Some Results on Acquisition Numbers

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Abstract In this paper we shall consider acquisition sequences of a graph. The formation of each acquisition sequence is a process that creates an independent set. Each acquisition sequence is a sequence of "acquisitions" which are defined on a graph G for which each vertex originally has a value of one associated with it. In an *acquisition* a vertex transfers all of its value to an adjacent vertex with equal or greater value. For an *acquisition sequence*, one continues until no more acquisitions are possible. The parameter $a(G)$ is defined to be the minimum possible number of vertices with a nonzero value at the conclusion of such an acquisition sequence. Clearly, if S is a set of vertices with nonzero values at the end of some acquisition sequence, then S is independent, and we call such a set S an *acquisition set*. We show that for a given graph G, "Is $a(G) = 1$?" is NP-complete, and describe a linear time algorithm to determine the acquisition number of a caterpillar.

1 Introduction

The acquisition number $a(G)$ is related to the operation "acquisition" which is defined on a graph G for which each vertex originally has a value of one associated with it. In an *acquisition*, a vertex transfers all of its value to an adjacent vertex with equal or greater value. For an *acquisition sequence*, one continues until no more acquisitions are possible. The acquisition number $a(G)$ is defined to be the minimum number of vertices with a nonzero value at the conclusion of such an acquisition sequence. Clearly for each graph G, $a(G) \geq 1$.

If S is a set of vertices with nonzero values at the end of some acquisition sequence, then we call such a set S an *acquisition set*. Clearly an acquisition set is independent. It is easy to verify the following proposition.

Proposition 1. *Every maximal independent set is an acquisition set. Thus $a(G) \leq i(G) \leq \beta(G)$, where the independence number $\beta(G)$ and lower indepen-*

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dence number $i(G)$ are the maximum and minimum cardinalities of maximal independent sets in G.

For any two given graphs G_1 and G_2 with disjoint vertex sets $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$, the Cartesian graph product $G = G_1 \square G_2$ of graphs G_1 and G_2 is the graph with the vertex set $X \times Y$, and $u = (x_i, y_j)$ is adjacent to $v = (x_a, y_b)$ whenever $x_i = x_a$ and y_j is adjacent to y_b in G_2, or $y_j = y_b$ and x_i is adjacent to x_a in G_1. For example, $K_2 \square K_2 = C_4$.

Proposition 2. For any two given graphs G_1 and G_2 with disjoint vertex sets,

$$a(G_1 \square G_2) \leq a(G_1) \times a(G_2).$$

Proof. We use the same notation mentioned above in this proof. Let $X_i = \{(x_i, y_j) \,|\, 1 \leq j \leq n\}$ for $1 \leq i \leq m$, and $Y_j = \{(x_i, y_j) \,|\, 1 \leq i \leq m\}$ for $1 \leq j \leq n$. Clearly the subgraph induced by Y_j in G is isomorphic to G_1, while the subgraph induced by X_i in G is isomorphic to G_2, where $1 \leq i \leq m$ and $1 \leq j \leq n$. Let S_1 be the acquisition set in G_1 at the end of some acquisition sequence $\mathcal{S}_1$ such that $|S_1| = a(G_1)$, and S_2 be the acquisition set in G_2 at the end of some acquisition sequence $\mathcal{S}_2$ such that $|S_2| = a(G_2)$. Without loss of generality, we can assume that $S_2 = \{y_1, y_2, \ldots, y_{a(G_2)}\}$. Describe an acquisition sequence on G as follows. First we can perform the acquisition sequence $\mathcal{S}_2$ on each subgraph induced by X_i, where $1 \leq i \leq m$. After that, the set of vertices with nonzero values in G is $\cup_{j=1}^{a(G_2)} Y_j$. Furthermore, all vertices in each set Y_j have the same nonzero value for $1 \leq j \leq a(G_2)$. Then we can perform the acquisition sequence $\mathcal{S}_1$ on each subgraph induced by Y_j for $1 \leq j \leq a(G_2)$, which is G_1. The resulting set is an acquisition set having $a(G_1) \times a(G_2)$ vertices. The proposition follows. ∎

We note that $a(P_5) = 2$ and $a(P_5 \square P_5) = 3$, as shown in [2]. Hence, one can have $a(G_1 \square G_2) < a(G_1) \times a(G_2)$.

