Invariants of Fibonacci Graphs

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Abstract

The Fibonacci graph G_n is the graph whose vertex set is the collection of n-bit binary strings having no contiguous ones, and two vertices are adjacent if and only if their Hamming distance is one. Values of several graphical invariants are determined for these graphs, and bounds are found for other invariants.

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1 Introduction

The *Fibonacci graph* G_n, as defined by Munarini [4], is the graph whose vertex set is the collection of n-bit binary strings having no contiguous ones, and two vertices are adjacent if and only if their Hamming distance is one. Munarini studied the relations among Fibonacci graphs, Fibonacci semilattices, and Fibonacci lattices. In this paper we determine the values of several invariants for Fibonacci graphs and provide bounds for others. This kind of information for specific families of graphs can prove useful when developing and testing algorithms for general graphs. Graph theoretical concepts and results can be explored in Chartrand and Zhang [2], Gross and Yellen [3], and West [6].

The structure of G_n can be expressed in terms of the graphs G_{n-2} and G_{n-1}. To see this for $n \geq 3$, notice the vertex set of G_n can be partitioned into those vertices that end in 0 and those that end in 01. Moreover, the first $n-1$ bits that precede the 0 and the first $n-2$ bits that precede the 01 are labels of vertices in G_{n-1} and G_{n-2}, respectively. Thus G_{n-1} and G_{n-2}

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appear as disjoint induced subgraphs of G_n. Vertices $x_1x_2\ldots x_{n-1}0$ and $x_1x_2\ldots x_{n-2}01$ in G_n are said to be *lifts* of vertex $x_1x_2\ldots x_{n-1}$ in G_{n-1} and vertex $x_1x_2\ldots x_{n-2}$ in G_{n-2}, respectively. The vertices of the form $x_1x_2\ldots x_{n-2}00$ in the G_{n-1} subgraph of G_n induce a second copy of G_{n-2}. Since the vertex $x_1x_2\ldots x_{n-2}00$ is adjacent to the vertex $x_1x_2\ldots x_{n-2}01$, there is a matching M between the two copies. The structure of G_n is summarized as follows.

Remark 1 *The Fibonacci graph G_n contains G_{n-1} and G_{n-2} as disjoint induced subgraphs, and there is a matching M between the G_{n-2} and a second copy of G_{n-2} appearing as an induced subgraph of the G_{n-1}.*

The above inductive construction is the basis for several of the results in this paper and is depicted in Figure 1.

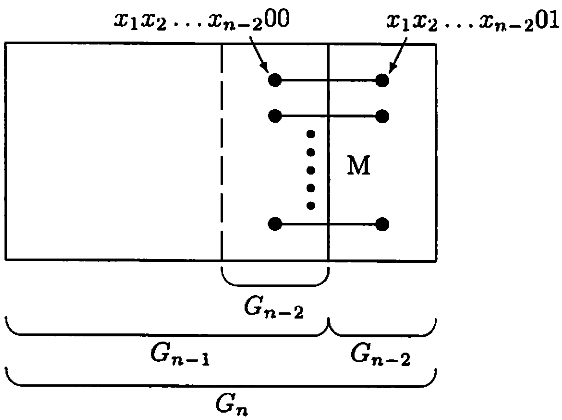

Figure 1: Structure of Fibonacci graph G_n

The following observation is apparent from the structure of Fibonacci graphs.

Observation 2 *Let $n \geq 3$. Then*

a. $|V(G_n)| = |V(G_{n-1})| + |V(G_{n-2})|$, *and*

b. $|E(G_n)| = |E(G_{n-1})| + |E(G_{n-2})| + |V(G_{n-2})|$.

The next observation follows immediately from the fact that any edge of G_n joins a vertex having an even number of 1's with a vertex having an odd number of 1's.

Observation 3 *The Fibonacci graph G_n is bipartite.*

Section 2 provides expressions for the order and size of Fibonacci graphs, Section 3 gives values for several graphical invariants, and Section 4 presents some open questions.

