Some multiple chromatic numbers of Kneser graphs

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Abstract

The Kneser graph $K(m, n)$ (when $m > 2n$) has the n-subsets of an m-set as its vertices, two vertices being adjacent in $K(m, n)$ whenever they are disjoint sets. The kth chromatic number of any graph G (denoted by $\chi_k(G)$) is the least integer t such that the vertices can be assigned k-subsets of $\{1, 2, \ldots, t\}$ with adjacent vertices receiving disjoint k-sets. S. Stahl has conjectured that, if $k = qn - r$ where $q \geq 1$ and $0 \leq r < n$, then $\chi_k(K(m, n)) = qm - 2r$. This expression is easily verified when $r = 0$; Stahl has also established its validity for $q = 1$, for $m = 2n + 1$ and for $n = 2, 3$. We show here that the expression is also valid for all $q \geq 2$ in the following further classes of cases:

(i) $2n + 1 < m \leq n(2 + r^{-1})$ $(0 < r < n$, all $n > 1)$;

(ii) $4 \leq n \leq 6$ and $1 \leq r \leq 2$ (all m);

(iii) $7 \leq n \leq 11$ and $r = 1$ (all m);

(iv) $(n, r, m) = (7, 2, 18), (12, 1, 37), (12, 1, 38)$ or $(13, 1, 40)$.

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Introduction

Fractional graph theory [7] has been widely studied in recent years. Many graph parameters have fractional analogues that bear the same relation to the original as a linear programming problem bears to its integer relaxation. One of the first parameters to be 'fractionalized' was the chromatic number.

Let [t] denote the set of integers $\{1, 2, \ldots, t\}$. Given a graph G, its fractional chromatic number $\chi_f(G)$ is defined as $\inf_{k\in Z^+}(\frac{\chi_k(G)}{k})$ where $\chi_k(G)$, the kth *chromatic number* of G, is the least integer t for which the vertices of G can be assigned k-subsets of $[t]$ such that adjacent vertices receive disjoint k-sets. It is shown in [3] that $\chi_f(G) = \lim_{k\to\infty}(\frac{\chi_k(G)}{k})$, for any graph G, and that this limit is always achieved for some $k \in Z^+$. Such a colouring is said to be a *k-fold colouring* of G.

Given such a colouring of G, it is useful to consider it from a dual point of view, as a set of t *colour sets* $C_1, \ldots, C_t$, each C_i being a set of pairwise non-adjacent vertices of G, such that each vertex belongs to k colour sets. We shall adopt this viewpoint.

For positive integers $m > 2n \geq 4$ the *Kneser graph* $K(m, n)$ has all the n-subsets of $[m]$ as its vertices, two such vertices being adjacent whenever they are disjoint subsets of $[m]$. These graphs play an important role in fractional graph theory, since an n-fold colouring of a graph G using at most m colours may be regarded as a homomorphism: $G \to K(m, n)$.

Often the definition of Kneser graph is extended to the case $m = 2n$; however, the graph is then bipartite and its chromatic properties are rather trivial, so we shall assume $m > 2n \geq 4$ throughout. In particular, this condition is assumed in the statements of both theorems below.

It is easily established (see [7], for example) that

$$\chi_f(K(m, n)) = m/n, \qquad (1)$$

but difficult to find $\chi_k(K(m, n))$ for general k except when k is a multiple of n. (In this case, an easy counting argument shows that

$$\chi_{qn}(K(m, n)) = qm \qquad (2)$$

for any positive integer q.)

Even the value $k = 1$ is interesting; Kneser [4] conjectured

$$\chi_1(K(m, n)) = m - 2n + 2, \tag{3}$$

and this was eventually proved by Lovász [5] using the first of a number of topological arguments. A purely combinatorial proof [6] was not published till 2004.

In [8], Stahl established the inequality

$$\chi_{k+1}(G) \geq \chi_k(G) + 2. \tag{4}$$

Repeated use of this inequality, together with (2), shows that if $q \geq 1$ and $0 < r < n$, then

$$\chi_{qn-r}(K(m, n)) \leq qm - 2r. \tag{5}$$

Stahl [8], [9] conjectured that this is actually an equality:

$$\chi_{qn-r}(K(m, n)) = qm - 2r. \tag{6}$$

In [8] he showed that the conjecture holds when $m = 2n + 1$, and in [9] he extended the result, showing that it holds when $n = 2, 3$. As he also noted, (3) and (4) show that

the conjecture holds for $q = 1$ and $0 < r < n$. Thus, we now assume $q \geq 2$.