The k-cube Q_k is the graph of order 2^k with vertex set $V(Q_k) = \{(a_1, a_2, \ldots, a_k) \,|\, a_i \in \{0,1\}$ for $1 \leq i \leq k\}$, and two vertices $u = (b_1, b_2, \ldots, b_k)$ and $v = (c_1, c_2, \ldots, c_k)$ are adjacent in Q_k if and only if there exists a positive integer j such that $b_j \neq c_j$ and $b_i = c_i$ for all $i \neq j$, where $1 \leq j \leq k$. Notice that $Q_k = Q_{k-1} \square K_2$ for $k \geq 2$. By Proposition 2, $a(Q_k) \leq a(Q_{k-1}) \times a(K_2) = a(Q_{k-1})$ for $k \geq 2$. That, together with the fact that $a(Q_2) = 1$, implies the following corollary.

Corollary 3. *$a(Q_k) = 1$, where k is any positive integer.*

From the definition of acquisition, each acquisition to a vertex at most doubles its value. Therefore we have the following results. We use $deg(v)$ to denote the degree of vertex v.

Theorem 4. [2] *The maximum value that may be concentrated at any one vertex v by any sequence of acquisitions is $2^{deg(v)}$.*

Theorem 5. *For a vertex v not in the acquisition set S, the maximum value attained by v at any point during the acquisition sequence is $2^{deg(v)-1}$.*

From Theorem 4, it easily follows for path P_n and cycle C_n on n vertices that $a(P_n) = a(C_n) = \lceil \frac{n}{4} \rceil$. The following Theorem gives us an upper bound for $a(G)$.

Theorem 6. [2] *For any connected graph G with the number of vertices $n \geq 2$, $a(G) \leq \frac{n+1}{3}$.*

From [2], the upper bound is achievable by the tree illustrated in Figure 1.

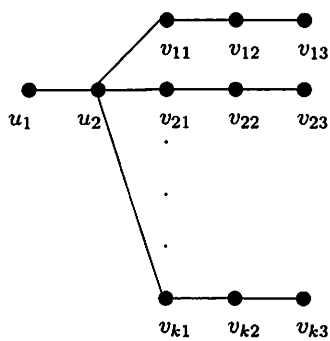

Figure 1. Tree T_{3k+2} with $a(T_{3k+2}) = k + 1$

In Section 2, we show that deciding, for a given graph G, "Is $a(G) = 1$?" is NP-complete. In Section 3, we consider caterpillars and describe a linear time algorithm to determine the acquisition number of a caterpillar.

2 NP-completeness

Although Theorem 6 gives an upper bound for $a(G)$, it can be seen from this section that determining $a(G)$ for an arbitrary graph is believed to be very difficult. In this section, we shall prove that deciding, for a given graph G, "Is $a(G) = 1$?" is NP-complete. For a formal definition of NP-completeness and a list of NP-complete problems, see [1].

Theorem 7. *Deciding, for a given graph G, "Is $a(G) = 1$?" is NP-complete.*

Proof. We shall relate the problem "Is $a(G) = 1$?" for a given graph G to the three-dimension matching problem (3DM), which has been shown to be NP-complete (see [1], page 221). Recall that the 3DM problem is stated as follows:

3DM Problem: Let $X = \{x_1, x_2, \ldots, x_m\}, Y = \{y_1, y_2, \ldots, y_m\}$, $Z = \{z_1, z_2, \ldots, z_m\}$ and let $M \subseteq X \times Y \times Z$. Does there exist a subset of M of size m such that each pair of elements of the subset disagree in all three coordinates?

We demonstrate that a solution to the problem 3DM is equivalent to a solution of the problem "Is $a(G) = 1$?" for a specified graph G. Here G can be constructed from the set M in time polynomial in m. In other words, there is a polynomial transformation from the problem 3DM to the problem "Is $a(G) = 1$?". Next we shall describe the construction of G from a given 3DM problem.

For a given 3DM problem, let $|M| = t$. Clearly we can assume that $t \geq m$. Each element in X, Y, Z and M is represented by a vertex in G. X, Y, Z and M are independent subsets of $V(G)$. Each vertex in X is adjacent to two end-vertices, while each vertex in Y is adjacent to one end-vertex. If $(x_i, y_j, z_k) \in M$, then the corresponding vertex of M in G is adjacent to the vertices x_i of X, y_j of Y and z_k of Z. All the vertices in M are adjacent to a vertex $u \notin X \cup Y \cup Z \cup M$ and the subgraph induced by $N[u]$ is $K_{1,t+8}$, a star of $t + 9$ vertices. Note that every vertex in M has degree four. The graph G is illustrated in Figure 2. For illustrative purposes (x_1, y_1, z_1) is assumed to be an element of M. All other edges are precisely as depicted.

