ON σ-LABELING THE UNION OF THREE CYCLES

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ABSTRACT. Let G be a graph of size n with vertex set $V(G)$ and edge set $E(G)$. A *σ-labeling* of G is a one-to-one function $f : V(G) \to \{0, 1, \dots, 2n\}$ such that $\{|f(u)-f(v)| : \{u, v\} \in E(G)\} = \{1, 2, \dots, n\}$. Such a labeling of G yields cyclic G-decompositions of K_{2n+1} and of $K_{2n+2} - F$, where F is a 1-factor of K_{2n+2}. It is conjectured that a 2-regular graph of size n has a σ-labeling if and only if $n \equiv 0$ or $3 \pmod 4$. We show that this conjecture holds when the graph has at most three components.

1. INTRODUCTION

If a and b are integers we denote $\{a, a+1, \dots, b\}$ by $[a, b]$ (if $a > b$, $[a, b] = \emptyset$). Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{Z}_n$ the group of integers modulo n. For a graph G, let $V(G)$ and $E(G)$ denote the vertex set of G and the edge set of G, respectively. The *order* and the *size* of a graph G are $|V(G)|$ and $|E(G)|$, respectively. Let k be a positive integer and let $V(K_k) = [0, k-1]$. The *length* of an edge $\{i, j\}$ in K_k is defined as $\min\{|i-j|, k - |i-j|\}$. It is easy to see that if k is odd, then K_k consists of k edges of length i for $i = 1, 2, \dots, \frac{k-1}{2}$. Similarly, if k is even, then K_k consists of k edges of length i for $i = 1, 2, \dots, \frac{k}{2} - 1$ and $\frac{k}{2}$ edges of length $\frac{k}{2}$; moreover, in this case, the edges of length $\frac{k}{2}$ constitute a 1-factor in K_k.

Let $V(K_k) = \mathbb{Z}_k$ and let G be a subgraph of K_k. By *clicking* G, we mean applying the isomorphism $i \to i+1$ to $V(G)$. Note that clicking preserves edge lengths. Let H and G be graphs such that G is a subgraph of H. A *G-decomposition* of H is a set $\Gamma = \{G_1, G_2, \dots, G_t\}$ of pairwise disjoint subgraphs of H each of which is isomorphic to G and such that $E(H) = \bigcup_{i=1}^{t} E(G_i)$. If H is K_k, a G-decomposition Γ of H is *cyclic* if clicking is a permutation of Γ. If G is a graph and r is a positive integer, rG denotes the vertex disjoint union of r copies of G.

For any graph G, a one-to-one function $f : V(G) \to \mathbb{N}$ is called a *labeling* (or a *valuation*) of G. In [19], Rosa introduced a hierarchy of labelings. We add a few items to this hierarchy. Let G be a graph with n edges and no isolated vertices and let f be a labeling of G. Let $f(V(G)) = \{f(u) : u \in V(G)\}$. Define a function $\bar{f} : E(G) \to \mathbb{Z}^+$ by $\bar{f}(e) = |f(u) - f(v)|$, where $e = \{u, v\} \in E(G)$. Let $\bar{E}(G) = \{\bar{f}(e) : e \in E(G)\}$. Consider the following conditions:

(ℓ1) $f(V(G)) \subseteq [0, 2n]$,

(ℓ2) $f(V(G)) \subseteq [0, n]$,

(ℓ3) $\bar{E}(G) = \{x_1, x_2, \ldots, x_n\}$, where for each $i \in [1, n]$ either $x_i = i$ or $x_i = 2n + 1 - i$,

(ℓ4) $\bar{E}(G) = [1, n]$.

If in addition G is bipartite with bipartition $\{A, B\}$ of $V(G)$ (with every edge in G having one endvertex in A and the other in B) such that

(ℓ5) for each $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$, we have $f(a) < f(b)$,

(ℓ6) there exists an integer λ (called the *boundary value* of f) such that $f(a) \leq \lambda$ for all $a \in A$ and $f(b) > \lambda$ for all $b \in B$.

Then a labeling satisfying the conditions:

(ℓ1), (ℓ3) is called a *ρ-labeling*;

(ℓ1), (ℓ4) is called a *σ-labeling*;

(ℓ2), (ℓ4) is called a *β-labeling*.

A β-labeling is necessarily a σ-labeling which in turn is a ρ-labeling. If G is bipartite and a ρ, σ or β-labeling of G also satisfies (ℓ5), then the labeling is *ordered* and is denoted by ρ^+, σ^+ or β^+, respectively. If in addition (ℓ6) is satisfied, the labeling is *uniformly-ordered* and is denoted by ρ^{++}, σ^{++} or β^{++}, respectively.

A β-labeling is better known as a *graceful* labeling and a uniformly-ordered β-labeling is an *α-labeling* as introduced in [19].

Labelings are critical to the study of cyclic graph decompositions as seen in the following two results from [19] and [11], respectively.

Theorem 1. Let G be a graph with n edges. There exists a cyclic G-decomposition of K_{2n+1} if and only if G has a ρ-labeling.

Theorem 2. Let G be a graph with n edges that has a ρ^+-labeling. Then there exists a cyclic G-decomposition of K_{2nx+1} for all positive integers x.

Note that a ρ-labeling f of a graph G with n edges is an embedding of G in K_{2n+1} (with $V(K_{2n+1}) = [0, 2n]$) so that there is exactly one edge in G of length i for $i = 1, 2, \ldots, n$. It is easy to see that if f is a σ-labeling, then G can be embedded in K_{2n+2} so that there is exactly one edge in G of length i for $i = 1, 2, \ldots, n$. Thus the following holds for σ-labelings (but not necessarily for ρ-labelings, in general).

Theorem 3. If G with n edges has a σ-labeling, then there exists a cyclic G-decomposition of $K_{2n+2} - F$, where F is a 1-factor of K_{2n+2}.

A non-bipartite graph G is *almost-bipartite* if G contains an edge e whose removal renders the remaining graph bipartite (for example, odd cycles are almost-bipartite). In [5], Blinco et al. introduced a variation of a ρ-labeling of an almost-bipartite graph G of size n that yields cyclic G-decompositions of K_{2nx+1}. They called this labeling a γ-labeling. Rather than restate the (lengthy) definition of a γ-labeling here, we direct the interested reader to [5]. We do note however that a γ-labeling is necessarily a ρ-labeling.