2 Order and Size

Observation 2a states that $|V(G_n)|$ satisfies the Fibonacci recursion. For the initial conditions, note that $V(G_1) = \{0,1\}$ and $V(G_2) = \{00, 01, 10\}$. Thus, if F_n is the n^{th} Fibonacci number, starting with $F_1 = F_2 = 1$, then $|V(G_1)| = F_3$ and $|V(G_2)| = F_4$. The following observation is now immediate.

Observation 4 *For the Fibonacci graph G_n, $|V(G_n)| = F_{n+2}$.*

The situation is not quite as straightforward for the number of edges of G_n. Three expressions are derived, one combinatorial in nature, one in terms of Fibonacci numbers, and a closed form. The combinatorial expression is a consequence of the following result.

Proposition 5 *The number of vertices of the Fibonacci graph G_n that have exactly k 1's is $\binom{n-k+1}{k}$ for $k \leq \lceil n/2 \rceil$.*

Proof: Every n-bit binary string with k 1's, no two adjacent, can be constructed by (1) starting with a string of length $2k - 1$ of the form $10101\ldots101$, and (2) distributing the remaining $n - 2k + 1$ 0's into the $k + 1$ zones delineated by the k 1's. This is a standard combinatorial problem (see Roberts [5], page 42), and the number of ways to do this is $\binom{k+1+(n-2k+1)-1}{n-2k+1} = \binom{n-k+1}{n-2k+1} = \binom{n-k+1}{k}$. □

Using Proposition 5 it is possible to obtain an expression for the number of edges in G_n. It is based on the observation that a vertex with k 1's, $k \geq 1$, has exactly k neighbors with $k - 1$ 1's.

Corollary 6 *For the Fibonacci graph G_n, $|E(G_n)| = \sum_{k=1}^{\lceil \frac{n}{2} \rceil} k\binom{n-k+1}{k}$.*

An alternative expression for $|E(G_n)|$ is based on the recurrence of Observation 2b. For convenience, let E_n denote $|E(G_n)|$. With this notation, $E_1 = F_2 = 1$ and $E_2 = F_3 = 2$.

Proposition 7 *For the Fibonacci graph G_n, $|E(G_n)| = \sum_{i=1}^{n} F_i F_{n+1-i}$.*

Proof: We use induction on n. The facts that $E_1 = 1$, $\sum_{i=1}^{1} F_i F_{n+1-i} = F_1 F_1 = 1$, $E_2 = 2$, and $\sum_{i=1}^{2} F_i F_{n+1-i} = F_1 F_2 + F_2 F_1 = 2$ establishes the base case. Now assume that the result holds for all values of the index up to some $n \geq 1$. By Observaation 2b and Observation 4, $E_{n+1} = E_n + E_{n-1} + F_{n+1}$. Hence, the inductive hypothesis yields

$$\begin{aligned} E_{n+1} &= \sum_{i=1}^{n} F_i F_{n+1-i} + \sum_{i=1}^{n-1} F_i F_{n-i} + F_{n+1} \\ &= \sum_{i=1}^{n-1} (F_i F_{n+1-i} + F_i F_{n-i}) + F_n F_1 + F_{n+1}. \end{aligned}$$

Since $F_1 = F_2 = 1$,

$$\begin{aligned} E_{n+1} &= \sum_{i=1}^{n-1} F_i (F_{n+1-i} + F_{n-i}) + F_n F_2 + F_{n+1} F_1 \\ &= \sum_{i=1}^{n-1} F_i F_{n+2-i} + (F_n F_2 + F_{n+1} F_1) \\ &= \sum_{i=1}^{n+1} F_i F_{n+2-i}, \end{aligned}$$

as desired. □

Solving the recurrence given in Observation 2b, the following closed form for E_n can be obtained.

Theorem 8 *For the Fibonacci graph G_n,*

$$E_n = \frac{2}{5\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right] + \frac{n}{5} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} + \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right]$$

3 Invariant Properties

Various properties and values of invariants of Fibonacci graphs are developed in this section. The first indicates, unsurprisingly, that most Fibonacci graphs are nonplanar.