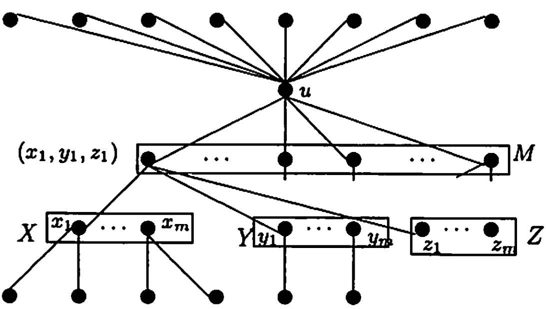

Figure 2. The graph G corresponding to the problem 3DM

Clearly, G has $6m + t + 9$ vertices and can be constructed from a given 3DM problem in polynomial time.

Claim 1. If there exists a solution to the problem 3DM, then $a(G) = 1$.

Suppose that we have a solution to the problem 3DM. It follows that there exists an m-subset A of M such that each pair of elements of the subset disagree in all three coordinates. By applying appropriate permutations of the indices on Y and Z, we can assume that $A = \{a_i = \{x_i, y_i, z_i\} | 1 \le i \le m\}$. Next we shall describe an acquisition sequence such that there will be only one vertex with nonzero value left at the end of this sequence, the vertex u.

First transfer the value of all end-vertices in $V(G) - Z$ to the vertices that are adjacent to them. Then each vertex in X has value 3, each vertex in Y has value 2, and u has value 9.

Second, performing the following acquisitions: $z_i \rightarrow (x_i, y_i, z_i) \in A$ for $1 \le i \le m$, will make each vertex in A have value 2 and each vertex in Z have value 0.

Third, by the fact that each vertex in $A \cup Y$ has value 2, performing the following acquisitions: $y_i \rightarrow (x_i, y_i, z_i) \in A$ for $1 \le i \le m$, will make each vertex in A have value 4 and each vertex in Y have value 0.

Fourth, noticing that each vertex in X has value 3, we can do the following acquisitions: $x_i \rightarrow (x_i, y_i, z_i) \in A$ for $1 \le i \le m$, to make the values of each vertex in A and X, respectively, be 7 and 0.

Finally, notice that the value of each vertex in M is either 7 or 1, which is smaller than 9, the value of u. And $M \cup \{u\}$ is the set of vertices with nonzero value at this time. So performing the following acquisitions: $v \rightarrow u$ for all $v \in M$, will leave us only one vertex with nonzero value, namely u. By the above discussion, $a(G) = 1$.

Claim 2. $a(G) = 1$ implies that there exists a solution to the problem 3DM.

The case when $m = 1$ is trivial. Consider $m \ge 2$. Then $|V(G)| \ge 23$. Let S be any acquisition sequence that will achieve $a(G) = 1$, and S be the corresponding acquisition set. Let $S = \{v\}$. Suppose that $v \ne u$. Note that the degree of every vertex in M is four. By Theorem 4, $v \in X \cup Y \cup Z$. To achieve $a(G) = 1$, the acquisition sequence S can be assumed to transfer the value of all end-vertices in $V(G) - Z$ to the vertices that are adjacent to them first. So now the value of u is 9, each vertex in X has a value of 3 and each vertex in Y has a value of 2. By Theorem 5 and the assumption that $v \in X \cup Y \cup Z$, the maximum value of all vertices in M during the acquisition sequence S is at most 8, which makes $u \rightarrow w$ impossible, where $w \in M$. It follows that v cannot acquire u, a contradiction. Therefore $v = u$, that is, $S = \{u\}$.

Notice that in order to transfer the value of any vertex in X to u, S has to transfer the value of that vertex to a vertex in M at some point. Without loss of generality, we can assume that the value of x_i is transferred to the ith element in M in S, say m_i. Let $A' = \{m_i \in M,$ where $1 \le i \le m\}$. Next we shall show that A' is a solution to the problem 3DM.