Let G be a graph with n edges and Eulerian components and let f be a β-labeling of G. It is well-known (see [19]) that we must have $n \equiv 0$ or 3 (mod 4). Moreover, if such a G is bipartite, then $n \equiv 0 \pmod 4$. These conditions hold since for such a G, $\sum_{e \in E(G)} \bar{f}(e) = n(n+1)/2$. This sum must in turn be even, since each vertex is incident with an even number of edges and $\bar{f}(e) = |f(u) - f(v)|$, where u and v are the end vertices of e. Thus we must have $4|n(n+1)$. Clearly, the same will hold if such a G admits a σ-labeling. We shall refer to this restriction as the *parity condition*. There are no such restrictions on $|E(G)|$ if f is a ρ-labeling.

Theorem 4. (Parity Condition) If a graph G with Eulerian components and n edges has a σ-labeling, then $n \equiv 0$ or 3 (mod 4). If such a G is bipartite, then $n \equiv 0 \pmod 4$.

In [19], Rosa presented α- and β-labelings of C_{4m} and of C_{4m+3}, respectively. It is also known that both C_{4m+1} and C_{4m+2} admit ρ-labelings. It was also shown in [11] that there exists a ρ^{+}-labeling of C_{4m+2}, for all positive integers m. It can be easily checked that this labeling is actually a ρ^{++}-labeling.

In this manuscript, we will focus on labelings of 2-regular graphs (i.e., the vertex-disjoint union of cycles). If a 2-regular graph G is bipartite, then it is known that G admits a σ^{+}-labeling if the parity condition is satisfied (see [11]) and a ρ^{++}-labeling otherwise (see [4]). Such a G need not admit an α-labeling, even if the parity condition is satisfied. It is well-known for example that $3C_4$ does not have an α-labeling (see [15]). Similarly, if G is not bipartite, then G need not admit a β-labeling even if the parity condition is satisfied. For example, it is shown in [16] that rC_3 does not admit a β-labeling for all $r > 1$ and rC_5 never admits a β-labeling. Moreover, it is known that $C_3 \bigcup C_3 \bigcup C_5$ is the smallest 2-regular graph that satisfies the parity condition, yet fails to have a β-labeling (see [2]). It is thus reasonable to focus on labelings that are less restrictive than β-labelings when studying 2-regular graphs.

Here, we shall show that every 2-regular graph G consisting of three components has a σ-labeling (or a more restricted labeling) if and only if the parity condition is satisfied. In a companion article [3], it is shown that if the parity condition is not satisfied, then such a G necessarily admits a ρ-labeling. These results provide further evidence in support of a conjecture of El-Zanati and Vanden Eynden that every 2-regular graph admits a σ-labeling if the parity condition is satisfied and a ρ-labeling otherwise.

Let r, s and t be positive integers ≥ 3 and let $G = C_r \bigcup C_s \bigcup C_t$. If we consider the congruences of r, s and t modulo 4, then G then belongs to one of 20 types of graphs (see Table 1). In each of the ten cases where the parity condition is satisfied, we will show that G has a σ-labeling (or a more restricted labeling). If G does not satisfy the parity condition, then G necessarily admits a ρ-labeling (see [3]).

mod 4			Labeling of $C_r \cup C_s \cup C_t$	Reference
r	s	t		
0	0	0	σ^+ if $r = s = t = 4$ α otherwise	[11] [12]
0	0	1	γ (thus ρ)	[5]
0	0	2	ρ^{++}	[4]
0	0	3	σ	[14]
0	1	1	ρ	[3]
0	1	2	σ	[14]
0	1	3	σ	[14]
0	2	2	α	[12]
0	2	3	γ (thus ρ)	[7]
0	3	3	ρ	[3]
1	1	1	σ	This paper
1	1	2	σ	This paper
1	1	3	ρ	[3]
1	2	2	γ (Thus ρ)	[5]
1	2	3	ρ	[3]
1	3	3	σ	This paper
2	2	2	ρ^{++}	[4]
2	2	3	σ	[14]
2	3	3	σ	This paper
3	3	3	ρ	[3]

Table 1. Labelings of $C_r \cup C_s \cup C_t$, $r, s, t \geq 3$

2. Summary of Some of the Known Results

As stated in the previous section, the following is known for cycles (see [18], [19] and [11]).

Theorem 5. Let $m \geq 3$ be an integer. Then, C_m admits an α-labeling if $m \equiv 0 \pmod 4$, a ρ-labeling if $m \equiv 1 \pmod 4$, a ρ^{++}-labeling if $m \equiv 2 \pmod 4$, and a β-labeling if $m \equiv 3 \pmod 4$.

For 2-regular graphs with two components, we have the following important result from Abrham and Kotzig [2].

Theorem 6. Let $m \geq 3$ and $n \geq 3$ be integers. Then the graph $C_m \bigcup C_n$ has a β-labeling if and only if $m+n \equiv 0$ or $3 \pmod 4$. Moreover, $C_m \bigcup C_n$ has an α-labeling if and only if both m and n are even and $m+n \equiv 0 \pmod 4$.

If the parity condition is not satisfied, then $C_m \bigcup C_n$ has a ρ^{++}-labeling if both m and n are even [4] and a ρ-labeling otherwise [10].

For 2-regular graphs with more than two components, the following is known. In [15], Kotzig shows that if $r > 1$, then rC_3 does not admit a β-labeling. Similarly, he shows that rC_5 does not admit a β-labeling for any r. In [16], Kotzig shows that $3C_{4k+1}$ admits a β-labeling for all $k \geq 2$. From results in [8], it can be shown that rC_3 admits a ρ-labeling for all $r \geq 1$. The ρ-labeling in [8] can be modified to produce a σ-labeling of rC_3 when the parity condition is satisfied. In [12], Eshghi shows that $C_{2m} \bigcup C_{2n} \bigcup C_{2k}$ has an α-labeling for all m, n, and $k \geq 2$ with $m+n+k \equiv 0 \pmod 2$ except when $m = n = k = 2$. In [1], Abrham and Kotzig show that rC_4 has an α-labeling for all positive integers $r \neq 3$. In [9], it is shown that $3C_m$ and $4C_m$ admit σ-labelings if the parity condition is satisfied and ρ-labelings otherwise. An additional result follows by combining results from [11] and from [4].