Proposition 9 *The Fibonacci graph G_n is planar if and only if $n \leq 5$.*

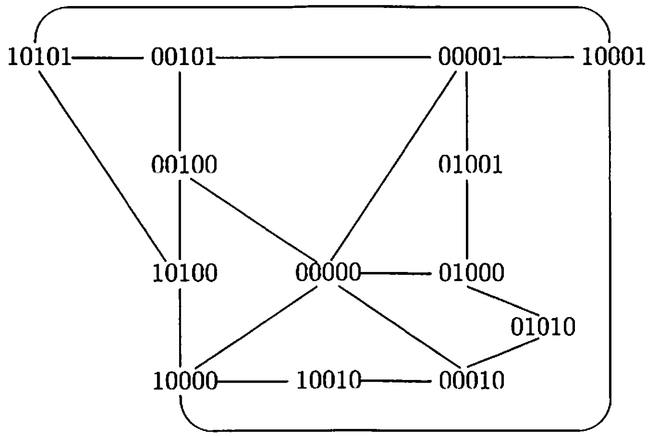

Figure 2: Planar representation of G_5

Proof: Figure 2 gives a planar representation of G_5. Since G_n is contained in G_{n+1}, it follows that G_i is planar for $1 \le i \le 5$. The same inclusion will show the nonplanarity of the remaining Fibonacci graphs once it is shown that G_6 is nonplanar. For that we show that a homeomorphic copy of $K_{3,3}$ exists in G_6. The partite sets are $\{100000, 010000, 000001\}$ and $\{000000, 101010, 010101\}$. Vertex 000000 is adjacent to all the vertices in the first set. Vertex 101010 is adjacent to them, in order, by paths having the following internal vertices: $\langle 101000\rangle$, $\langle 100010, 000010, 010010\rangle$, and $\langle 001010, 001000, 001001\rangle$. Vertex 010101 is adjacent to them, in order, by paths having the following internal vertices: $\langle 010100, 000100, 100100\rangle$, $\langle 010001\rangle$, and $\langle 000101\rangle$. □

Let $r(G)$ and $d(G)$ be the *radius* and *diameter*, respectively, of graph G.

Proposition 10 *For the Fibonacci graph G_n, $d(G_n) = n$ and $r(G_n) = \lceil \frac{n}{2} \rceil$.*

Proof: Clearly $d(G_n) \le n$. Vertices 1010... and 0101... exist in G_n and have distance n. Since $d(G) \le 2r(G)$ for any graph, $r(G_n) \ge \lceil \frac{n}{2} \rceil$. Equality follows from the fact that the vertex of all 0's is within distance $\lceil \frac{n}{2} \rceil$ of every other vertex. □

We next determine minimum degree, denoted $\delta(G)$ for graph G, which will then lead to results about connectivity. The following lemma provides a starting point.

Lemma 11 *Let x be a vertex with k 1's. Then*

$$deg(x) \geq \begin{cases} k & \text{if } n \leq 3k \\ n-2k & \text{if } n > 3k \end{cases},$$

and the bound is achieved for some such vertex.

Proof: Vertex x has k neighbors with $k-1$ 1's and a number of neighbors with $k+1$ 1's equal to the number of 0's in x that can be changed to 1. This value is minimized when $x = 010010010\ldots010000\ldots000$ in which the pattern "010" is repeated k times followed by $n-3k$ 0's. In this case the number of neighbors with $k+1$ 1's is $n-3k$ if $n > 3k$. If $n \leq 3k$, there are no neighbors with $k+1$ 1's. The result follows. □

Corollary 12 *For the Fibonacci graph G_n, $\delta(G_n) = \lceil \frac{n}{3} \rceil$.*

Proof: From Lemma 11, there is a vertex of degree k if $n \leq 3k$, that is, if $k \geq \lceil \frac{n}{3} \rceil$. Thus, the minimum degree for all vertices having at least $\lceil \frac{n}{3} \rceil$ 1's equals $\lceil \frac{n}{3} \rceil$. Suppose there is a $k < \lceil \frac{n}{3} \rceil$ such that $n-2k < \lceil \frac{n}{3} \rceil$, that is, $2k > n - \lceil \frac{n}{3} \rceil = \lfloor \frac{2n}{3} \rfloor$ so $2k \geq \lfloor \frac{2n}{3} \rfloor + 1$ or $k \geq \frac{1}{2} \lfloor \frac{2n}{3} \rfloor + \frac{1}{2} = \frac{1}{2} \lceil \frac{2n-2}{3} \rceil + \frac{1}{2} \geq \frac{2n-2+3}{6} = \frac{2n+1}{6}$. This implies $k > \frac{n}{3}$, and hence, $k \geq \lceil \frac{n}{3} \rceil$. In other words, there is no value of $k < \lceil \frac{n}{3} \rceil$ that gives a smaller number. □

