CHANGE IN IRREGULARITY STRENGTH BY AN EDGE

KM. KATHIRESAN,
DEPARTMENT OF MATHEMATICS,
AYYA NADAR JANAKI AMMAL COLLEGE,
SIVAKASI – 626 124,
INDIA, kathir2esan@yahoo.com.

K. MUTHUGURUPACKIAM,
DEPARTMENT OF MATHEMATICS,
ARULMIGU KALASALINGAM COLLEGE OF ARTS AND SCIENCE,
KRISHNANKOIL – 626 190,
INDIA, gurupackiam@yahoo.com.

Revised : July 22, 2006.

ABSTRACT. In this paper we discuss how the addition of a new edge affects the irregularity strength of a graph.

1. INTRODUCTION

In this paper we consider simple undirected and connected graphs with no K_2 components and at most one isolated vertex. Let $G = (V, E)$ be a graph. A network $G(w)$ consists of the graph together with an assignment $w : E(G) \to Z^+$. The sum of the weights of the edges incident with a vertex is called the label of that vertex. If all the labels are pairwise distinct, $G(w)$ is called an irregular network. The strength of the network $G(w)$ is defined by $s(G(w)) = \max_{e \in E}\{w(e)\}$. The irregularity strength $s(G)$ of G is defined as $s(G) = \min\{s(G(w)) / G(w) \text{ is irregular}\}$.

The problem of finding irregularity strength of graphs was proposed by Chartrand et al., [2] and has proved to be difficult, in general. There are not many graphs for which the irregularity strength is known. The readers are advised to refer the survey of Lehel [6] and the papers [3,8]. Stanislar Jendrol and Michal Tkac [8] studied the irregularity strength of the union of t copies of the complete graph K_p. Gyarfas [4] determined the irregularity strength of $K_n - mK_2$. Jeffrey H. Dinitz [5] determined the irregularity strength of the $m \times n$ grid for certain m and n. Olivier Togni [7] studied the irregularity strength of the toroidal grid. D. Amar and O. Togni [1] established that the irregularity strength of any tree with no vertices of degree 2 is its number of pendent vertices.

Definition 1.1. Let G be any graph which is not complete, e be any edge of $\overline{G}$, then e is called the positive edge if $s(G+e) > s(G)$, e is called a negative edge if $s(G+e) < s(G)$ and e is called a stable edge if $s(G+e) = s(G)$.

Example 1.2. In P_3, the edge joining the end points is a positive edge . In C_4, the edge joining any two diametrically opposite vertices is a negative edge.
In P_4, the edge joining one pendent vertex and one internal vertex with distance 2 from this pendent vertex is a stable edge.

Definition 1.3. If all the edges of $\overline{G}$ are positive, (negative, stable) edges of G, then G is called a positive (negative, stable) graph. Otherwise G is called a mixed graph.

Example 1.4. Star graphs are negative graphs.

In this paper we discuss positive, negative and stable edges of two families of graphs namely paths and cycles.

2. Positive, Negative and Stable edges of certain families of graphs.

We shall use the following functions to simply the later notations.

$$\text{Let } \alpha(i) = \begin{cases} 1 & \text{if } i \equiv 1 \pmod 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\beta(i) = \begin{cases} 1 & \text{if } i \equiv 1 \pmod 4 \\ 0 & \text{otherwise} \end{cases}$$

$$\gamma(i) = \begin{cases} 1 & \text{if } i \equiv 2 \pmod 4 \\ 0 & \text{otherwise} \end{cases}$$

First we determine the positive, negative and stable edges of the path P_n.

Theorem 2.1 Let $v_1, v_2, v_3, \ldots, v_n$,$n \geq 3$ be the n consecutive vertices of P_n .If $n \equiv 0 \pmod 4$,then the edge $v_1 v_n$ is positive and all other edges of P_n are stable. If $n \equiv 1 \pmod 4$, then $v_1 v_n$, $v_2 v_n$ and $v_1 v_{n-1}$ are stable and all other edges are negative. If $n \equiv 2 \pmod 4$, then $v_1 v_n$ is stable and all other edges are negative. If $n \equiv 3 \pmod 4$, then $v_1 v_n$ is positive and all other edges are negative.

Proof. Let $v_1, v_2, v_3, ..., v_n$ be the n consecutive vertices of P_n. Now add any edge of $\overline{P_n}$ to P_n. We get either a cycle or a cycle of length m with a path of length m-n (the tail) attached to one vertex of the cycle or a cycle of length m with two paths attached to two consecutive vertices of the cycle.

Suppose the edge is v_1v_n then the graph becomes C_n. By[3],

$$s(C_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil & \text{for } n \equiv 1(\text{mod} 4) \\ \left\lceil \frac{n}{2} \right\rceil + 1 & \text{otherwise.} \end{cases}$$

It is easy to verity that $s(P_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil + 1 & \text{if n} \equiv 2(\text{mod4}) \\ \left\lceil \frac{n}{2} \right\rceil & \text{otherwise} \end{cases}$

Hence v_1v_n is a positive edge if $n \not\equiv 1,2(\text{mod} 4)$, otherwise it is a stable edge.

Assume that the edge is v_mv_1 where $2 < m < n$. Since the graph $P_n + v_mv_1$ and $P_n + v_nv_{n-m+1}$ are isomorphic, it is enough to verify the theorem for $P_n + v_nv_{n-m+1}$.