Theorem 7. Let G be a 2-regular bipartite graph of order n. Then G has a σ^+-labeling if $n \equiv 0 \pmod 4$ and a ρ^{++}-labeling if $n \equiv 2 \pmod 4$.

A result by Hevia and Ruiz [14] proves very useful.

Theorem 8. The disjoint union of a graph with a β-labeling, together with a collection of graphs with α-labelings, has a σ-labeling.

When applied to 2-regular graphs and combined with the results of Abrham and Kotzig [2], Theorem 8 yields the following.

Corollary 9. Let $G_1 \in \{C_{4x+3}, C_{4x+3} \bigcup C_{4y+1}, C_{4x+1} \bigcup C_{4y+2}\}$, where $x \geq 0$ and $y \geq 1$ are integers. If G_2 is a 2-regular bipartite graph of order $0 \pmod 4$, then $G_1 \bigcup G_2$ admits a σ-labeling.

In [5], it is shown that if G admits an α-labeling and $j > 1$, then $G \cup C_{2j+1}$ admits a γ-labeling. Thus for example, both $C_{4x} \cup C_{4y} \cup C_{4z+1}$ and $C_{4x+1} \cup C_{4y+2} \cup C_{4z+2}$ admit γ-labelings. These results are generalized in [7], where it is shown that every 2-regular almost-bipartite graph $G \neq C_3 \cup (kC_4)$, $k \in \{0, 1\}$, has a γ-labeling.

3. Main results

Let r, s and t be positive integers ≥ 3 and let $G = C_r \bigcup C_s \bigcup C_t$. We shall show that G admits a σ-labeling (or a more restricted labeling) if and only if $r+s+t \equiv 0$ or $3 \pmod 4$. If $r+s+t \equiv 1$ or $2 \pmod 4$, then G admits a ρ-labeling (see [3]). Table 1 summarizes the results for labeling $C_r \bigcup C_s \bigcup C_t$.

Before proceeding, some additional definitions and notational conventions are necessary. Denote the path with consecutive vertices $a_1, a_2, \ldots, a_k$ by $(a_1, a_2, \ldots, a_k)$. By $(a_1, a_2, \ldots, a_k) + (b_1, b_2, \ldots, b_j)$, where $a_k = b_1$, we mean the path $(a_1, \ldots, a_k, b_2, \ldots, b_j)$.

To simplify our consideration of various labelings, we will sometimes consider graphs whose vertices are named by distinct nonnegative integers, which are also their labels. Let a, b, and k be integers with $0 \le a \le b$ and $k > 0$. Set $d = b - a$. We define the path

$$P(a,k,b) = (a, a+k+2d-1, a+1, a+k+2d-2, a+2, \ldots, b-1, b+k, b).$$

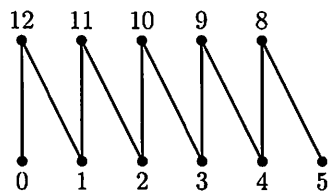

Figure 1. The path $P(0,3,5)$.

We note that the labeling of $P(a,k,b)$ is a translation of a k-graceful labeling of the path P_{2d+1} (as introduced in 1982 by Slater [20] and by Maheo and Thuillier [17]). It is easily checked that $P(a,k,b)$ is simple and

$$V(P(a,k,b)) = [a,b] \cup [b+k, b+k+d-1].$$

Furthermore, the edge labels of $P(a,k,b)$ are distinct and

$$\bar{E}(P(a,k,b)) = [k, k+2d-1].$$

These formulas will be used extensively in the proofs that follow.

As can be seen from Table 1, $G = C_r \bigcup C_s \bigcup C_t$ satisfies the parity condition in 10 of the 20 possible cases. We shall present the new results in four theorems, followed by our main theorem.

Theorem 10. Let x, y, z be positive integers with $x \le y \le z$, and let $G = C_{4x+1} \bigcup C_{4y+1} \bigcup C_{4z+1}$. Then G has a σ-labeling.

Proof. The three cycles $G_1 = C_{4x+1}, G_2 = C_{4y+1}$, and $G_3 = C_{4z+1}$ are defined as follows:

$$
\begin{aligned}
G_1 &= P(4x+4y+4z+3, 2x+4y+4z+3, 5x+4y+4z+3) \\
&+ P(5x+4y+4z+3, 4y+4z+3, 6x+4y+4z+2) \\
&+ (6x+4y+4z+2, 6x+4y+4z+5, 8x+8y+8z+6, 4x+4y+4z+3), \\
G_2 &= P(0, 2x+2y+4z+3, y-x) + P(y-x, 4z+4, 2y-1) \\
&+ (2y-1, 2y, 2x+4y+4z+2, 0), \\
G_3 &= P(6x+4y+4z+6, 2z+3, 6x+4y+5z+6) \\
&+ P(6x+4y+5z+6, 4, 6x+4y+6z+5) \\
&+ (6x+4y+6z+7, 6x+4y+8z+9, 6x+4y+4z+6).
\end{aligned}
$$

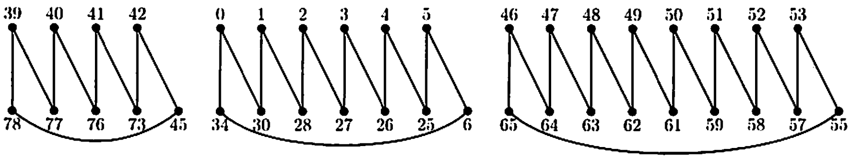

Figure 2. A σ-labeling of $C_9 \cup C_{13} \cup C_{17}$

Now we compute

$$
\begin{aligned}
V(G_1) &= [4x+4y+4z+3, 6x+4y+4z+2] \cup [7x+8y+8z+6, 8x+8y+8z+5] \\
&\cup [6x+8y+8z+5, 7x+8y+8z+3] \cup \{6x+4y+4z+5, 8x+8y+8z+6\}, \\
V(G_2) &= [0, 2y-1] \cup [x+3y+4z+3, 4y+4z+2] \cup [2y+4z+3, x+3y+4z+1] \\
&\cup \{2y, 2x+4y+4z+2\}, \\
V(G_3) &= [6x+4y+4z+6, 6x+4y+6z+5] \cup [6x+4y+7z+9, 6x+4y+8z+8] \\
&\cup [6x+4y+6z+9, 6x+4y+7z+7] \cup \{6x+4y+6z+7, 6x+4y+8z+9\}.
\end{aligned}
$$

We can order these as follows.