Our first result has a straightforward proof:

Theorem 1 *Let* $q \geq 2$ *and* $0 < r < n$. If $2 < \frac{m}{n} < 2 + \frac{1}{r}$, *then* (6) *holds.*

Proof. By (5), we need only show that $qm - 2r$ is a lower bound. Now by (1), together with the definition of fractional chromatic number,

$$\chi_{qn-r}(K(m, n)) \geq \frac{m(qn - r)}{n} = qm - \frac{mr}{n} > qm - 2r - 1,$$

since $\frac{m}{n} < 2 + \frac{1}{r}$. ■

In the next section, we extend the argument of [9] to obtain a rather technical proposition, which we then use to prove the main theorem:

Theorem 2 *Stahl's conjecture* (6) *holds for* $q \geq 2$ *and* $m \geq 2n + 1$, *for the following values of* n, r:

(i) $4 \leq n \leq 6$: $1 \leq r \leq 2$;

(ii) $7 \leq n \leq 11$: $r = 1$.

In addition, the conjecture holds for $q \geq 2$ and the following four sets of values for n, r, m:

$$(n, r, m) = (7, 2, 18),\ (12, 1, 37),\ (12, 1, 38),\ (13, 1, 40).$$

2. Generalizing an argument by Stahl

Proposition 3 *Let $q \geq 2$, $n \geq 4$, $1 \leq r < n$. If, for some integer $M > n(2 + r^{-1})$, we have*

$$(m-2r)\left(\binom{m-1}{n-1} - \binom{m-n-1}{n-1} + 1\right) + ((q-1)m-1)\binom{m-1}{n-1} < (qn-r)\binom{m}{n}$$
$$(2n < m \leq M), \tag{7}$$

then $\chi_{qn-r}(K(m, n)) = qm - 2r \quad (2n < m \leq M)$. (8)

Proof. Note first that (7) always holds when $2n < m < n(2 + r^{-1})$; indeed, the stronger inequality

$(m-2r)\binom{m-1}{n-1} + ((q-1)m-1)\binom{m-1}{n-1} < (qn-r)\binom{m}{n}$ holds under this condition since $\binom{m}{n} = \frac{m}{n}\binom{m-1}{n-1}$.

Assume, then, that (7) holds also for $n(2 + r^{-1}) \leq m \leq M$. Suppose for a contradiction that $\chi_{qn-r}(K(m, n)) < qm - 2r$ for some $m \leq M$, and let m be the least such value; thus $m \geq n(2 + r^{-1})$.

We now follow the method of proof of the main result of [9]. By supposition, there exists a $(qn - r)$-fold colouring of $K(m, n)$ using at most $qm - 2r - 1$ colour classes. We may assume that exactly $qm - 2r - 1$ classes, $C_1, \ldots, C_{qm-2r-1}$, are used, some of which may be empty.

The vertices of $K(m, n)$ are the n-subsets of $[m]$, so that the colour classes are families of such n-subsets, and the pairwise non-adjacency requirement implies that any two n-subsets belonging to the same colour class must intersect. For $1 \leq i \leq m$, a colour class all of whose vertices contain i is said to be *centred at i*.

Suppose that at least q of the colour classes are centred at some fixed i; we may assume $i = m$ and that the last q colour classes, $C_{q(m-1)-2r}, \ldots, C_{qm-2r-1}$, are so centred. Let X denote the set of vertices of $K(m, n)$ that contain m, and let

$$D_i = C_i \setminus X \quad (i = 1, \ldots, q(m-1)-2r-1).$$

Then $D_1, \ldots, D_{q(m-1)-2r-1}$ is a $(qn-r)$-fold colouring of $K(m-1, n)$ using $q(m-1)-2r-1$ colour classes, contradicting the minimality assumption.