Case(1)

Assume that $m \equiv 0(\text{mod} 4)$ and n-m $\neq 1$. Let $k = \left\lfloor \frac{n}{4} \right\rfloor$ and $t = \left\lfloor \frac{m}{4} \right\rfloor$

Now define the weight function $w : E(P_n + v_nv_{n-m+1}) \to Z^+$ by

(i) $w(v_{n-m+i}\ v_{n-m+1+i}) = \left\lceil \frac{n}{2} \right\rceil - \alpha(n)\left\lceil \frac{i}{2} \right\rceil - \alpha(n+1)\left\lfloor \frac{i}{2} \right\rfloor$, $\quad i = 1,2,3,...,2t$.

(ii) $w(v_{n-2t+i}\ v_{n-2t+1+i}) = \left\lfloor \frac{n}{2} \right\rfloor - t - \alpha(n)\left\lfloor \frac{i}{2} \right\rfloor - \alpha(n+1)\left\lceil \frac{i}{2} \right\rceil$,

$i = 1,2,3,...,2t - 1$

(iii) $w(v_i\ v_{i+1}) = \left\lceil \frac{i}{2} \right\rceil$, $i = 1,2,..,2k - 2t + 2\alpha(n) + \gamma(n)$

(iv) $w\left(v_{2k-2t+2\alpha(n)+\gamma(n)+i}\ v_{2k-2t+2\alpha(n)+\gamma(n)+1+i}\right)$

$$= k - t + \alpha(n)\left[\alpha(i)\left(2 + \left\lceil \frac{i}{2} \right\rceil\right) + \alpha(i+1)\left(1 + \frac{i}{2}\right)\right]$$

$$+ \gamma(n)\left(1 + \left\lceil \frac{i}{2} \right\rceil\right) + \gamma(n+2)\left(\alpha(i)\left(1 + \left\lceil \frac{i}{2} \right\rceil\right) + \alpha(i+1)\left(\frac{i}{2}\right)\right),$$

$$i = 1,2,3,\ldots,2k - 2t - \alpha\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)(2\alpha(n) + \alpha(n+1))$$

(v) $w(v_{n-m}\ v_{n-m+1}) = \left\lceil \frac{n}{2} \right\rceil - 1 + \gamma(n)$

(vi) $w(v_{n-m+1}\ v_n) = 1 + \beta(n)$

If n-m = 1, then define the weight function $w : E(P_n + v_n v_{n-m+1}) \to Z^+$ by

(i) $w(v_1\ v_2) = 1$, $w(v_{4k+1}\ v_2) = 2k + 1$,

(ii) $w(v_{i+1}\ v_{i+2}) = 2k + 1 - i$, $i = 1,2,\ldots,2k$

(iii) $w(v_{2k+1+i}\ v_{2k+2+i}) = i\alpha(i) + (i+1)\alpha(i+1)$, $i = 1,2,3,\ldots,2k-1$

By case (1), we have an irregular network $P_n + v_{n-m+1}v_n$ with maximum weight $\left\lceil \frac{n}{2} \right\rceil - \alpha(n) < s(P_n)$ if $n \not\equiv 0 (\mathrm{mod}\ 4)$, and $n - m \neq 1$. Hence the edge $v_{n-m+1}v_n$ is a negative edge. If n-m=1, then the maximum weight is $2k+1 = \left\lceil \frac{n}{2} \right\rceil$ and hence $v_2 v_n$ and $v_1 v_{n-1}$ are stable. If $n \equiv 0 (\mathrm{mod}\ 4)$, then the optimal labels are {1,2,…,4k-1,4k}. In $P_n + v_{n-m+1}v_n$,there is only one vertex having degree 3, so it is not possible to obtain the optimal labels with fewer than $2k = \left\lceil \frac{n}{2} \right\rceil$ weight. Hence $v_{n-m+1}v_n$ is stable a edge if $n \equiv 0 (\mathrm{mod}\ 4)$.

Case (2). If $m \equiv 1 (\mathrm{mod}\ 4)$, then define the weight function $w : E(P_n + v_n v_{n-m+1}) \to Z^+$ by

(i) $w(v_{n-m+i}\ v_{n-m+1+i}) = 2k - \beta(n)\left\lfloor \frac{i-1}{2} \right\rfloor + \beta(n+2)\left(2 - \left\lceil \frac{i}{2} \right\rceil\right)$

$$+\gamma(n)\left(1-\left\lfloor \frac{i}{2}\right\rfloor\right) - \gamma(n+2)\left\lfloor \frac{i}{2}\right\rfloor,$$

$i = 1,2,3,\ldots,2t + \gamma(n+2)$

(ii) $w\left(v_{n-2t+i-1+\gamma(n+2)}\ v_{n-2t+i+\gamma(n+2)}\right)$

$= 2k - t - \beta(n)\left\lfloor \frac{i}{2}\right\rfloor - \beta(n+2)\left(\left\lfloor \frac{i}{2}\right\rfloor - 1\right) + \gamma(n)\left(\left\lceil \frac{i}{2}\right\rceil - 1\right) + \gamma(n+2)\left(1 + \left\lfloor \frac{i}{2}\right\rfloor\right)$

, $i = 1,2,\ldots,2t - \gamma(n+2)$

(iii) $w(v_i\ v_{i+1}) = \left\lceil \frac{i}{2} \right\rceil$, $i = 1,2,\ldots,2k - 2t + 1 - \gamma(n+2)$