Since $S = \{u\}$ and each vertex in Z is adjacent to some vertices in M, where each vertex in M is adjacent to four vertices with values $1, 2, 3$ and 9 respectively, then S has to perform the following acquisitions first: for each $z \in Z$, $z \rightarrow w$ for some $w \in M$. After performing these acquisitions, we claim that all vertices in A' must have value 2. If the claim is not true, then there exists j such that $m_j \in A'$ has value 1 after doing the above acquisitions. Consider m_j. The vertex m_j with value 1 is adjacent to four vertices with values 0, 2, 3 and 9 respectively, and the subgraph induced by $N[m_j]$ is $K_{1,4}$ with m_j as the center of the star. It follows that $x_j \to m_j$ is not feasible, a contradiction. Thus all vertices in A' should have value 2 after performing the above acquisitions. Since any vertex in A' is adjacent to only one vertex in Z, each pair of elements in A' disagrees in z coordinates.

Since $S = \{u\}$ and each vertex in Y with a value of 2 is adjacent to some vertices in M, then S has to perform the following acquisitions: for each $z \in Z$, $z \to w$ for some $w \in M$. Notice that for $w \in A'$, w has a value of 2 and is adjacent to four vertices with values 0, 2, 3 and 9 respectively; while for $w \in M - A'$, w has a value of 1 and is adjacent to four vertices with values 0, 2, 3 and 9 respectively. It follows that $z \to w$, where $w \in A'$. As any vertex in A' is adjacent to only one vertex in Y, each pair of elements in A' disagrees in y coordinates.

By the above discussion, we know that each pair of elements in A' disagrees in the x, y, z coordinates. Thus A' is one solution to the problem 3DM.

By Claim 1 and Claim 2, we know that a solution to the problem 3DM is equivalent to a solution of the problem "Is $a(G) = 1$?" for a specified graph G. Hence Theorem 7 holds. ∎

3 Results on the Acquisition Number of Caterpillars

Given a tree T of order at least 3, we shall denote by T' the subtree of T obtained by removing all the end-vertices of T, where an end-vertex is a vertex of degree one. A tree T is called a *caterpillar* if T' is a path, and we call T' the spine of the caterpillar T. In this section we shall describe a linear time algorithm for determining the acquisition number of a caterpillar, and present two theorems regarding how to achieve the lower bound and the upper bound of $a(T)$.

First, we consider the tree T, on seven vertices illustrated in Figure 3.

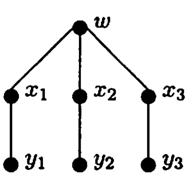

Figure 3. A tree T_7 with $a(T) = 2$

Doing the following acquisition sequence in T_7: $x_1 \to w, y_2 \to x_2, y_3 \to x_3, x_2 \to w, x_3 \to w$, will give us an acquisition set $\{y_1, w\}$ with cardinality 2. Thus $a(T_7) \leq 2$. It is easy to verify that w cannot acquire the values of all vertices in T_7. That, together with the facts that w is the only vertex in T_7 with the maximum degree 3 and the order of T_7 is 7, implies that $a(T_7) \geq 2$. By the above discussion, $a(T_7) = 2$. Notice that any acquisition sequence in T_7 that results in an acquisition set of cardinality 2 cannot have every acquisition in the form of $y_i \to x_i$, where $1 \leq i \leq 3$. Unlike the case for T_7, for any caterpillar T, we can assume that all the acquisitions of transferring the value of all end-vertices to the vertices that are adjacent to them can be included in an acquisition sequence. We shall restate this observation as follows.

Proposition 8. *Let T be a caterpillar. Then there is always an acquisition set $X \subset V(T')$ with $|X| = a(T)$.*

Proof. Let $\mathcal{S}$ be an acquisition sequence of T, and X be the corresponding acquisition set at the end of $\mathcal{S}$, where $|X| = a(T)$. Suppose that u is any end-vertex of T and $uv \in E(T)$. The proposition holds if we can assume that $\mathcal{S}$ has the acquisition $u \to v$. Consider the value of u at the end of $\mathcal{S}$. At the very beginning, u has a value of one. Besides the acquisition $u \to v$, there are only two other possibilities regarding the value of u. They are that $v \to u$ appears in $\mathcal{S}$ and then makes the value of u equal to two, or that the value of u remains unchanged. First, if $v \to u$ is in $\mathcal{S}$, then this acquisition will only affect the value of u and v because u is an end-vertex. At the end of $\mathcal{S}$, $u \in X$. We do as well by doing $u \to v$ to make $v \in X$ possible at the end of $\mathcal{S}$ instead of $u \in X$. Second, assume that the value of u is unchanged at the end of $\mathcal{S}$, then $u \in X$. Since u has value one at the end of $\mathcal{S}$ and u is adjacent to v, by the definition of acquisition sequences, we must have that $v \to w$ for some $w \in V(T)$ in $\mathcal{S}$. Notice that $deg(w) = 1$ implies that $w \in X$, which contradicts the fact that $|X| = a(T)$ because we can do better by doing the following acquisitions: $w \to v$ and $u \to v$, to make $v \in X$ possible instead of $\{u, w\} \subset X$. Thus $deg(w) \neq 1$. It follows that w is in the spine. We shall consider two cases.

