ (these arise from delete a block from 4-GDDs of type 3^4 and 3^5, respectively), to get a 4-IGDD of type $(9, 3)^h(3s)^1$ and then a $\{4, 3s + 1\}$-GDD of type $3^{2h+s}(3h)^1$. Give every point of the resulting GDD weight 2 and apply Construction 2.6. We form a (3,3)-frame of type $6^{2h+s}(6h)^1$. We then apply Construction 2.3 with such a frame to obtain the desired 3-resolvable $(18h + 6s + 1, 6h + 1)$-ISTS.

Lemma 4.6 There exists a 3-resolvable $(24h + 6s + 1, 8h - 1)$-ISTS for $h \equiv 1 \pmod{3}$, $h \geq 4$ and $0 \leq s \leq 4(h - 1)/3$.

Proof Start with a 4-RGDD of type 4^h (whose existence is shown in [3]), where $h \equiv 1 \pmod{3}$ and $h \geq 4$. Adjoin infinite points to s of the 1-resolution classes of the GDD. We get a $\{4, 5\}$-GDD of type 4^hs^1 in which every block of size 5 hits the group of size s, where $0 \leq s \leq h - 1$. Assign weight (3,1) to every point of the original GDD, and assign weight (3,0) to the s infinite points. Apply Construction 2.5, using 4-IGDD of types

$(3,1)^4(3,0)^1$ and $(3,1)^4$, to get a 4-IGDD of type $(12,4)^h(3s)^1$ and then a $\{4,3s+1\}$-GDD of type $3^{(8h+3s+1)/3}(4h-1)^1$. Give every point of the resulting GDD weight 2 and apply Construction 2.6. We form a (3,3)-frame of type $6^{(8h+3s+1)/3}(8h-2)^1$. We then apply Construction 2.3 with such a frame to obtain the desired 3-resolvable $(24h+6s+1, 8h-1)$-ISTS.

From Lemma 4.5 and Lemma 4.6, we have the following result.

Lemma 4.7 There exists a 3-resolvable (v,u)-ISTS for $(v,u) \in \{(3u+k,u) : k = 16,40,\ u = 49,55,73,79,97,103\} \cup \{(3u+22,u) : u = 31,55,103\} \cup \{(3u+34,u) : u = 49,97\} \cup \{3u+46,u) : u = 55,79\} \cup \{(109,31),(283,73)\}$.

Proof Apply Lemma 4.5 with $h = 8,12,16$ and $s = 3,6,7$ or 11, we know that the result is true for $u = 49,73$ and 97. For the other values of u, we apply Lemma 4.6 with $h = 4,7,10,13$ and $s = 2,3,6$ or 7.

Lemma 4.8 Suppose there exists a $TD(6,m)$, a 3-resolvable $(6m+a,a)$-ISTS and a 3-resolvable $(6s+a,a)$-ISTS, where $m \le s \le 2m$. Then there exists a 3-resolvable $(36m+6s+a, 12m+a)$-ISTS.

Proof Start with the $TD(6,m)$ and give every point of the last group weight 6, every point of the second last group weight 3 or 6, and every point of the remaining groups weight 3, using 4- GDDs of type $3^4 6^2$ and $3^5 6^1$ (whose existence, see [2]), we get a 4-GDD of type $(3m)^4(6m)^1(3s)^1$. Give every point of the resulting GDD weight 2 and apply Construction 2.6. We form a (3,3)-frame of type $(6m)^4(12m)^1(6s)^1$. We then apply Construction 2.3 with such a frame to obtain the desired 3-resolvable $(36m+6s+a, 12m+a)$-ISTS.

Lemma 4.9 There exists a 3-resolvable (v,u)-ISTS for $(v,u) \in F_1 \cup F_2$.

Proof For the case $(v,u) = (175,43)$, see Lemma 4.4. For the cases $(v,u) = (397,121),(403,127)$, see Theorem 3.14. For the case $(v,u) = (127,37)$, we apply Construction 2.3 with a (3,3)-frame of type 30^4 in Lemma 2.2 to obtain the desired design.

For the case $(v,u) = (142,43)$, start with a $\{4,7\}$-GDD of type $3^{14}9^1 21^1$ (whose existence is shown in [11]). Give every point of the GDD weight 2 and apply Construction 2.6. We form a (3,3)-frame of type $6^{14}18^1 42^1$. We then apply Construction 2.3 with such a frame to obtain the desired design.

For the case $(v,u) = (199,61)$, start with a 4-frame of type 6^5 (whose existence is shown in [11]) and adjoin two infinite points to a hole of the frame, we get a $\{4,5\}$-IGDD of type $6^4(8,2)^1$. Assign weight (3,1) to every point of the original GDD and assign weight (3,0) to the two infinite points, we obtain a 4-IGDD of type $(18,6)^4(24,6)^1$, using 4-IGDD of types $(3,1)^4(3,0)^1$ and $(3,1)^4$. Give every point of the resulting IGDD weight 2 and apply Construction 2.6. We form an infinite (3,3)-frame of type $(36,12)^4(48,12)^1$. We then apply Construction 2.4 with such a frame, using 3-resolvable $(43;13,7;1)$-$\diamond$-ISTS in Lemma 3.5, to obtain the desired design.