(iv) $w\left(v_{2k-2t+1-\gamma(n+2)+i}\ v_{2k-2t+2-\gamma(n+2)+i}\right)$

$= k - t + \alpha(n)\left(1 + \left\lceil \frac{i}{2}\right\rceil\right) + \gamma(n)\left(1 + \left\lceil \frac{i}{2}\right\rceil\right) + \gamma(n+2)\left(\alpha(i)\left(1 + \left\lceil \frac{i}{2}\right\rceil\right) + \alpha(i+1)\left(\frac{i}{2}\right)\right)$

$i = 1,2,3,\ldots,2k - 2t - 2\beta(n) - \gamma(n) - 2\gamma(n+2)$

(v) $w(v_{n-m}\ v_{n-m+1}) = \begin{cases} 2k+1, & \text{if } n \equiv 2 \pmod 4 \text{ and if } t < k \\ 2k + \beta(n+2) - \gamma(n+2), & \text{otherwise} \end{cases}$

(vi)a) If $n \not\equiv 2 \pmod 4$ $w(v_{n-m+1}\ v_n) = 1 + \beta(n)$

(vi)b)If $n \equiv 2 \pmod 4$ $w(v_{n-m+1}\ v_n) = \begin{cases} 1 & \text{if } t < k \\ 2k+1 & \text{if } t = k. \end{cases}$

By case(2), we have an irregular network $P_n + v_{n-m+1} v_n$ with maximum weight $2k + \beta(n+2) + \gamma(n) < s(P_n)$ if $n \not\equiv 0 \pmod 4$. Hence the edge $v_{n-m+1} v_n$ is negative if $n \not\equiv 0 \pmod 4$ and is stable if $n \equiv 0 \pmod 4$.

Case (3). If $m \equiv 2 \pmod 4$, then define the weight function $w : E(P_n + v_n v_{n-m+1}) \rightarrow Z^+$ by

(i) $w(v_{n-m+i}\ v_{n-m+1+i})$

$$= 2k - \beta(n)\left\lfloor \frac{i-1}{2} \right\rfloor - \beta(n+2)\left(\left\lceil \frac{i}{2} \right\rceil - 2\right) + \gamma(n)\left(\left\lfloor \frac{i}{2} \right\rfloor - 1\right) + \gamma(n+2)\left\lfloor \frac{i}{2} \right\rfloor$$

$i = 1,2,3,\ldots,2t+1$

(ii) $w(v_{n-2t-1+i}\ v_{n-2t+i})$

$$= 2k - t - \beta(n)\left[\alpha(i)\left(\frac{i-1}{2}\right) + \alpha(i+1)\left(1 + \frac{i}{2}\right)\right] - \beta(n+2)\left(\left\lceil \frac{i}{2} \right\rceil - 1\right)$$

$$+ \gamma(n)\left[\alpha(i)\left(\frac{i+1}{2}\right) + \alpha(i+1)\left(\frac{i}{2} - 1\right)\right] + \gamma(n+2)\left(1 + \left\lfloor \frac{i}{2} \right\rfloor\right), \quad i = 1,2,3,\ldots,2t$$

(iii) $w(v_i\ v_{i+1}) = \left\lceil \frac{i}{2} \right\rceil, i = 1,2,3,\ldots,2k - 2t + \beta(n+2) + \gamma(n) - \gamma(n+2)$

(iv) $w\left(v_{2k-2t+\beta(n+2)+\gamma(n)-\gamma(n+2)+i}\ v_{2k-2t+1+\beta(n+2)+\gamma(n)-\gamma(n+2)+i}\right)$

$$= k - t + \left\lceil \frac{i}{2} \right\rceil + \alpha(n)\big(\beta(n)\alpha(i) + \beta(n+2)\big) + \gamma(n),$$

$i = 1,2,3,\ldots,2k - 2t - 2 + \beta(n+2)$

(v)a)If $n \equiv 3(\mathrm{mod}\,4)$ $\quad w(v_{n-m}\ v_{n-m+1}) = 2k+1$ when $t < k$.

(b) If $n \not\equiv 3(\mathrm{mod}\,4)$ $\quad w(v_{n-m}\ v_{n-m+1}) = 2k + \gamma(n) - \gamma(n+2)$

(vi)a)If $n \equiv 3(\mathrm{mod}\,4)$ $\quad w(v_{n-m+1}\ v_n) = \begin{cases} 1 & \text{if } t < k \\ 2k+1 & \text{if } t = k \end{cases}$

b) If $n \not\equiv 3(\mathrm{mod}\,4)$ $\quad w(v_{n-m+1}\ v_n) = 2\alpha(n) + \alpha(n+1)$

By case(3), we have an irregular network $P_n + v_{n-m+1}v_n$ with maximum weight $2k + \beta(n+2) + \gamma(n) < s(P_n)$ if $n \not\equiv 0(\mathrm{mod}\,4)$. Hence the edge $v_{n-m+1}v_n$ is negative if $n \not\equiv 0(\mathrm{mod}\,4)$ and is stable if $n \equiv 0(\mathrm{mod}\,4)$.