Let $\kappa_v(G)$ be the vertex connectivity of graph G and $\kappa_e(G)$ be its edge connectivity. It is well-known that $\kappa_v(G) \leq \kappa_e(G) \leq \delta(G)$ for any graph G. The next theorem shows these three invariants have the same value for Fibonacci graphs.

Theorem 13 *For the Fibonacci graph G_n, $\kappa_v(G_n) = \kappa_e(G_n) = \delta(G_n) = \lceil \frac{n}{3} \rceil$.*

Proof: We proceed by induction on n, the result easily checked for $1 \leq n \leq 6$. Corollary 12 shows that $\delta(G_n) = \lceil \frac{n}{3} \rceil$. The remainder of the proof relies on the structure described in Remark 1. Let $n \geq 7$ and S be a minimum disconnecting set of vertices of G_n. We consider three cases.

1. $S \cap V(G_{n-1})$ does not disconnect the G_{n-1} and $S \cap V(G_{n-2})$ does not disconnect the G_{n-2}. It follows that all edges of M must be removed so that G_n is disconnected by S, that is, S must include at least one vertex incident with each edge of M. Thus $|S| \geq |M| = |V(G_{n-2})| = F_n > \lceil \frac{n}{3} \rceil$.

2. $S \cap V(G_{n-2})$ disconnects G_{n-2}. It follows, by the inductive hypothesis, that $|S \cap V(G_{n-2})| \geq \lceil \frac{n-2}{3} \rceil$. Now the two or more components of G_{n-2} must not be joined using edges of M and paths in the G_{n-1} subgraph. Thus S must contain at least one vertex other than those separating the G_{n-2}, so $|S| \geq \lceil \frac{n-2}{3} \rceil + 1 = \lceil \frac{n+1}{3} \rceil \geq \lceil \frac{n}{3} \rceil$.

3. $S \cap V(G_{n-1})$ disconnects G_{n-1}. There are two subcases to consider.

   (a) Every component remaining in G_{n-1} contains a vertex incident to an edge of M. Then an argument similar to that of Case 2 can be used to show $|S| \geq \lceil \frac{n+2}{3} \rceil \geq \lceil \frac{n}{3} \rceil$.

   (b) A component remaining in the G_{n-1} lies entirely within the G_{n-3} embedded in the G_{n-1}. Note that this G_{n-3} is joined to another copy of G_{n-3} in the G_{n-1} by a matching M'. If S does not separate the first mentioned G_{n-3}, this component must be separated from the rest of G_{n-1} by removing a vertex incident with each edge of M'. Thus $|S| \geq |V(G_{n-3})| = F_{n-1} > \lceil \frac{n}{3} \rceil$. Otherwise S separates the G_{n-3}. An argument similar to that of Case 2 shows that at least one vertex must be in S to prevent the component in the G_{n-3} from being joined via the edges of M' to vertices outside the G_{n-3}. Thus at least $\lceil \frac{n-3}{3} \rceil$ vertices are required to separate the G_{n-3} and $|S| \geq \lceil \frac{n-3}{3} \rceil + 1 = \lceil \frac{n}{3} \rceil$.

In all cases, $|S| \geq \lceil \frac{n}{3} \rceil$. Since $\kappa_e(G_n) \leq \delta(G_n) = \lceil \frac{n}{3} \rceil$, the result follows. □

Because G_n is bipartite, the vertex chromatic number $\chi_v(G_n) = 2$. The next result shows that the edge chromatic number $\chi_e(G_n) = \Delta(G_n)$, where $\Delta(G_n)$ is the maximum degree of G_n.