G_i	Vertex Labels	G_i	Vertex Labels
G_2	$[0, 2y-1]$	G_3	$6x+4y+6z+7$
G_2	$2y$	G_3	$[6x+4y+6z+9, 6x+4y+7z+7]$
G_2	$[2y+4z+3, x+3y+4z+1]$	G_3	$[6x+4y+7z+9, 6x+4y+8z+8]$
G_2	$[x+3y+4z+3, 4y+4z+2]$	G_3	$6x+4y+8z+9$
G_2	$2x+4y+4z+2$	G_1	$[6x+8y+8z+5, 7x+8y+8z+3]$
G_1	$[4x+4y+4z+3, 6x+4y+4z+2]$	G_1	$[7x+8y+8z+6, 8x+8y+8z+5]$
G_1	$6x+4y+4z+5$	G_1	$8x+8y+8z+6$
G_3	$[6x+4y+4z+6, 6x+4y+6z+5]$		

The vertices of the three cycles are distinct and contained in $[0, 2(4x+4y+4z+3)] = [0, 8x+8y+8z+6]$. Note that if $z = 1$, the set $[6x+4y+6z+9, 6x+4y+7z+7]$ is empty. If in addition $y = x$, then the set

$[x+3y+4z+3, 4y+4z+2]$. Finally, if $x = 1$, the set $[6x+8y+8z+5, 7x+8y+8z+3]$ will also be empty. This however does not change the proof.

Likewise we compute

$$\begin{aligned}
\overline{E}(G_1) &= [2x+4y+4z+3, 4x+4y+4z+2] \cup [4y+4z+3, 2x+4y+4z] \\
&\cup \{3, 2x+4y+4z+1, 4x+4y+4z+3\}, \\
\overline{E}(G_2) &= [2x+2y+4z+3, 4y+4z+2] \cup [4z+4, 2x+2y+4z+1] \\
&\cup \{1, 2x+2y+4z+2, 2x+4y+4z+2\}, \\
\overline{E}(G_3) &= [2z+3, 4z+2] \cup [4, 2z+1] \cup \{2, 2z+2, 4z+3\}.
\end{aligned}$$

We can order these as follows.

Cycle	Edge Labels	Cycle	Edge Labels
G_2	1	G_2	$2x+2y+4z+2$
G_3	2	G_2	$[2x+2y+2z+3, 4y+4z+2]$
G_1	3	G_1	$[4y+4z+3, 2x+4y+4z]$
G_3	$[4, 2z+1]$	G_1	$2x+4y+4z+1$
G_3	$2z+2$	G_2	$2x+4y+4z+2$
G_3	$[2z+3, 4z+2]$	G_1	$[2x+4y+4z+3, 4x+4y+4z+2]$
G_3	$4z+3$	G_1	$4x+4y+4z+3$
G_2	$[4z+4, 2x+2y+4z+1]$		

Hence $\overline{E}(G) = [1, 4x+4y+4z+3]$. Then we have a σ-labeling.

As with the vertex labels, note that if $z = 1$, then $[4, 2z+1]$ will be empty. If in addition $y = x$, then $[2x+2y+4z+3, 4y+4z+2]$ is empty. Finally, if $x = 1$, the set $[4y+4z+3, 2x+4y+4z]$ is empty. Neither condition would however change the proof. □

Theorem 11. Let x, y, z be positive integers with $y \geq z$, and let $G = C_{4x+2} \bigcup C_{4y+1} \bigcup C_{4z+1}$. Then G has a σ-labeling.

Proof. The three cycles $G_1 = C_{4x+2}$, $G_2 = C_{4y+1}$,and $G_3 = C_{4z+1}$ are defined as follows:

$$\begin{aligned}
G_1 &= P(0, 2x+4y+4z+4, x) + P(x, 4y+4z+5, 2x-1) \\
&+ (2x-1, 2x+4y+4z+3, 2x+1, 4x+4y+4z+4, 0), \\
G_2 &= P(4x+4y+4z+5, 2y+4z+4, 4x+5y+4z+4) \\
&+ P(4x+5y+4z+4, 4z+3, 4x+6y+4z+4) \\
&+ (4x+6y+4z+4, 4x+6y+4z+5, 4x+8y+8z+8, 4x+4y+4z+5), \\
G_3 &= P(2x+2, 2z+2, 2x+z+2) + P(2x+z+2, 3, 2x+2z+1) \\
&+ (2x+2z+1, 2x+2z+3, 2x+4z+4, 2x+2).
\end{aligned}$$

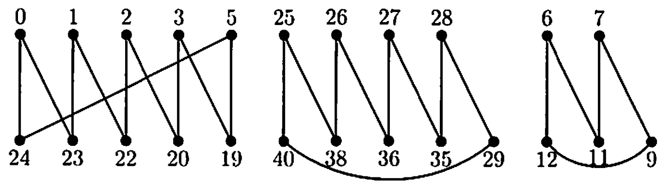

Figure 3. A σ-labeling of $C_{10} \cup C_9 \cup C_5$

Now we compute

$$\begin{aligned} V(G_1) &= [0, 2x-1] \cup [3x+4y+4z+4, 4x+4y+4z+3] \\ &\cup [2x+4y+4z+4, 3x+4y+4z+2] \\ &\cup \{2x+4y+4z+3, 2x+1, 4x+4y+4z+4\}, \\ V(G_2) &= [4x+4y+4z+5, 4x+6y+4z+4] \cup [4x+7y+8z+8, 4x+8y+8z+6] \\ &\cup [4x+6y+8z+7, 4x+7y+8z+6] \cup \{4x+6y+4z+5, 4x+8y+8z+8\}, \\ V(G_3) &= [2x+2, 2x+2z+1] \cup [2x+3z+4, 2x+4z+3] \\ &\cup [2x+2z+4, 2x+3z+2] \cup \{2x+2z+3, 2x+4z+4\}. \end{aligned}$$

We can order these as follows.