Case 1. $w \in X$. Then $\{u, w\} \subseteq X$. Consider all acquisitions involving w in $\mathcal{S}$. If there exists an acquisition $w' \to w$ in $\mathcal{S}$ and w' is in the spine, noticing that each element in $N(w) - \{v, w'\}$ is an end-vertex, we do as well by doing $u \to v$, and $x \to w$ for all $x \in N(w) - \{v, w'\}$, and doing either $w' \to w$ or $w \to w'$ at the end, because by doing it this way, instead of $\{u, w\} \subseteq X$ we still have $|\{u, v, w, w'\} \cap X| \leq 2$. If such an acquisition does not exist, then either $v \to w$ is the only acquisition involving w in $\mathcal{S}$, or all other acquisitions involving w are in the form of $w' \to w$ in $\mathcal{S}$ with the property that w' is an end-vertex. Under this circumstance, we can do better by performing $u \to v$ first, because by doing it this way, we shall have $|\{u, v, w\} \cap X| = 1$ instead of $\{u, w\} \subseteq X$.

Case 2. $w \notin X$. Then $w \to w'$ for some $w' \in V(T)$ in S. Since now w has a value of at least 2, w' is also in the spine. Clearly the value of w' is at least the value of w. If $w' \notin X$, we do as well by doing $u \to v$, and $w \to w'$ in S and at the end of S, v may be in X instead of $u \in X$. Otherwise we have $w' \in X$. Consider all acquisitions involving w' in S. By a similar argument as in Case 1, we do as well by doing $u \to v$ first.

By the above discussion, we can assume that S has the acquisition $u \to v$. The proof of Proposition 8 is complete. ■

The following observation gives a lower bound and an upper bound of $a(T)$, where T is a caterpillar.

Observation 9. *Let T be a caterpillar with $T' = v_1 v_2 \cdots v_k$. Then $1 \le a(T) \le \lceil \frac{k}{2} \rceil$.*

Proof. By Proposition 8, to determine $a(T)$, first we can transfer the value of all end-vertices to the vertices that are adjacent to them. Let X be an acquisition set of T obtained by doing it this way. Then $X \subset \{v_1, v_2, \ldots, v_{k-1}, v_k\}$. That, together with the fact that $\beta(P_k) = \lceil \frac{k}{2} \rceil$, implies that $|X| \le \lceil \frac{k}{2} \rceil$. Thus $a(T) \le \lceil \frac{k}{2} \rceil$. This completes the proof of Observation 9. ■

We next present a linear time algorithm to determine $a(T)$ for caterpillar T.

Algorithm ACQUISITION_CATERPILLAR (To determine an acquisition set X of a caterpillar T such that $|X| = a(T)$.)
Input: a caterpillar T with $T' = v_1 v_2 \cdots v_n$.
Step 1. Transfer the value of each end-vertex to the vertex that is adjacent to it. Then after step 1, a weighted path $P = v_1 v_2 \cdots v_n$ is obtained, where v_i has the value $w(v_i)$ for $1 \le i \le n$.
Step 2. Iterate the following on the weighted path P until $|V(P)| \le 2$.

2.1 For as long as possible, transfer the value of the first remaining left endpoint v_i to its neighbor v_{i+1} and delete v_i from P. That is, if v_i is the left endpoint of P and $w(v_i) \le w(v_{i+1})$, then $w(v_{i+1}) := w(v_{i+1}) + w(v_i)$ and $P := P - v_i$, where the operator ":=" means that the value of the right side of an expression is assigned to the left side of the expression.