Proposition 14 *For the Fibonacci graph G_n, $\chi_e(G_n) = \Delta(G_n) = n$.*

Proof: The result is true for $n = 1$ and $n = 2$. Proceeding by induction, edge color G_{n-2} and G_{n-1} with $n-2$ and $n-1$ colors, respectively, where the colors for G_{n-2} are a subset of those for G_{n-1}. Employ a new n^{th} color for all edges of the matching M. This gives a proper edge coloring of G_n with n colors. Since maximum degree is n, the result follows. □

The next theorem, on hamiltonian paths and cycles of Fibonacci graphs, is used to find values for several invariants. Its proof uses the following notation. Let P be a path in G_{n-1}. Then appending a 0 to the label of each vertex of P produces vertices that form a corresponding path in G_n. We say this path is the *lift* of P and denote it by $P0$. Other lifts (for example, lifting P to G_{n+1}) use corresponding notation (for example, $P01$). Similarly, the lift of an edge e is denoted $e0$, $e01$, etc. Diagrams will be presented to illustrate the proof. Unlabeled line segments on the diagrams denote paths. Labeled line segments denote single edges, and a label M indicates an edge in the matching. These matching edges are referred to as *M-edges*.

Theorem 15 *For $k \geq 1$, the Fibonacci graphs G_{3k-1} and G_{3k} have hamiltonian paths, and G_{3k+1} has a hamiltonian circuit.*

Proof: The proof is by induction on k. We actually prove somewhat more than what is stated in the theorem: There is a hamiltonian path P of G_{3k} whose lift $P0$ is a portion of a hamiltonian circuit C of G_{3k+1}.

The Fibonacci graphs G_2 and G_3 have hamiltonian paths $\langle 10, 00, 01\rangle$ and $\langle 010, 000, 001, 101, 100\rangle$, respectively. Furthermore, G_4 has hamiltonian circuit $\langle 0100, 0000, 0010, 1010, 1000, 1001, 0001, 0101, 0100\rangle$, which contains the lift of a hamiltonian path in G_3, $\langle 0100, 0000, 0010, 1010, 1000\rangle$. This establishes the result for $k = 1$. Now assume for some $k \geq 1$ that G_{3k-1} has a hamiltonian path and G_{3k} has a hamiltonian path P whose lift, $P0$, is contained in hamiltonian circuit C of G_{3k+1}. Our construction for G_{3k+2}, G_{3k+3}, and G_{3k+4} uses Observation 2a.

1. **A hamiltonian path in G_{3k+2}** (refer to Figure 3).

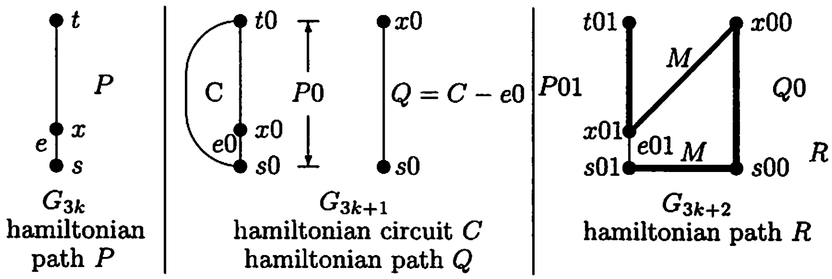

Figure 3: Constructing a hamiltonian path in G_{3k+2}

Let the end vertices of P be s and t, and $e = \{s, x\}$ be the edge of P incident to s. The lift $e0 = \{s0, x0\}$ of e lies in C. Let $Q = C - e0$, so Q is a hamiltonian path in G_{3k+1} with end vertices $s0$ and $x0$. Consider the lifts $P01$ and $Q0$ in G_{3k+2}. The following trace yields a hamiltonian path R in G_{3k+2}:

$s01$ to $s00$ using an M-edge
$s00$ to $x00$ along $Q0$
$x00$ to $x01$ using an M-edge
$x01$ to $t01$ along $P01$

2. **A hamiltonian path in G_{3k+3}** (refer to Figure 4).