Thus, for each integer $i \in [m]$, at most $q-1$ of the colour classes are centred at i. Hence, any vertex of $K(m, n)$ must be contained in at most $n(q-1)$ centred colour classes. But the colour classes constitute a $(qn-r)$-fold colouring, and thus each vertex is contained in at least $n-r$ non-centred colour classes. That is to say, the non-centred colour classes constitute an $(n-r)$-fold colouring of $K(m, n)$. But (as noted above) (6) holds for $q = 1$, and so there are at least $m-2r$ non-centred colour classes.

Now by the Erdős-Ko-Rado Theorem [1] and the Hilton-Milner Theorem [2], each centred colour class has size at most $\binom{m-1}{n-1}$ and each non-centred colour class has size at most $\binom{m-1}{n-1} - \binom{m-n-1}{n-1} + 1$. Therefore,

$$(m-2r)\left(\binom{m-1}{n-1} - \binom{m-n-1}{n-1} + 1\right) + ((q-1)m-1)\binom{m-1}{n-1} \geq (qn-r)\binom{m}{n}$$

contrary to assumption.

Thus there is no $m \in [2n+1, M]$ such that $\chi_{qn-r}(K(m, n)) < qm-2r$, and the proposition is established. ∎

3. The proof of the main theorem

In this proof, we assume $n \geq 4$ and make a careful study of the inequality (7).

Again using the identity $\binom{m}{n} = \frac{m}{n}\binom{m-1}{n-1}$, (7) may be recast as follows:

$$f(m, n, r) < g(m, n, r), \tag{9}$$

where

$$f(m,n,r) = \left(\frac{rm}{n} - 2r - 1\right)\binom{m-1}{n-1}; \quad g(m,n,r) = (m-2r)\left(\binom{m-n-1}{n-1} - 1\right).$$

We have $g(m, n, r) > 0$ for all m, n, r under consideration, and so (7) holds whenever $f(m, n, r) \leq 0$ (that is, whenever $m \leq n(2 + r^{-1})$).

Let $\mu(n, r) = \min\{m\colon\ m > n(2 + r^{-1})\} = 1 + \lfloor n(2 + r^{-1}) \rfloor$. It is arithmetically straightforward to verify that (9) holds when $m = \mu(n, r)$, for all values of n, r considered in parts (i) and (ii) of the theorem, and also when $(n, r) = (7, 2)$, $(12, 1)$ and $(13, 1)$. These last three pairs of values give $\mu(n, r) = 18$, 37 and 40 respectively, and it is also easy to verify that $f(12, 1, 38) < g(12, 1, 38)$. Thus the four particular cases in the final assertion of the theorem are verified; but in order to establish the more important parts (i) and (ii), we must consider $m > \mu(n, r)$.

For the remainder of the argument, then, we are concerned with parts (i) and (ii) of the theorem, and we assume $f(m, n, r) > 0$.

Let $h(m,n,r) = (m-2r)\binom{m-n-1}{n-1}$, and consider the functions

$$G(m,n,r) = \frac{f(m+1,n,r)g(m,n,r)}{g(m+1,n,r)f(m,n,r)},$$

$$H(m,n,r) = \frac{f(m+1,n,r)h(m,n,r)}{h(m+1,n,r)f(m,n,r)}.$$

Since $\dfrac{\binom{m-n-1}{n-1}-1}{\binom{m-n}{n-1}-1} < \dfrac{\binom{m-n-1}{n-1}}{\binom{m-n}{n-1}}$, it follows that $G(m, n, r) < H(m, n, r)$. In particular, $G(m, n, r) < 1$ whenever $H(m, n, r) < 1$; that is, whenever

$$f(m+1, n, r)h(m, n, r) < h(m+1, n, r)f(m, n, r).$$

This inequality reduces to

$$(r(m+1) - (2r+1)n)m(m-2r)(m-2n+1) <$$
$$(rm - (2r+1)n)(m-n+1)(m-2r+1)(m-n). \qquad (10)$$

The terms of order 3 and 4 in m cancel, yielding the following quadratic inequality in m:

$$m^2(-rn^2 - 2r^2 + 3rn + n) +$$
$$m(2r^2n^2 + 2r^2n - 7rn^2 + 3rn - 2r^2 - 3n^2 + n^3 + 2rn^3 + n) +$$
$$(n^3 - n^2 + 4r^2n^2 - 4r^2n^3) < 0. \qquad (11)$$