2.2 If v_i is the first remaining left endpoint and $w(v_i) > w(v_{i+1})$, then let k be the largest integer with $k \ge i+1$ such that we can acquire the values of $v_{i+1}, \ldots, v_k$ onto v_{i+1}. Let $w(v_{i+1}) := w(v_{i+1}) + \cdots + w(v_k)$. If $w(v_i) \ge w(v_{i+1})$, then $v_i \in X$. Otherwise, $v_{i+1} \in X$. Now let $P := P - \{v_i, v_{i+1}, \ldots, v_k\}$.
Step 3. If $|V(P)| = 1$, then $X := X \cup V(P)$. If $|V(P)| = 2$, then $X := X \cup \{u\}$, where u is the vertex with larger value on P.

For example, for tree T_{33} in Figure 4a, after the execution of Step 1, we have $w(v_1) = 2$, $w(v_2) = 3$, $w(v_3) = 5$, $w(v_4) = 6$, $w(v_5) = 3$, $w(v_6) = 2$, $w(v_7) = 1$, $w(v_8) = 3$, $w(v_9) = 2$, $w(v_{10}) = 2$, $w(v_{11}) = 1$, $w(v_{12}) = 1$ and $w(v_{13}) = 2$, which is shown in Figure 4b. Notice that we can transfer the values of v_1 and v_2 to v_3 and we can acquire the values of v_5, v_6, v_7 onto v_4. Thus after the first execution of Step 2, $w(v_1) = w(v_2) = w(v_3) = 0$, $w(v_4) = 22$, $w(v_5) = w(v_6) = w(v_7) = 0$, $w(v_8) = 3$, $w(v_9) = 2$, $w(v_{10}) = 2$, $w(v_{11}) = 1$, $w(v_{12}) = 1$ and $w(v_{13}) = 2$, and $X = \{v_4\}$, and the weighted path changes to $v_8v_9v_{10}v_{11}v_{12}v_{13}$, which is shown in Figure 4c. Similarly after the second execution of Step 2, $w(v_8) = 0$, $w(v_9) = 7$, $w(v_{10}) = 0$, $w(v_{11}) = 1$, $w(v_{12}) = 1$ and $w(v_{13}) = 2$, and $X = \{v_4, v_9\}$, and the weighted path changes to $v_{11}v_{12}v_{13}$, which is shown in Figure 4d. Finally after the third execution of Step 2 and the execution of Step 3, $w(v_{11}) = w(v_{12}) = 0$, and $w(v_{13}) = 4$, and the acquisition set $X = \{v_4, v_9, v_{13}\}$. Therefore $a(T_{33}) = 3$.

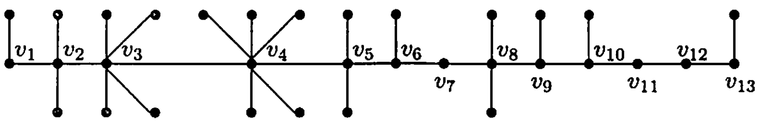

Figure 4a. A caterpillar T_{33}

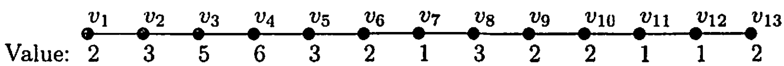

Figure 4b. The weighted path obtained after the execution of Step 1 on T_{33}

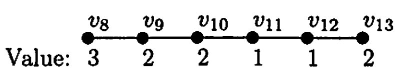

Figure 4c. The weighted path obtained after the first execution of Step 2, where $X = \{v_4\}$ and $k = 7$

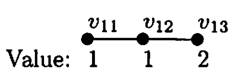

Figure 4d. The weighted path obtained after the second execution of Step 2, where $X = \{v_4, v_9\}$ and $k = 10$

Theorem 10. *Algorithm ACQUISITION_CATERPILLAR is a correct linear time algorithm.*

Proof. Let S be an acquisition sequence of T, and X be the corresponding acquisition set such that $|X| = a(T)$. Note that Proposition 8 justifies the first step of the algorithm. After Step 1, the vertices of nonzero values will form a path, say $P = v_1 v_2 \cdots v_n$. Let $w(v_i)$ be the value of v_i, where $1 \leq i \leq n$.

Claim 1. If $w(v_1) \leq w(v_2)$, then we can assume that $\mathcal{S}$ has the acquisition $v_1 \to v_2$.