Cycle	Vertex Labels	Cycle	Vertex Labels
G_1	$[0, 2x-1]$	G_1	$[2x+4y+4z+4, 3x+4y+4z+2]$
G_1	$2x+1$	G_1	$[3x+4y+4z+4, 4x+4y+4z+3]$
G_3	$[2x+2, 2x+2z+1]$	G_1	$4x+4y+4z+4$
G_3	$2x+2z+3$	G_2	$[4x+4y+4z+5, 4x+6y+4z+4]$
G_3	$[2x+2z+4, 2x+3z+2]$	G_2	$4x+6y+4z+5$
G_3	$[2x+3z+4, 2x+4z+3]$	G_2	$[4x+6y+8z+7, 4x+7y+8z+6]$
G_3	$2x+4z+4$	G_2	$[4x+7y+8z+8, 4x+8y+8z+6]$
G_1	$2x+4y+4z+3$	G_2	$4x+8y+8z+8$

The vertices of the three cycles are distinct and contained in $[0, 2(4x+4y+4z+4)] = [0, 8x+8y+8z+8]$. Note that if $z=1$, the set $[2x+2z+4, 2x+3z+2]$ will be empty. If in addition $y=1$, then the set $[4x+7y+8z+8, 4x+8y+8z+6]$ will be empty. Finally, if in addition $x=1$, the set $[2x+4y+4z+4, 3x+4y+4z+2]$ will also be empty. This however does not change the proof.

Likewise we compute

$$\begin{aligned} \overline{E}(G_1) &= [2x+4y+4z+4, 4x+4y+4z+3] \cup [4y+4z+5, 2x+4y+4z+2] \\ &\cup \{4y+4z+4, 4y+4z+2, 2x+4y+4z+3, 4x+4y+4z+4\}, \\ \overline{E}(G_2) &= [2y+4z+4, 4y+4z+1] \cup [4z+3, 2y+4z+2] \\ &\cup \{1, 2y+4z+3, 4y+4z+3\}, \\ \overline{E}(G_3) &= [2z+2, 4z+1] \cup [3, 2z] \cup \{2, 2z+1, 4z+2\}. \end{aligned}$$

We can order these as follows.

Cycle	Edge Labels	Cycle	Edge Labels
G_2	1	G_2	$[2y+4z+4, 4y+4z+1]$
G_3	2	G_1	$4y+4z+2$
G_3	$[3, 2z]$	G_2	$4y+4z+3$
G_3	$2z+1$	G_1	$4y+4z+4$
G_3	$[2z+2, 4z+1]$	G_1	$[4y+4z+5, 2x+4y+4z+2]$
G_3	$4z+2$	G_1	$2x+4y+4z+3$
G_2	$[4z+3, 2y+4z+2]$	G_1	$[2x+4y+4z+4, 4x+4y+4z+3]$
G_2	$2y+4z+3$	G_1	$4x+4y+4z+4$

Hence $\overline{E}(G) = [1, 4x+4y+4z+4]$. Then we have a σ-labeling.

As with the vertex labels, note that if $z = 1$, the set $[3, 2z]$ is empty. If in addition $y = 1$, then $[2y+4z+4, 4y+4z+1]$ is empty. Finally, if in addition $x = 1$, then $[4y+4z+5, 2x+4y+4z+2]$ will also be empty. Neither condition would however change the proof. □

Theorem 12. Let $x \geq 1$, $y \geq z$ be nonnegative integers and let $G = C_{4x+1} \bigcup C_{4y+3} \bigcup C_{4z+3}$. Then G has a σ-labeling.

Proof. We will distinguish two cases according to whether $y = 0$ or $y \geq 1$.
Case 1: $y = 0$.
If $y = 0$, then z must be 0. The three cycles $G_1 = C_{4x+1}$, $G_2 = C_3$, and $G_3 = C_3$ are defined as follows:

$$\begin{aligned} G_1 &= P(4x+7, 2x+7, 5x+7) + P(5x+7, 8, 6x+6) \\ &+ (6x+6, 6x+8, 8x+14, 4x+7), \\ G_2 &= (0, 3, 7, 0), \\ G_3 &= (4, 5, 10, 4). \end{aligned}$$

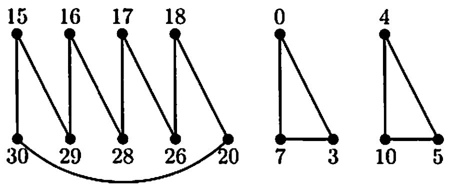

Figure 4. A σ-labeling of $C_9 \cup C_3 \cup C_3$

Now we compute

$$\begin{aligned} V(G_1) &= [4x+7, 6x+6] \cup [7x+14, 8x+13] \cup [6x+14, 7x+12] \cup \{6x+8, 8x+14\}, \\ V(G_2) &= \{0, 3, 7\}, \\ V(G_3) &= \{4, 5, 10\}. \end{aligned}$$

We can order these as $0, 3$ from G_2, then $4, 5$ from G_3, then 7 from G_2 and 10 from G_3, and finally

$$[4x+7, 6x+6], 6x+8, [6x+14, 7x+12], [7x+14, 8x+13], 8x+14$$

from G_1.

The vertices of the three cycles are distinct and contained in $[0, 8x+8y+8z+14]$. Note that if $x = 1$, the set $[6x+14, 7x+12]$ will be empty. This however does not change the proof.

