An Upper Bound for the k-Tuple Domination Number

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Abstract

We show that the double domination number of an n-vertex, isolate-free graph with minimum degree δ is bounded above by $n(\ln(\delta+1)+\ln\delta+1)/\delta$. This result improves a previous bound obtained by J. Harant and M. A. Henning [On double domination in graphs, *Discuss. Math. Graph Theory* **25** (2005), 29-34]. Further, we show that for fixed k and large δ the k-tuple domination number is at most $n(\ln\delta+(k-1+o(1))\ln\ln\delta)/\delta$, a bound that is essentially best possible.

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1 Introduction

A vertex subset D of a (simple) graph G is *k-tuple dominating* if the closed neighbourhood of each vertex contains at least k vertices in D; that is, each vertex in D has at least $k-1$ neighbours in D and each vertex outside D has at least k neighbours in D. Such sets exist only for graphs of minimum degree at least $k-1$. The *k-tuple domination number* of G, denoted by $\gamma_{\times k}(G)$, is the smallest cardinality of a k-tuple dominating set of G.

The study of k-tuple domination was initiated by Harary and Haynes [3]. References to subsequent work on this topic may be found in [2, 5]. For background on domination the reader is referred to [4].

For the case $k = 2$, when k-tuple domination is also called *double domination*, Harant and Henning [2] proved that $\gamma_{\times 2}(G) \leq n(\ln(\overline{d}+1) + \ln \delta + 1)/\delta$ for a graph G of order n, minimum degree $\delta \geq 1$ and average degree $\overline{d}$. Here we improve this bound, replacing $\overline{d}$ by δ. We also give a bound on $\gamma_{\times k}(G)$ that is asymptotically correct.

2 The bound

Here is the main result of the paper.

Theorem 1 *Let G be an n-vertex graph of minimum degree δ.*

- *If $\delta \geq 1$, then $\gamma_{\times 2}(G) \leq n(\ln(\delta+1) + \ln \delta + 1)/\delta$.*
- *If k is fixed and δ is large, then $\gamma_{\times k}(G) \leq n(\ln \delta + (k-1+o(1)) \ln \ln \delta)/\delta$.*

It is easily checked, by considering suitable random graphs, that the ordinary domination number (equivalent to $\gamma_{\times 1}$) can be at least of order $(n/\delta) \ln \delta$ (see Alon [1]), and so in this sense the second bound in the theorem is tight.

The theorem follows from the next lemma, which has a probabilistic proof, as do those given in [1] and [2].

Lemma 2 *Let G be a graph of minimum degree $\delta \geq k-1$ and let $0 \leq p \leq 1$. Then*

$$\gamma_{\times k}(G) \leq n \left\{ p + \sum_{i=0}^{k-1} (k-i) \binom{\delta+1}{i} p^i (1-p)^{\delta+1-i} \right\}.$$

Proof. For each vertex $v \in G$, pick a set N_v comprising v and δ of its neighbours, so $|N_v| = \delta + 1$. Any set D of vertices with $|D \cap N_v| \geq k$ for every v is then k-tuple dominating. Form a random subset X of the vertices of G by independently placing each vertex into X with probability p. Now define the sets $V_i = \{v \in G : |N_v \cap X| = i\}$ for $0 \leq i \leq k-1$. Form the set X_i by placing within it $k-i$ members of $N_v \setminus X$ for each $v \in V_i$. Note that $|X_i| \leq (k-i)|V_i|$. Now the set $D = X \cup \bigcup_{i=0}^{k-1} X_i$ is a k-tuple-dominating set. Since there is a choice of X for which $|D|$ has at most its expected value $E(|D|)$, it follows that $\gamma_{\times k}(G) \leq E(|D|)$, and by the linearity of expectation we have

$$\gamma_{\times k}(G) \leq E(|D|) \leq E(|X|)+\sum_{i=0}^{k-1} E(|X_i|) \leq E(|X|)+\sum_{i=0}^{k-1}(k-i)E(|V_i|) . \quad (1)$$

If M is a random subset of a finite set N, then the expectation of the random variable $|M|$ is given by $E(|M|) = \sum_{j\in M} P(j \in M)$. Now for each vertex $v \in G$, we have $P(v \in X) = p$, $P(v \in V_i) = \binom{\delta+1}{i}p^i(1-p)^{\delta+1-i}$. We conclude that

$$E(|X|) \leq np \quad \text{and} \quad E(|V_i|) \leq n\binom{\delta + 1}{i}p^i(1-p)^{\delta+1-i} ,$$

which together with (1) completes the proof of the lemma. ■

Proof of Theorem 1. Since $1 - p \leq e^{-p}$, Lemma 2 implies

$$\gamma_{\times 2}(G) \leq np + n(2 + p(\delta - 1))(1 - p)^{\delta} \leq np + n(1 + \delta)e^{-p\delta} ,$$

and taking $p = (\ln(\delta+1)+\ln \delta)/\delta$ we obtain the first bound of the theorem.

For the second bound, let $\epsilon > 0$ and $p = (\ln \delta + (k - 1 + \epsilon) \ln \ln \delta)/(\delta - k + 2)$. Then from Lemma 2 we obtain

$$\begin{aligned}\gamma_{\times k}(G) &\leq n \left(p + k\sum_{i=0}^{k-1}((\delta + 1)p)^i(1 - p)^{\delta+1-i}\right) \\ &\leq n \left(p + k^2((\delta + 1)p)^{k-1}e^{-p(\delta-k+2)}\right) ,\end{aligned}$$

and since $((\delta+1)p)^{k-1}e^{-p(\delta-k+2)} = (1+o(1))(\ln \delta)^{k-1}(\ln \delta)^{-(k-1+\epsilon)}\delta^{-1} < \epsilon/\delta$ if δ is large, we have $\gamma_{\times k}(G) \leq n(p + k^2\epsilon/\delta)$. Now this holds for any given $\epsilon > 0$ provided δ is large, so the theorem is proved. ■

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