Case (4). If $m \equiv 3(\mathrm{mod}\,4)$, then define the weight function $w : E(P_n + v_n v_{n-m+1}) \to Z^+$ by

(i) $w(v_{n-m+i}\ v_{n-m+1+i}) = \left\lfloor \frac{n}{2} \right\rfloor - \alpha(n)\left\lfloor \frac{i-1}{2} \right\rfloor - \alpha(n+1)\left\lfloor \frac{i}{2} \right\rfloor$,

$i = 1,2,3,\ldots,2t + 2\alpha(n) + \alpha(n+1)$

(ii)a)If $n \equiv 1(\bmod 4)$,

$$w(v_{n-2t-1+i}\ v_{n-2t+i}) = 2k - t - \alpha(i)\left(1 + \left\lceil \frac{i}{2} \right\rceil\right) - \alpha(i+1)\left\lceil \frac{i}{2} \right\rceil$$

$$i = 1,2,3,\ldots,2t - \left\lfloor \frac{t}{k-1} \right\rfloor$$

$w(v_{n-1}\ v_n) = 1$ if $t = k-1$

(ii)b)If $n \not\equiv 1(\bmod 4)$, $w\left(v_{n-2t-1-\alpha(n+1)+i}\ v_{n-2t-\alpha(n+1)+i}\right)$

$$= 2k - t - \alpha(n)\left[\alpha(i)\left\lceil \frac{i}{2} \right\rceil + \alpha(i+1)\left(\left\lceil \frac{i}{2} \right\rceil - 1\right)\right] - \gamma(n)\left\lfloor \frac{i}{2} \right\rfloor - \gamma(n+2)\left(1 + \left\lfloor \frac{i}{2} \right\rfloor\right),$$

$$i = 1,2,3,\ldots,2t + \alpha(n+1)$$

(iii) $w(v_i\ v_{i+1}) = \left\lceil \frac{i}{2} \right\rceil$,

$$i = 1,2,\ldots,2k - 2t + \beta(n)\left(1 - 2\left\lfloor \frac{k-1}{t} \right\rfloor\right) + \beta(n+2) - \gamma(n+2)$$

(iv) $w\left(v_{2k-2t+\alpha(n)-\gamma(n+2)+i}\ v_{2k-2t+\alpha(n)-\gamma(n+2)+1+i}\right)$

$$= k - t + \left\lceil \frac{i}{2} \right\rceil + \alpha(n) + \gamma(n)\alpha(i),$$

$$i = 1,2,3,\ldots,2k - 2t - 2\alpha\left(\left\lfloor \frac{n}{2} \right\rfloor\right) - \alpha\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)\left(4\alpha(n) + 3\alpha(n+1)\right)$$

(v)a) If $n \not\equiv 0(\bmod 4)$, $w(v_{n-m}\ v_{n-m+1}) = 2k + \alpha\left(\left\lfloor \frac{n}{2} \right\rfloor\right)$

b) If $n \equiv 0(\bmod 4)$, $w(v_{n-m}\ v_{n-m+1}) = 2k - 1$ if $t < k-1$

(vi)a) If $n \not\equiv 0(\bmod 4)$, $w(v_{n-m+1}\ v_n) = \alpha(n)\left[2\beta(n) + \beta(n+2)\right] + \alpha(n+1)$

b) If $n \equiv 0(\bmod 4)$, $w(v_{n-m+1} v_n) = \begin{cases} 1 & \text{if } t < k-1 \\ 2k-1 & \text{if } t = k-1 \end{cases}$

By case(4), we have an irregular network $P_n + v_{n-m+1} v_n$ with maximum weight $\left\lfloor \frac{n}{2} \right\rfloor < s(P_n)$ if $n \not\equiv 0(\bmod 4)$. Hence the edge $v_{n-m+1} v_n$ is negative if $n \not\equiv 0(\bmod 4)$ and is stable if $n \equiv 0(\bmod 4)$.

Now let us discuss the case in which the graph consists of a cycle of length m with two paths attached to two consecutive vertices of the cycle. Add the edge $v_{n-m+1-j} v_{n-j}$, since the graphs $P_n + v_{n-m+1-j} v_{n-j}$ and $P_n + v_{j+1} v_{m+j}$ are isomorphic, it is enough to verify the theorem for $P_n + v_{n-m+1-j} v_{n-j}$, where $1 \le j \le \left\lceil \frac{n-m-1}{2} \right\rceil$.

Define the weight function $w : E(P_n + v_{n-j} v_{n-m+1-j}) \to Z^+$ by

A. $w(v_{n-i}\ v_{n+1-i}) = (i+1)\alpha(i) + i\alpha(i+1),\ i = 1,2,3,\dots, j-1$.

B. $w(v_i\ v_{i+1}) = i,\ i = 1,2,3,\dots, j-1$.

Case(1) If $n \equiv 0(\bmod 4)$, then

(i) $w(v_{n-j}\ v_{n+1-j}) = \left\lfloor \frac{n-m}{2} \right\rfloor + \beta(n+2) + \gamma(n)$

(ii) $w(v_{n-j}\ v_{n-m+1-j}) = 2t+1$.

(iii)a) If $n \not\equiv 3(\bmod 4)$, $w(v_{n-j-i}\ v_{n+1-j-i}) = \left\lfloor \frac{n}{2} \right\rfloor - i\alpha(i) - (i-1)\alpha(i+1)$,

$i = 1,2,3,\dots,2t$

b) (1) If $n \equiv 3(\bmod 4)$, $w(v_{n-j-1}\ v_{n-j}) = 2k$

(2) $w(v_{n-j-1-i}\ v_{n-j-i}) = \left\lfloor \frac{n}{2} \right\rfloor + \alpha(i)(1-i) - i\alpha(i+1)$,

$i = 1,2,3,\dots,2t-1$.