Let $f = \{u, v\}$ be an edge of Q and S be the hamiltonian path $C - f$ in G_{3k+1} with end vertices u and v. By construction, the hamiltonian path R in G_{3k+2} contains $Q0$ and hence the lifted edge $f0 = \{u0, v0\}$. Consider the lifts $R0$ and $S01$. The following trace yields a hamiltonian path T of G_{3k+3}:

$s010$ to $u00$ along $R0$
$u00$ to $u01$ using an M-edge

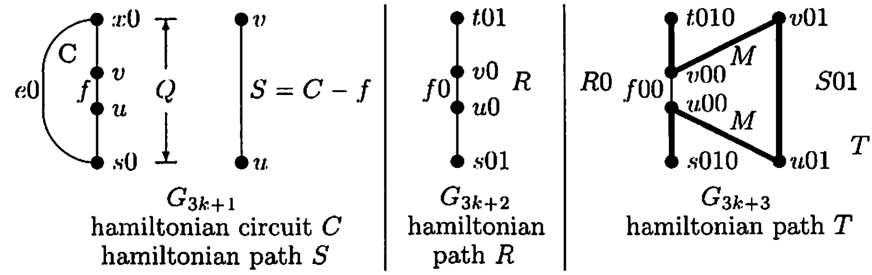

Figure 4: Constructing a hamiltonian path in G_{3k+3}

$u01$ to $v01$ along $S01$
$v01$ to $v00$ using an M-edge
$v00$ to $t010$ along $R0$

3. **A hamiltonian circuit in** G_{3k+4} that contains $T0$, the lift of T (refer to Figure 5).

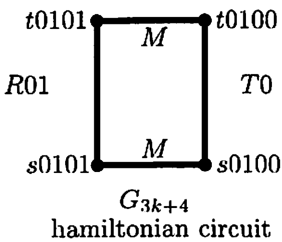

Figure 5: Constructing a hamiltonian circuit in G_{3k+4}

Consider the lifts $R01$ and $T0$ in G_{3k+4}. The following trace yields a hamiltonian circuit in G_{3k+4} that contains $T0$:

$s0101$ to $s0100$ using an M-edge
$s0100$ to $t0100$ along $T0$
$t0100$ to $t0101$ using an M-edge
$t0101$ to $s0101$ along $R01$

□

The *edge independence number*, $\beta_e(G)$, of graph G is the cardinality of a maximum matching. If $\beta_e(G) = \left\lfloor \frac{|V(G)|}{2} \right\rfloor$, the matching is said to be *perfect* if $|V(G)|$ is even and *near perfect* if $|V(G)|$ is odd. The *vertex independence number*, $\beta_v(G)$, of graph G is the cardinality of a largest set of independent vertices.

Corollary 16 *For the Fibonacci graph G_n, $\beta_e(G_n) = \left\lfloor \frac{|V(G_n)|}{2} \right\rfloor$.*

Corollary 17 *For the Fibonacci graph G_n, $\beta_v(G_n) = \left\lceil \frac{|V(G_n)|}{2} \right\rceil$.*

Proof: Since G_n has a hamiltonian path, $\beta_v(G_n) \leq \left\lceil \frac{|V(G_n)|}{2} \right\rceil$. Since G_n is bipartite, $\beta_v(G_n) \geq \left\lceil \frac{|V(G_n)|}{2} \right\rceil$. □

The *vertex cover number*, $\alpha_v(G)$, of graph G is the minimum number of vertices such that every edge is incident to at least one of them. Using Gallai's theorem (see Chartrand and Zhang [2], page 191) $\beta_v(G)+\alpha_v(G) = |V(G)|$ for any graph G yields the following.

Corollary 18 *For the Fibonacci graph G_n, $\alpha_v(G_n) = \left\lfloor \frac{|V(G_n)|}{2} \right\rfloor$.*

The *vertex clique cover number*, $\theta_v(G)$, of graph G is the minimum number of complete subgraphs required to include all the vertices of G.