Consider the m^2 coefficient of (11). For $r \geq 1$, $n \geq 4$ we have

$$-rn^2 - 2r^2 + 3rn + n \leq -r(n^2 + 2r - 4n) < 0,$$

and so inequality (11) holds whenever m exceeds the larger root, $\rho(n, r)$, of the corresponding quadratic equation,

$$m^2(-rn^2 - 2r^2 + 3rn + n) +$$
$$m(2r^2n^2 + 2r^2n - 7rn^2 + 3rn - 2r^2 - 3n^2 + n^3 + 2rn^3 + n) +$$
$$(n^3 - n^2 + 4r^2n^2 - 4r^2n^3) = 0. \qquad (12)$$

Let $m_0 = \lceil \rho(n,r) \rceil$. Then $\dfrac{f(m+1,n,r)g(m,n,r)}{g(m+1,n,r)f(m,n,r)} < 1$ whenever $m \geq m_0$, and thus $f(m, n, r) < g(m, n, r)$ $(m > m_0)$ provided that $f(m_0, n, r) < g(m_0, n, r)$. It is straightforward to check that the latter condition holds for the values of n, r given in the statement of the theorem. More explicit information is given in Table 1 of the Appendix.

For each value of n, r in the statement of the theorem, we must now consider $\dfrac{f(m,n,r)}{g(m,n,r)}$ $(2n < m < m_0)$. Let $\lambda(n, r)$ be the lesser root of (10). We note that $\lambda(n, r) < 2$ when $r = 1$ and $4 \leq n \leq 11$, while

$\lambda(n, r) \leq 4$ when $r = 2$ and $4 \leq n \leq 6$. Thus, $H(m, n, r)$ is positive for $2n + 1 \leq m \leq m_0$, so that $\dfrac{f(m,n,r)}{h(m,n,r)}$ is increasing in this range.

Clearly, $\dfrac{g(m,n,r)}{h(m,n,r)}$ is an increasing function of m, and so to check that $\dfrac{f(m,n,r)}{g(m,n,r)} < 1$ $(2n < m < m_0(n, r))$, it is sufficient to let $\mu = \mu(n, r)$, $m_0 = m_0(n, r)$ and verify that

$$\frac{g(\mu,n,r)}{h(\mu,n,r)} = \frac{\binom{\mu-n-1}{n-1} - 1}{\binom{\mu-n-1}{n-1}} > \frac{f(m_0,n,r)}{h(m_0,n,r)}.$$

Table 1 of the Appendix shows the quadratic equation (12), and the values of $\mu(n, r)$ and $m_0(n, r)$, for all relevant n, r. It is straightforward to check that the inequality holds comfortably in all these cases.

Thus (7), and therefore (8), holds for all $m > 2n$ (for the given values of n, r); and this proves parts (i) and (ii). ■

4. Appendix

In Table 1, we give explicitly, for the values of n, r required by the statement of Theorem 2: the quadratic equation (12) (multiplied by -1); $\mu(n, r)$; and $m_0(n,r) = \lceil \rho(n,r) \rceil$.

n and r	equation	$\mu(n, r)$	$m_0(n, r)$
$n = 4, r = 1$	$2m^2 - 86m + 144 = 0$	13	42
$n = 4, r = 2$	$12m^2 - 228m + 720 = 0$	11	15
$n = 5, r = 1$	$7m^2 - 203m + 300 = 0$	16	28
$n = 5, r = 2$	$23m^2 - 467m + 1500 = 0$	13	17
$n = 6, r = 1$	$14m^2 - 394m + 540 = 0$	19	27
$n = 6, r = 2$	$38m^2 - 838m + 2700 = 0$	16	19
$n = 7, r = 1$	$23m^2 - 677m + 882 = 0$	22	29
$n = 8, r = 1$	$34m^2 - 1070m + 1344 = 0$	25	31
$n = 9, r = 1$	$47m^2 - 1591m + 1944 = 0$	28	33
$n = 10, r = 1$	$62m^2 - 2258m + 2700 = 0$	31	36
$n = 11, r = 1$	$79m^2 - 3089m + 3630 = 0$	34	38

Table 1

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