Finally we compute

$$\begin{aligned}
\overline{E}(G_1) &= [2x+7, 4x+6] \cup [8, 2x+5] \cup \{2, 2x+6, 4x+7\}, \\
\overline{E}(G_2) &= \{3, 4, 7\}, \\
\overline{E}(G_3) &= \{1, 5, 6\}.
\end{aligned}$$

We can order these as edge label 1 from G_3, 2 from G_1, 3 and 4 from G_2, 5 and 6 from G_3, 7 from G_2, and

$$[8, 2x+5], 2x+6, [2x+7, 4x+6], 4x+7$$

from G_1. Thus $\overline{E}(G) = [1, 4x+7]$. Then we have a σ-labeling. Again, if $x = 1$ the set $[8, 2x+5]$ will be empty. This however does not change the proof.

Case 2: $y \geq 1$.
The three cycles $G_1 = C_{4x+1}, G_2 = C_{4y+3}$, and $G_3 = C_{4z+3}$ are defined as follows:

$$\begin{aligned}
G_1 &= P(4x+4y+4z+7, 2x+4y+4z+7, 5x+4y+4z+7) \\
&+ P(5x+4y+4z+7, 4y+4z+8, 6x+4y+4z+6) \\
&+ (6x+4y+4z+6, 6x+4y+4z+8, 8x+8y+8z+14, 4x+4y+4z+7), \\
G_2 &= P(0, 2y+4z+5, y+1) + P(y+1, 4z+6, 2y) + (2y, 2y+3, 4y+4z+7, 0), \\
G_3 &= P(6x+4y+4z+9, 2z+5, 6x+4y+5z+9) \\
&+ P(6x+4y+5z+9, 4, 6x+4y+6z+9) \\
&+ (6x+4y+6z+9, 6x+4y+6z+10, 6x+4y+8z+14, 6x+4y+4z+9).
\end{aligned}$$

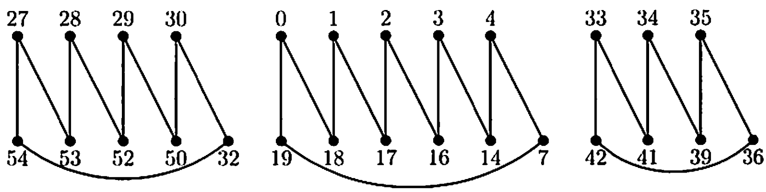

Figure 5. A σ-labeling of $C_9 \cup C_{11} \cup C_7$

Now we compute

$$\begin{aligned} V(G_1) &= [4x+4y+4z+7, 6x+4y+4z+6] \cup [7x+8y+8z+14, 8x+8y+8z+13] \\ &\cup [6x+8y+8z+14, 7x+8y+8z+12] \cup \{6x+4y+4z+8, 8x+8y+8z+14\}, \\ V(G_2) &= [0, 2y] \cup [3y+4z+6, 4y+4z+6] \cup [2y+4z+6, 3y+4z+4] \\ &\cup \{2y+3, 4y+4z+7\}, \\ V(G_3) &= [6x+4y+4z+9, 6x+4y+6z+9] \cup [6x+4y+7z+14, 6x+4y+8z+13] \\ &\cup [6x+4y+6z+13, 6x+4y+7z+12] \cup \{6x+4y+6z+10, 6x+4y+8z+14\}. \end{aligned}$$

We can order these as follows.

G_i	Vertex Labels	G_i	Vertex Labels
G_2	$[0, 2y]$	G_3	$6x+4y+6z+10$
G_2	$2y+3$	G_3	$[6x+4y+6z+13, 6x+4y+7z+12]$
G_2	$[2y+4z+6, 3y+4z+4]$	G_3	$[6x+4y+7z+14, 6x+4y+8z+13]$
G_2	$[3y+4z+6, 4y+4z+6]$	G_3	$6x+4y+8z+14$
G_2	$4y+4z+7$	G_1	$[6x+8y+8z+14, 7x+8y+8z+12]$
G_1	$[4x+4y+4z+7, 6x+4y+4z+6]$	G_1	$[7x+8y+8z+14, 8x+8y+8z+13]$
G_1	$6x+4y+4z+8$	G_1	$8x+8y+8z+14$
G_3	$[6x+4y+4z+9, 6x+4y+6z+9]$		

The vertices of the three cycles are distinct and contained in $[0, 2(4x+4y+4z+7)] = [0, 8x+8y+8z+14]$. Note that if $z = 0$, the sets $[6x+4y+6z+13, 6x+4y+7z+12]$ and $[6x+4y+7z+14, 6x+4y+8z+13]$ are empty. If in addition $y = 1$, then the set $[2y+4z+6, 3y+4z+4]$ is empty. Finally, if in addition $x = 1$, the set $[6x+8y+8z+14, 7x+8y+8z+12]$ will also be empty. This however does not change the proof.

Likewise we compute

$$\begin{aligned} \overline{E}(G_1) &= [2x+4y+4z+7, 4x+4y+4z+6] \cup [4y+4z+8, 2x+4y+4z+5] \\ &\cup \{2, 2x+4y+4z+6, 4x+4y+4z+7\}, \\ \overline{E}(G_2) &= [2y+4z+5, 4y+4z+6] \cup [4z+6, 2y+4z+3] \\ &\cup \{3, 2y+4z+4, 4y+4z+7\}, \\ \overline{E}(G_3) &= [2z+5, 4z+4] \cup [4, 2z+3] \cup \{1, 2z+4, 4z+5\}. \end{aligned}$$

We can order these as follows.

Cycle	Edge Labels	Cycle	Edge Labels
G_3	1	G_2	$2y+4z+4$
G_1	2	G_2	$[2y+4z+5, 4y+4z+6]$
G_2	3	G_2	$[4y+4z+7]$
G_3	$[4, 2z+3]$	G_1	$[4y+4z+8, 2x+4y+4z+5]$
G_3	$2z+4$	G_1	$2x+4y+4z+6$
G_3	$[2z+5, 4z+4]$	G_1	$[2x+4y+4z+7, 4x+4y+4z+6]$
G_3	$4z+5$	G_1	$4x+4y+4z+7$
G_2	$[4z+6, 2y+4z+3]$		

Hence $\overline{E}(G) = [1, 4x + 4y + 4z + 7]$. Then we have a σ-labeling.