(iv) $w(v_{n-m-j+i}\ v_{n-m-j+1+i})$

$= 2k+1-i+\beta(n+2)(\alpha(i+1)-\alpha(i))+\gamma(n)(\alpha(i+1)-\alpha(i))-2\gamma(n+2)\alpha(i),$

$i = 1,2,3,\ldots,2t-1.$

(v) $w(v_{j-1+i}\ v_{j+i}) = j+\left\lfloor \frac{i-1}{2} \right\rfloor, \quad i = 1,2,\ldots,2k-2t+1-2\left\lceil \frac{j}{2} \right\rceil + 2\alpha\left(\left\lfloor \frac{n}{2} \right\rfloor\right)$

(vi) $w\left(v_{2k-2t-2\left\lceil \frac{j}{2} \right\rceil+2\alpha\left(\left\lfloor \frac{n}{2} \right\rfloor\right)+j+i}\ v_{2k-2t-2\left\lceil \frac{j}{2} \right\rceil+2\alpha\left(\left\lfloor \frac{n}{2} \right\rfloor\right)+j+i+1}\right)$

$= k-t+\left\lfloor \frac{j}{2} \right\rfloor+\left\lceil \frac{i}{2} \right\rceil+\alpha\left(\left\lfloor \frac{n}{2} \right\rfloor\right), \quad i = 1,2,3,\ldots,\left\lceil \frac{n-m}{2} \right\rceil-2\left\lfloor \frac{j}{2} \right\rfloor-\alpha\left(\left\lfloor \frac{n}{2} \right\rfloor\right)$

Case (2). If $m \equiv 1 \pmod 4$, then

(i) $w(v_{n-j}\ v_{n+1-j}) = \left\lfloor \frac{n-m}{2} \right\rfloor + \alpha\left(\left\lceil \frac{n}{2} \right\rceil + 1\right)$

(ii) $w(v_{n-j}\ v_{n-m+1-j}) = 2t+\beta(n)+2\gamma(n)+\gamma(n+2)$

(iii)a) $w(v_{n-j-1}\ v_{n-j}) = 2k+\beta(n+2)-\alpha(n+1)$

b) $w(v_{n-j-i}\ v_{n-j+1-i}) = 2k+1-i+\alpha(i)\left[\alpha(n)-\alpha(n+1)\right], \ i = 2,3,\ldots,2t$

(iv)a) $w(v_{n-m-j+1}\ v_{n-m-j+2}) = 2k+\beta(n+2)-2\beta(n)-2\gamma(n+2)$

b) $w(v_{n-m-j+i}\ v_{n-m-j+1+i})$

$= 2k-i+\beta(n)\left[\alpha(i+1)-\alpha(i)\right]-2\beta(n+2)+2\gamma(n)+\gamma(n+2)\left[\alpha(i+1)-\alpha(i)\right],$

$i = 2,3,\ldots,2t-1$

c) $w(v_{n-2t-j-1}\ v_{n-2t-j}) = 2k-2t+\beta(n)+2\beta(n+2)+2\gamma(n)$

(v) $w(v_{j-1+i}\ v_{j+i}) = j+\left\lfloor \frac{i-1}{2} \right\rfloor, \ i = 1,2,3,\ldots,2k-2t+1-2\left\lceil \frac{j}{2} \right\rceil + 2\beta(n+2)$

(vi) $w\left(v_{2k-2t-2\left\lceil \frac{j}{2} \right\rceil+2\beta(n+2)+j+i}\ v_{2k-2t-2\left\lceil \frac{j}{2} \right\rceil+2\beta(n+2)+j+i+1}\right)$

$$= k - t + \left\lfloor \frac{j}{2} \right\rfloor + \left\lceil \frac{i}{2} \right\rceil + \beta(n+2),\ i = 1,2,3,\ldots,\left\lfloor \frac{n-m}{2} \right\rfloor - \beta(n+2) - 2\left\lfloor \frac{j}{2} \right\rfloor + \gamma(n)$$

Case (3). If $m \equiv 2 (\mathrm{mod}\, 4)$, then

(i) $w(v_{n-j}\ v_{n+1-j}) = 2k - 2t$

(ii) $w(v_{n-j}\ v_{n-m+1-j}) = 2t + 2 - \gamma(n+2)$

(iii)a) $w(v_{n-j-1}\ v_{n-j}) = 2k + \gamma(n) - \gamma(n+2)$

b) $w(v_{n-j-i}\ v_{n-j+1-i})$

$$= \alpha(i)\left[2k + 2 - i - \beta(n+2) - 2\gamma(n+2)\right] + \alpha(i+1)\left[2k + 1 - i + \beta(n+2)\right],$$

$i = 2,3,\ldots,2t$

c) $w(v_{n-j-2t-1}\ v_{n-j-2t}) = 2k - 2t + \beta(n) - \gamma(n+2)$

(iv)a) $w(v_{n-m-j+i}\ v_{n-m-j+i+1})$

$$= \alpha(i)\left[2k - i - \beta(n) + \beta(n+2)\right] + \alpha(i+1)\left[2k - i + \beta(n) + 3\beta(n+2) + 2\gamma(n)\right],$$