For the case $(v,u)=(271,85)$, start with a $\{4,5\}$-IGDD of type $(9,3)^4$ $(8,2)^1$ (whose existence is shown in [11]), give every point of the IGDD weight 6 and apply Construction 2.6, we form an incomplete (3,3)-frame of type $(54,18)^4(48,12)^1$. We then apply Construction 2.4 with such a frame, using 3-resolvable $(61;19,7;1)$-$\diamond$-ISTS in Lemma 3.5, to obtain the desired design.

For the case $(v,u)=(289,85)$, start with a $\{4,7\}$-GDDs of type $3^{14}9^1 21^1$ (whose existence is shown in [11]). Give every point of the GDD weight 4 and apply Construction 2.6. We form a (3,3)-frame of type $12^{14}36^1 84^1$. We then apply Construction 2.3 with such a frame to obtain the desired design.

For the cases $(v,u)=(289,91),(343,109)$, start with 4-GDDs of type $9^{11}45^1$, 39^4 (whose existence, the first comes from a 3-RGDD of type 9^{11} and adjoin infinite points to 45 1-resolution classes of the GDD, the second see Lemma 2.1). Give every point of these GDDs weight 2 and apply Construction 2.6. We form (3,3)-frames of type $18^{11}90^1$, 78^4, respectively. We then apply Construction 2.3 with such frames to obtain the desired designs.

For the cases $(v,u)=(223,61),(217,67),(223,67),241,67)$, we can apply Lemma 3.10 with $m=n_1=5$ and $(n_2,a)=(2,1),(0,7),(1,7),(4,7)$ to obtain the desired designs.

From Lemma 4.7, we only need to consider the cases $(v,u)=(313,91)$, $(319,91),(367,109),(361,115),(385,115)$, which can be obtained by applying Lemma 4.8 with $(m,s,a)=(7,9,7),(7,10,7),(8,11,13),(8,9,19),(8,13,19)$, respectively. The required 3-resolvable $(67,19)$-ISTS and 3-resolvable $(6s+19,19)$-ISTSs for $s=9$ and 13 are provided in Lemma 4.11 below.

This completes the proof.

Up to now, we have established the following result.

Theorem 4.10 Let $v\equiv u\equiv 1 \pmod 6$ and $v\geq 3u+4$, then there exists a 3-resolvable STS(v) containing a 3-resolvable sub-STS(u) for $31\leq u<121$.

Lemma 4.11 There exists a 3-resolvable $(v,19)$-ISTS for $v\equiv 1 \pmod 6$ and $v\geq 61$.

Proof From Lemma 4.3, we only need to consider the cases $v=57+k$, $k\equiv 16 \pmod{24}$. Note that there exists a 3-resolvable $(v,61)$-ISTS for $v\equiv 1 \pmod 6$ and $v\geq 187$, we only need to consider the cases

$v = 73, 97, 121, 145, 169$.

For the cases $v = 73, 145$, we apply Construction 2.3 with (3,3)-frames of type 18^4, 18^8 in Lemma 2.2 to obtain the desired designs. For the case $v = 97$, start with a 4-GDD of type $9^4 12^1$ in Lemma 2.1. Give every point of the GDD weight 2 and apply Construction 2.6. We form a (3,3)-frame of type $18^4 24^1$. We then apply Construction 2.3 with such a frame to obtain the desired design. For the case $v = 121$, start with a $TD(5, 5)$ and truncate two groups to sizes 4 and 1, we get a $\{4, 5\}$-GDD of type $3^4 4^2$. Give every point of the resulting GDD weight 6 and apply Construction 2.6. We form a (3,3)-frame of type $18^4 24^2$. We then apply Construction 2.3 with such a frame to obtain the desired design. For the case $v = 169$, start with a $TD(6, 5)$ and truncate one group to size 3, we get a $\{5, 6\}$-GDD of type $5^5 3^1$. Give every point of the resulting GDD weight 6 and apply Construction 2.6. We form a (3,3)-frame of type $30^5 18^1$. We then apply Construction 2.3 with such a frame to obtain the desired design.

Lemma 4.12 There exists a 3-resolvable $(v, 25)$-ISTS for $v \equiv 1 \pmod 6$ and $v \geq 79$.

Proof From Lemma 4.3, we only need to consider the cases $v = 75 + k$, $k \equiv 16 \pmod{24}$. Note that there exists a 3-resolvable $(v, 79)$-ISTS for $v \equiv 1 \pmod 6$ and $v \geq 247$, we only need to consider the cases $v = 91, 115, 139, 163, 187, 211, 235$.

For the case $v = 91$, start with a 4-GDD of type $3^9 6^1 12^1$ (whose existence is shown in [10]). Give every point of the GDD weight 2 and apply Construction 2.6. We form a (3,3)-frame of type $6^9 12^1 24^1$. We then apply Construction 2.3 with such a frame to obtain the desired design. For the other values of v, start with 4-GDDs of type $12^h m^1$ (whose existence is shown in [4]), where $m = 9$ and $4 \leq h \leq 9$. Give every point of the GDDs weight 2 and apply Construction 2.6. We form (3,3)-frames of type $24^h (2m)^1$. We then apply Construction 2.3 with such frames to obtain the desired designs.

We are now in a position to prove Theorem 1.3.

The proof of Theorem 1.3 From Theorem 3.14 and Theorem 4.10, we know that the result is true for $u \geq 31$. For the cases $7 \leq u < 31$, we know that the result is true from Lemma 3.3, Lemma 3.4, Lemma 4.11 and Lemma 4.12.

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