As with in the vertex labels, if $z = 0$ the sets $[4, 2z+3]$ and $[2z+5, 4z+4]$ will be empty. If in addition $y = 1$, then $[4z+6, 2y+4z+3]$ will be empty. Finally, if in addition $x = 1$, then $[4y + 4z + 8, 2x + 4y + 4z + 5]$ will also be empty. Neither condition would change the proof. □

Theorem 13. Let $x \geq 1$, $y \geq z$ be nonnegative integers, and let $G = C_{4x+2} \bigcup C_{4y+3} \bigcup C_{4z+3}$. Then G has a σ-labeling.

Proof. The three cycles $G_1 = C_{4x+2}, G_2 = C_{4y+3}$, and $G_3 = C_{4z+3}$ are defined as follows:

$$\begin{aligned}
G_1 &= P(0, 2x + 4y + 4z + 8, x) + P(x, 4y + 4z + 9, 2x - 1) \\
&+ (2x - 1, 2x + 4y + 4z + 7, 2x + 1, 4x + 4y + 4z + 8, 0), \\
G_2 &= P(4x + 4y + 4z + 9, 2y + 4z + 6, 4x + 5y + 4z + 9) \\
&+ P(4x + 5y + 4z + 9, 4z + 5, 4x + 6y + 4z + 9) \\
&+ (4x + 6y + 4z + 9, 4x + 6y + 4z + 11, 4x + 8y + 8z + 16, 4x + 4y + 4z + 9), \\
G_3 &= P(2x + 2, 2z + 4, 2x + z + 2) + P(2x + z + 2, 3, 2x + 2z + 2) \\
&+ (2x + 2z + 2, 2x + 2z + 3, 2x + 4z + 6, 2x + 2).
\end{aligned}$$

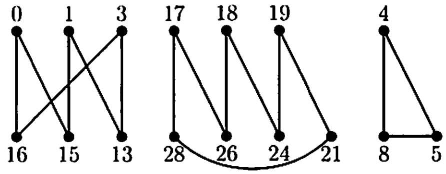

Figure 6. A σ-labeling of $C_6 \cup C_7 \cup C_3$

Now we compute

$$\begin{aligned}
V(G_1) &= [0, 2x - 1] \cup [3x + 4y + 4z + 8, 4x + 4y + 4z + 7] \\
&\cup [2x + 4y + 4z + 8, 3x + 4y + 4z + 6] \\
&\cup \{2x + 1, 2x + 4y + 4z + 7, 4x + 4y + 4z + 8\}, \\
V(G_2) &= [4x + 4y + 4z + 9, 4x + 6y + 4z + 9] \\
&\cup [4x + 7y + 8z + 15, 4x + 8y + 8z + 14] \\
&\cup [4x + 6y + 8z + 14, 4x + 7y + 8z + 13] \\
&\cup \{4x + 6y + 4z + 11, 4x + 8y + 8z + 16\}, \\
V(G_3) &= [2x + 2, 2x + 2z + 2] \cup [2x + 3z + 6, 2x + 4z + 5] \\
&\cup [2x + 2z + 5, 2x + 3z + 4] \cup \{2x + 2z + 3, 2x + 4z + 6\}.
\end{aligned}$$

We can order these as follows.

Cycle	Vertex Labels	Cycle	Vertex Labels
G_1	$[0, 2x-1]$	G_1	$[2x+4y+4z+8, 3x+4y+4z+6]$
G_1	$2x+1$	G_1	$[3x+4y+4z+8, 4x+4y+4z+7]$
G_3	$[2x+2, 2x+2z+2]$	G_1	$4x+4y+4z+8$
G_3	$2x+2z+3$	G_2	$[4x+4y+4z+9, 4x+6y+4z+9]$
G_3	$[2x+2z+5, 2x+3z+4]$	G_2	$4x+6y+4z+11$
G_3	$[2x+3z+6, 2x+4z+5]$	G_2	$[4x+6y+8z+14, 4x+7y+8z+13]$
G_3	$2x+4z+6$	G_2	$[4x+7y+8z+15, 4x+8y+8z+14]$
G_1	$2x+4y+4z+7$	G_2	$4x+8y+8z+16$

The vertices of the three cycles are distinct and contained in $[0, 2(4x+4y+4z+8)] = [0, 8x+8y+8z+8]$. Note that if $z=0$, the sets $[2x+2z+5, 2x+3z+4]$ and $[2x+3z+6, 2x+4z+5]$ will be empty. If in addition $y=0$, then the sets $[4x+6y+8z+14, 4x+7y+8z+13]$ and $[4x+7y+8z+15, 4x+8y+8z+14]$ will be empty. Finally, if $x=1$, the set $[2x+4y+4z+8, 3x+4y+4z+6]$ will also be empty. This however does not change the proof.

Likewise we compute

$$
\begin{aligned}
\overline{E}(G_1) &= [2x+4y+4z+8, 4x+4y+4z+7] \cup [4y+4z+9, 2x+4y+4z+6] \\
&\cup \{4y+4z+8, 4y+4z+6, 2x+4y+4z+7, 4x+4y+4z+8\}, \\
\overline{E}(G_2) &= [2y+4z+6, 4y+4z+5] \cup [4z+5, 2y+4z+4] \\
&\cup \{2, 2y+4z+5, 4y+4z+7\}, \\
\overline{E}(G_3) &= [2z+4, 4z+3] \cup [3, 2z+2] \cup \{1, 2z+3, 4z+4\}.
\end{aligned}
$$

We can order these as follows.

Cycle	Edge Labels	Cycle	Edge Labels
G_3	1	G_2	$[2y+4z+6, 4y+4z+5]$
G_2	2	G_1	$4y+4z+6$
G_3	$[3, 2z+2]$	G_2	$4y+4z+7$
G_3	$2z+3$	G_1	$4y+4z+8$
G_3	$[2z+4, 4z+3]$	G_1	$[4y+4z+9, 2x+4y+4z+6]$
G_3	$4z+4$	G_1	$2x+4y+4z+7$
G_2	$[4z+5, 2y+4z+4]$	G_1	$[2x+4y+4z+8, 4x+4y+4z+7]$
G_2	$2y+4z+5$	G_1	$4x+4y+4z+8$

Hence $\overline{E}(G) = [1, 4x+4y+4z+4]$. Then we have a σ-labeling.