$i = 1,2,3,\ldots,2t-1$

b) $w(v_{n-m-j+2t}\ v_{n-m-j+2t+1}) = 2k - 2t + 3\beta(n+2) + 2\gamma(n)$

(v) $w(v_{j-1+i}\ v_{j+i}) = j + \left\lfloor \frac{i-1}{2} \right\rfloor,\ i = 1,2,3,\ldots,2k - 2t + 1 - 2\left\lceil \frac{j}{2} \right\rceil$

(vi) $w\left(v_{2k-2t-2\left\lceil \frac{j}{2} \right\rceil + j + i}\ v_{2k-2t-2\left\lceil \frac{j}{2} \right\rceil + j + i + 1} \right) = k - t + \left\lfloor \frac{j}{2} \right\rfloor + \left\lceil \frac{i}{2} \right\rceil,$

$$i = 1,2,3,\ldots,2k - 2t - 2\left\lfloor \frac{j}{2} \right\rfloor - \beta(n) + \beta(n+2) - 2\gamma(n+2)$$

Case(4). If $m \equiv 3 (\mathrm{mod}\, 4)$, then

(i) $w(v_{n-j}\ v_{n-j+1}) = \left\lfloor \frac{n-m}{2} \right\rfloor + \alpha\left(\left\lceil \frac{n}{2} \right\rceil\right)$

(ii) $w(v_{n-j}\ v_{n-m+1-j}) = 2t + 2$

(iii)a) $w(v_{n-j-1}\ v_{n-j}) = 2k + \beta(n+2) + \gamma(n) - \gamma(n+2)$

b) $w(v_{n-j-i}\ v_{n+1-j-i}) = \alpha(i)\left[2k + 2 - i - 2\gamma(n+2)\right] + \alpha(i+1)(2k + 1 - i)$,

$i = 2,3,\ldots,2t+1$

(iv) $w\left(v_{n-m-j+i}\ v_{n-m-j+1+i}\right)$

$= \alpha(i)\left[2k - i + \beta(n+2)\right] + \alpha(i+1)\left[2k - i + 3\beta(n+2) + 2\gamma(n)\right],\quad i = 1,2,3,\ldots,2t+1$

(v) $w(v_{j-1+i}\ v_{j+i}) = j + \left\lfloor \frac{i-1}{2} \right\rfloor,\ i = 1,2,3,\ldots,2k - 2t + 1 - 2\left\lceil \frac{j}{2} \right\rceil - 2\gamma(n+2)$

(vi) $w\left(v_{2k-2t-2\left\lceil \frac{j}{2} \right\rceil - 2\gamma(n+2)+j+i}\ v_{2k-2t-2\left\lceil \frac{j}{2} \right\rceil - 2\gamma(n+2)+j+i+1} \right)$

$= k - t + \left\lfloor \frac{j}{2} \right\rfloor + \left\lceil \frac{i}{2} \right\rceil - \gamma(n+2),\ i = 1,2,3,\ldots,2k - 2t - 2\left\lfloor \frac{j}{2} \right\rfloor - 2\beta(n) - \alpha(n+1)$

By cases (1),(2),(3) and (4), we have an irregular network of $P_n + v_{n-j}\ v_{n-m+1-j}$ with consecutive labels 1,2,3,…,n or almost consecutive labels 1,2,3,…,n-1,n+1 with maximum weight $\left\lceil \frac{n}{2} \right\rceil < s(P_n)$ if $n \equiv 0 (\mathrm{mod}\, 4)$ and $\left\lceil \frac{n}{2} \right\rceil + 1 < s(P_n)$ if $n \not\equiv 0 (\mathrm{mod}\, 4)$. Hence the edge $v_{n-j}\ v_{n-m+1-j}$ is negative. ∎

Now we shall prove that all cycles are negative graphs.

Theorem 2.2 The cycle C_n is a negative graph for any $n \geq 4$.

Proof. Let $v_1, v_2, \ldots, v_n$ be the n vertices of C_n. Add any edge of $\overline{C_n}$ to C_n, we get a union of two cycles with a common edge.

Case (1). Suppose $n \equiv 0 (\mathrm{mod}\, 4)$. Let n=4k. Then the two cycles are of the same parity.

If we add the edge $v_{2k+1}\ v_{2k-2t+1}$ to C_n, we get the union of two odd cycles of length $m = 2t+1$ and $n - 2t + 1$ with a common edge $v_{2k+1}\ v_{2k-2t+1}$, which is isomorphic to the graph $C_n + e$ where two odd cycles are of length m and n-m+2, where e is arbitrary. So it is enough to verify the theorem for the graph $C_n + v_{2k+1}v_{2k-2t+1}$ for $t = 1,2,3,\ldots,k-1$.

If we add the edge $v_{2k+1}\ v_{2k-2t+2}$ to C_n, we get the union of two even cycles of length $m = 2t$ and $n - 2t + 2$ with a common edge $v_{2k+1}\ v_{2k-2t+2}$, which is isomorphic to the graph $C_n + e$ where two even cycles are of length m and $n-m+2$ respectively, for an arbitrary edge *e*. So it is enough to verify the theorem for the graph $C_n + v_{2k+1}v_{2k-2t+2}$ for $t = 2,3,\ldots,k$.