Proposition 19 *For the Fibonacci graph G_n, $\theta_v(G_n) = \left\lceil \frac{|V(G_n)|}{2} \right\rceil$.*

Proof: Since G_n is bipartite, $\theta_v(G_n) \geq \left\lceil \frac{|V(G_n)|}{2} \right\rceil$. The reverse inequality follows from the fact that G_n has a hamiltonian path. □

4 Open Problems

The following problems seem to be difficult. We present them with some partial results.

Problem 1

Determine the *domination number*, $\gamma(G_n)$, that is, the cardinality of a smallest set $S \subseteq V(G_n)$ such that every vertex not in S is adjacent to at least one vertex of S. Results so far give only fragmentary information. The following exact values and bounds have been established[1] (here and in other tables below the values were obtained from a combination of direct computations and computer determinations):

[1] Some of these bounds were obtained by a genetic algorithm developed by Rollins College undergraduates Sarah Connelly, Sócrates Pérez, and Dana Singer during a summer research project.

n	$\gamma(G_n)$
1	1
2	1
3	2
4	3
5	4
6	5
7	8
8	≤ 12
9	≤ 18
10	≤ 29
11	≤ 46
12	≤ 75

We have only crude general bounds, given in the next two observations.

Observation 20 *For the Fibonacci graph G_n, $\left\lceil \frac{F_{n+2}}{3} \right\rceil \geq \gamma(G_n) \geq \frac{F_{n+2}}{n+1}$.*

Proof: The maximum degree of G_n is n, so one vertex can dominate at most $n+1$ vertices. The lower bound follows since F_{n+2} is the number of vertices of G_n. The upper bound is a direct consequence of Theorem 15. □

The bounds of Observation 20 seem weak. The lower bound is based on a maximum degree of n. But only one vertex, $00\ldots0$, has this degree and most other vertices have much smaller degree.

Observation 21 *For the Fibonacci graph G_n, $n \geq 8$, $\gamma(G_n) \leq F_{n-1} - F_{n-7}$.*

Proof: From the above table, $\gamma(G_7) = 8 = F_6$ and $\gamma(G_8) \leq 12 = F_7 - 1$. The result follows inductively by recognizing, for $n \geq 9$, that $\gamma(G_n) \leq \gamma(G_{n-1}) + \gamma(G_{n-2})$. □

Problem 2

Determine the *packing number*, $P(G_n)$, that is, the cardinality of a largest set $S \subseteq V(G_n)$ such that the closed neighborhoods of the vertices of S are pairwise disjoint. The following values have been determined:

n	$P(G_n)$
1	1
2	1
3	2
4	2
5	3
6	5
7	6
8	9
9	14

A bound similar to the lower bound in Observation 20 is given in the following.

Observation 22 *For the Fibonacci graph G_n, $P(G_n) \geq \frac{F_{n+2}}{n^2+1}$.*

Proof: Let $N_2[x]$ be the distance 2 closed neighborhood of vertex x, that is, the set of vertices distance at most 2 from x. Using the fact that the maximum degree of G_n is n, we have for any vertex x that $|N_2[x]| \leq 1 + n + n(n-1) = n^2 + 1$. Let X be a maximum packing set. Then the number of vertices of distance at most two from at least one of the vertices of X is at most $|\cup_{x \in X} N_2[x]| \leq (n^2+1)|X|$. Any vertices not in this union are distance at least three from every vertex of X. But any such vertex then could be added to X, a contradiction. Thus, $(n^2+1)|X| \geq |\cup_{x \in X} N_2[x]| = |V(G_n)|$ so $|X| \geq \frac{|V(G_n)|}{n^2+1} = \frac{F_{n+2}}{n^2+1}$. □

This bound probably can be improved significantly.

Problem 3

Determine the *bandwidth*, $B(G_n)$, that is, $\min_f \max_{e=vw} |f(w) - f(v)|$ where f ranges over all bijections $f : V(G_n) \rightarrow \{1, 2, \ldots, |V(G_n)|\}$ and e ranges over all edges of G_n. Some values and bounds are known.

n	$B(G_n)$
1	1
2	1
3	2
4	3
5	4
6	6
7	≤ 9
8	≤ 15
9	≤ 21
10	≤ 33
11	≤ 48
12	≤ 76
13	≤ 112
14	≤ 179

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