As with the vertex labels, note that if $z=0$, then $[3, 2z+2]$ and $[2z+4, 4z+3]$ will be empty. If in addition $y=0$, then $[4z+5, 2y+4z+4]$ and $[2y+4z+6, 4y+4z+5]$ are empty. Finally, if $x=1$, the set $[4y+4z+9, 2x+4y+4z+6]$ is empty. Neither condition would however change the proof. □

We conclude this section with our main result.

Theorem 14. Let G be a 2-regular graph of size n and at most three components. Then G admits a σ-labeling if and only if $n \equiv 0$ or $3 \pmod 4$.

Proof. The condition $n \equiv 0$ or $3 \pmod 4$ is necessary by Theorem 4 (the parity condition). If G has at most two components, then sufficiency is obtained from Theorems 5 and 6. Now let r, s and t be positive integers ≥ 3 and let $G = C_r \bigcup C_s \bigcup C_t$ (thus $n = r + s + t$). If $r \equiv s \equiv t \equiv 0 \pmod 4$, or $r \equiv 0 \pmod 4$ and $s \equiv t \equiv 2 \pmod 4$, then G admits an α-labeling by Eshghi's results [12], unless $r = s = t = 4$ in which case G has a σ^+-labeling by [11]. If $r \equiv s \equiv 0 \pmod 4$ and $t \equiv 3 \pmod 4$, then G admits a σ-labeling by Theorem 8. If $r \equiv 0 \pmod 4$, $s \equiv 1 \pmod 4$ and $t \equiv 2$ or $3 \pmod 4$, then G admits a σ-labeling by Corollary 9. The case $r \equiv s \equiv 2 \pmod 4$, $t \equiv 3 \pmod 4$, is obtained similarly. The rest of the cases are done by the previous four theorems. □

4. Concluding Remarks

The study of graph decompositions is a popular branch of modern combinatorial design theory (see [6] for an overview). In particular, the study of G-decompositions of K_{2n+1} (and of K_{2nx+1}) when G is a graph with n edges (and x is a positive integer) has attracted considerable attention. The study of graph labelings is also quite popular (see Gallian [13] for a dynamic survey). Theorems 1 and 2 provide powerful links between the two areas. Much of the attention on labelings has been on graceful labelings (i.e., β-labelings). Unfortunately, the parity condition "disqualifies" large classes of graphs from admitting graceful labelings. This difficulty is compounded by the fact that certain classes of graphs with ρ-labelings meet the parity condition, yet fail to be graceful ($C_3 \bigcup C_3 \bigcup C_5$ is one such example).

In conclusion, we note that our results here, along with results from [3], [8] and [14] among others, provide further evidence in support of the following conjecture of El-Zanati and Vanden Eynden.

Conjecture 15. Every 2-regular G graph of size n has a ρ-labeling. Moreover, if $n \equiv 0$ or $3 \pmod 4$ then G has a σ-labeling.

As a final comment, we note that this work was done while the first author was enrolled in an undergraduate research program at Illinois State University.

References

[1] J. Abrham and A. Kotzig, All 2-regular graphs consisting of 4-cycles are graceful, *Discrete Math.* **135** (1994), 1 14.

[2] J. Abrham and A. Kotzig, Graceful valuations of 2-regular graphs with two components, *Discrete Math.* **150** (1996), 3–15.

[3] A. Aguado, S.I. El-Zanati, H. Hake, J. Stob, and H. Yayla, On ρ-labeling the union of three cycles, *Australas. J. Combin.*, to appear.

[4] A. Blinco and S.I. El-Zanati, A note on the cyclic decomposition of complete graphs into bipartite graphs, *Bull. Inst. Combin. Appl.* **40**, (2004), 77–82.

[5] A. Blinco, S.I. El-Zanati, and C. Vanden Eynden, On the cyclic decomposition of complete graphs into almost-bipartite graphs, *Discrete Math.* **284** (2004), 71–81.
[6] J. Bosák, *Decompositions of Graphs*, Kluwer Academic Publishers Group, Dordrecht, 1990.
[7] R.C. Bunge, S.I. El-Zanati, and C. Vanden Eynden, On γ-labelings of almost-bipartite graphs, in preparation.
[8] J.H. Dinitz and P. Rodney, Disjoint difference families with block size 3, *Util. Math.* **52** (1997), 153–160.
[9] D. Donovan, S.I. El-Zanati, C. Vanden Eynden, and S. Sutinuntopas, Labelings of unions of up to four uniform cycles, *Australas. J. Combin.* **29**(2004), 323–336.
[10] J. Dumouchel and S.I. El-Zanati, On labeling the union of two cycles, *J. Comb. Math. Comb. Comput.* **53** (2005), 3–11.
[11] S.I. El-Zanati, C. Vanden Eynden and N. Punnim, On the cyclic decomposition of complete graphs into bipartite graphs, *Australas. J. Combin.* **24** (2001), 209–219.
[12] K. Eshghi, The existence and construction of α-valuations of 2-regular graphs with 3 components, Ph.D. Thesis, Industrial Engineering Dept., University of Toronto, 1997.
[13] J.A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.*, Dynamic Survey 6, 148 pp.
[14] H. Hevia and S. Ruiz, Decompositions of complete graphs into caterpillars, *Rev. Mat. Apl.* **9** (1987), 55–62.
[15] A. Kotzig, β-valuations of quadratic graphs with isomorphic components, *Util. Math.*, **7** (1975), 263–279.
[16] A. Kotzig, Recent results and open problems in graceful graphs, *Congress. Numer.* **44** (1984), 197–219.
[17] M. Maheo and H. Thuillier, On d-graceful graphs, *Ars Combin.* **13** (1982), 181–192.
[18] A. Rosa, On the cyclic decomposition of the complete graph into polygons with odd number of edges, *Časopis Pěst. Mat.* **91** (1966), 53–63.
[19] A. Rosa, On certain valuations of the vertices of a graph, in: Théorie des graphes, journées internationales d'études, Rome 1966 (Dunod, Paris, 1967), 349–355.
[20] P.J. Slater, On k-graceful graphs, *Congr. Numer.* **36** (1982), 53–57.