Define the weight function $w : E(C_n + v_{2k+1}v_{2k-m+2}) \rightarrow Z^+$ by

(i) $w(v_i\ v_{i+1}) = i\alpha(i) + \alpha(m)[(i-1)\alpha(i+1)] + \alpha(m+1)[(i+1)\alpha(i+1)],$

$i = 1,2,3,\ldots,2k-2t+\alpha(m+1)$

(ii) $w(v_{2k-2t+\alpha(m+1)+i}\ v_{2k-2t+\alpha(m+1)+1+i})$

$= 2k-2t+i-\alpha(m)\alpha(i)+\alpha(m+1)[1+\alpha(i+1)],\quad i = 1,2,3,\ldots,2t-1+\alpha(m)$

(iii)a) If m is odd, $w(v_{2k+1}\ v_{2k+2}) = 2k$

b) $w(v_{2k+\alpha(m)+i}\ v_{2k+\alpha(m)+1+i}) = 2k+1-i,\quad i = 1,2,3,\ldots,2k-\alpha(m)$

(iv) $w(v_{2k+1}\ v_{2k-2t+1+\alpha(m+1)}) = 1$

Suppose the edge is $v_1\, v_{2k+1}$, then define the weight function $w : E(C_n + v_{2k+2}v_{2k-2t+2}) \rightarrow Z^+$ by

(i) $w(v_{2+i}\ v_{3+i}) = (i+1)\alpha(i) + (i+2)\alpha(i+1),\quad i = 1,2,3,\ldots,2k-1$

(ii) $w(v_{2k+1+i}\ v_{2k+2+i}) = 2k+1-i,\quad i = 1,2,3,\ldots,2k-1$

(iii) $w(v_1\ v_2) = w(v_2\ v_3) = w(v_1\ v_{2k+1}) = 1$

By the above weight function, we have an irregular network of $C_n + e$, with consecutive labels $2,3,4,\ldots,4k+1$ with maximum weight 2k, hence $s(C_n + e) \leq \left\lceil \frac{n}{2} \right\rceil < \left\lceil \frac{n}{2} \right\rceil + 1$. Thus e is a negative edge.

Case (2). Suppose $n \equiv 1 (\mathrm{mod}\, 4)$. Let n=4k+1. Then the two cycles are of different parity.

If we add the edge $v_{2k+1}\ v_{2k-m+2}$, to C_n, we get two cycles of length *m=2t+1(2t)* and $n-m+2$ respectively with a common edge $v_{2k+1}\ v_{2k-m+2}$ which is isomorphic to the graph $C_n + e$ where the two cycles are of length m and n-m+2 respectively for an arbitrary edge e. So, it is enough to verify the theorem for the graph $C_n + v_{2k+1}\ v_{2k-m+2}$ for $m = 3,4,5,\ldots,2k+1$.

Define the weight function $w : E(C_n + v_{2k+1}\ v_{2k-m+2}) \rightarrow Z^+$ by

(i) $w(v_i\ v_{i+1}) = k + \left\lceil \frac{i}{2} \right\rceil + \alpha(i),\ i = 1,2,\ldots,2k-m+1$

(ii) $w(v_{2k-m+1+i}\ v_{2k-m+2+i}) = 2k - \left\lfloor \frac{i-1}{2} \right\rfloor,\ i = 1,2,3,\ldots,m-1$

(iii) $w(v_{2k+1}\ v_{2k+2}) = 2k$

(iv) $w(v_{2k+1+i}\ v_{2k+2+i}) = \left\lceil \frac{i}{2} \right\rceil,\ i = 1,2,\ldots,2k$

(v) $w(v_{2k+1}\ v_{2k-m+2}) = \left\lceil \frac{m}{2} \right\rceil$

By the above weight function, we have an irregular network of $C_n + e$, with consecutive labels $2,3,\ldots,4k+2$ with maximum weight 2k, hence $s(C_n + e) \le 2k < \left\lceil \frac{n}{2} \right\rceil$. Thus e is a negative edge.

Case (3). Suppose $n \equiv 2(\mathrm{mod}\,4)$. Let n=4k+2. Then the two cycles are of the same parity.

If we add the edge $v_{2k+1}\ v_{2k-m+2}$ to C_n, we get two cycles of length m=2t+1(2t) and n-m+2 respectively with a common edge $v_{2k+1}\ v_{2k-m+2}$ which is isomorphic to the graph $C_n + e$ where two cycles are of length m and n-m+2 respectively, for an arbitrary edge e. So it is enough to verify the theorem for the graph $C_n + v_{2k+1}\, v_{2k-m+2}$ for m= 3,4,5,...2k+1.

Define the weight function $w : E(C_n + v_{2k+1}\ v_{2k-m+2}) \to Z^+$ by

(i) $w(v_i\ v_{i+1}) = k + 1 + \left\lceil \frac{i}{2} \right\rceil,\ i = 1,2,\ldots,2k-m+1$

(ii) $w(v_{2k-m+1+i}\ v_{2k-m+2+i}) = 2k - \alpha(i)\left(\frac{i-1}{2}\right) - \alpha(i+1)\left(\frac{i}{2} - 2\right),$

$i = 1,2,\ldots,m-3$

(iii) $w(v_{2k-2+i}\ v_{2k-1+i}) = 2k - t + i + \alpha(m+1),\quad$ i = 1,2

(iv) $w(v_{2k+1}\ v_{2k+2}) = 2k+1$

(v) $w(v_{2k+1+i}\ v_{2k+2+i}) = \left\lceil \frac{i}{2} \right\rceil$, $i = 1,2,3,...,2k+1$

(vi) $w(v_{2k+1}\ v_{2k-m+2}) = \left\lceil \frac{m}{2} \right\rceil$

If we add the edge $v_1\ v_{2k+2}$, we get two cycles of same length 2k+2 with a common edge $v_1\ v_{2k+2}$, which is isomorphic to the graph $C_n + e$ where two cycles are of same length 2k+2, for an arbitrary e. So it is enough to verify the theorem for $C_n + v_1 v_{2k+2}$.

Define the weight function $w : E(C_n + v_1\ v_{2k+2}) \to Z^+$ by

(i) $w(v_i\ v_{i+1}) = k - \left[\alpha(i)\left(\frac{i-1}{2}\right) + \alpha(i+1)\left(\frac{i}{2} - 2\right) \right]$, $\quad i = 1,2,3,..,2k-1$

(ii) $w(v_{2k}\ v_{2k+1}) = 1$, $w(v_{2k+1}\ v_{2k+2}) = 2$, $w(v_1\ v_{2k+2}) = 2k+1$

(iii) $w(v_{2k+1+i}\ v_{2k+2+i}) = 2k+1 - \left\lfloor \frac{i}{2} \right\rfloor$, $i = 1,2,3,...,2k+1$

By the above weight function, we have an irregular network of $C_n + e$, with almost consecutive labels $2,3,4,...,4k+1, 4k+2, 4k+4$ with maximum weight 2k+1, hence $s(C_n + e) \le 2k+1 < \left\lceil \frac{n}{2} \right\rceil$.

Thus e is a negative edge.

Case (4). Suppose $n \equiv 3 (\mathrm{mod}\ 4)$. Let n=4k+3. Then the cycles are of opposite parity. If we add the edge $v_{2k+2}\ v_{2k-m+3}$, we get two cycles of length m=2t+1(2t) and n-m+2 respectively with a common edge $v_{2k+2}\ v_{2k-m+3}$, which is isomorphic to the graph $C_n + e$, where the two cycles are of length m and n-m+2 respectively for an arbitrary edge e. So it is enough to verify the theorem for the graph $C_n + v_{2k+2} v_{2k-m+3}$ for m=3,4,5,…,2k+1.

Define the weight function $w : E(C_n + v_{2k+2}\ v_{2k-m+3}) \to Z^+$ by

(i) $w(v_i\ v_{i+1}) = \alpha(m)\left\{ \left[k+1+\left(\frac{i-1}{2}\right) \right]\alpha(i) + \left[k+3+\left(\frac{i-2}{2}\right) \right]\alpha(i+1) \right\}$

$$+\alpha(m+1)\left[k+1+\left\lceil\frac{i}{2}\right\rceil\right], \qquad i=1,2,\dots,2k-m+2$$

(ii) $w(v_{2k-m+2+i}\ v_{2k-m+3+i})$

$$=\alpha(m)\left(2k-\left\lfloor\frac{m}{2}\right\rfloor+2+\left\lfloor\frac{i}{2}\right\rfloor\right)+\alpha(m+1)\left(2k+2-\left\lceil\frac{i}{2}\right\rceil\right), \ i=1,2,\dots,m-3$$

(iii) $w(v_{2k-1+i}\ v_{2k+i})=\alpha(m)(2k+i)+\alpha(m+1)(2k+2i-t), \quad i=1,2$

(iv) $w(v_{2k+2}\ v_{2k+3})=2k+1+\alpha(m)$

(v) $w(v_{2k+2+i}\ v_{2k+3+i})=\left\lceil\frac{i}{2}\right\rceil, \ i=1,2,3,\dots,2k+1$

(vi) $w(v_{2k+2}\ v_{2k-m+3})=\alpha(m)+\alpha(m+1)\left\lfloor\frac{m}{2}\right\rfloor$

Add the edge $v_{2k+2}\ v_{4k+3}$ to get the cycles of lengths $2k+2$ and $2k+3$ and define the weight function $w: E(C_n+v_{2k+2}\ v_{4k+3})\to Z^+$ by

(i) $w(v_i\ v_{i+1})=k+\left\lceil\frac{i}{2}\right\rceil+\alpha(i), \ i=1,2,3,\dots,2k+1$

(ii) $w(v_{2k+1+i}\ v_{2k+2+i})=\left\lceil\frac{i}{2}\right\rceil, \ i=1,2,3,\dots,2k+1$

(iii) $w(v_1 v_{4k+3})=k$

(iv) $w(v_{2k+2}\ v_{4k+3})=2k+2$

By the above weight function, we have an irregular network of C_n+e, with almost consecutive labels $2,3,4,\dots,4k+3,4k+5$ with maximum weight 2k+2, hence $s(C_n+e)\le 2k+2<\left\lceil\frac{n}{2}\right\rceil+1$.

Thus e is a negative edge. ■

ACKNOWLEDGEMENTS. The second author wishes to thank Arumugam. S, Senior Professor, Arulmigu Kalasalingam College of Engineering, for many fruitful discussions. Many thanks to the anonymous referee for his valuable comments, which helped clarity and correct an earlier draft of the paper.

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