Consider the value of v_1 at the end of $\mathcal{S}$. Since $w(v_1) \leq w(v_2)$, by the definition of acquisition sequence, there are three possibilities regarding the value of v_1 in $\mathcal{S}$: $v_1 \to v_2$, or $w(v_1) = w(v_2)$ and $v_2 \to v_1$, or the value of v_1 is unchanged at the end of $\mathcal{S}$. If $w(v_1) = w(v_2)$ and $v_2 \to v_1$, then $v_1 \in X$. We do as well by doing $v_1 \to v_2$ because by doing it this way, v_2 may be in X instead of $v_1 \in X$. Assume that the value of v_1 is unchanged in $\mathcal{S}$, then $v_1 \in X$. Since v_1 is adjacent to v_2 that has nonzero values, we can assume that $v_2 \to v_3$ is in $\mathcal{S}$. Let k be the largest positive integer such that the following acquisitions are in $\mathcal{S}$: $v_2 \to v_3, v_3 \to v_4, \cdots, v_{k-1} \to v_k$. Then $\{v_1, v_k\} \subset X$. It is easy to verify that we do as well by doing $v_1 \to v_2$ and $v_3 \to v_4, \cdots, v_{k-1} \to v_k$ in this case. The proof of Claim 1 is complete.

Claim 2. Suppose that $w(v_1) > w(v_2)$. Let k be the largest positive integer such that the following acquisitions are feasible:

$$v_k \to v_{k-1}, v_{k-1} \to v_{k-2}, \cdots, v_3 \to v_2. \qquad (*)$$

Then we can assume that $\mathcal{S}$ has the above acquisitions $(*)$.

By performing $(*)$, $\{v_1, v_2, \ldots, v_k\} \cap X = v_1$ or $\{v_1, v_2, \ldots, v_k\} \cap X = v_2$. Consider the possible acquisitions that are related with v_k in $\mathcal{S}$.

Case 1. $v_{k+1} \to v_k$ is in $\mathcal{S}$. By the definition of k and the fact that $w(v_1) > w(v_2)$, at least two vertices among $v_1, v_2, \ldots, v_k$ have nonzero values at the end of $\mathcal{S}$. We do as well by doing $(*)$ and doing either $v_1 \to v_2$ or $v_2 \to v_1$ and keeping all the other acquisitions involving v_{k+1} except $v_{k+1} \to v_k$, because doing it this way gives us either $\{v_1, v_{k+1}\} \subseteq X$ or $\{v_2, v_{k+1}\} \subseteq X$.

Case 2. $v_k \to v_{k+1}$ and $v_{k+2} \to v_{k+1}$ are in $\mathcal{S}$. It follows that $|\{v_1, v_2\} \cap X| = 1$ and $v_{k+1} \in X$. We do as well by doing $(*)$ and do either $v_1 \to v_2$ or $v_2 \to v_1$, and at the end of $\mathcal{S}$, perform either $v_{k+1} \to v_{k+2}$ or $v_{k+2} \to v_{k+1}$, because doing it this way also gives us $|\{v_1, v_2, \ldots, v_k, v_{k+1}, v_{k+2}\} \cap X| = 2$.

Case 3. $v_k \to v_{k+1}$ and $v_{k+1} \to v_{k+2}$ are in $\mathcal{S}$. Let l be the largest positive integer such that the following acquisitions are in $\mathcal{S}$: $v_{k+1} \to v_{k+2}, v_{k+2} \to v_{k+3}, \cdots, v_{k+l} \to v_{k+l+1}$. Then $|\{v_1, v_2\} \cap X| = 1$ and $v_{k+l+1} \in X$. It is easy to verify that we do as well by doing $(*)$ and doing $v_{k+1} \to v_{k+2}, v_{k+2} \to v_{k+3}, \cdots, v_{k+l} \to v_{k+l+1}$, and doing either $v_1 \to v_2$ or $v_2 \to v_1$.

The proof of Claim 2 is complete.

Claim 1 and Claim 2 respectively prove the correctness of Step 2.1 and Step 2.2. Next we shall explain how to determine k with the property mentioned in Claim 2 in the algorithm ACQUISITION_CATERPILLAR.

Claim 3. Define a sequence $\{x_i \mid i \geq 1\}$ as follows:

$$x_1 = w(v_2)$$
$$x_i = \min\{x_{i-1} - w(v_{i+1}), w(v_{i+1})\} \text{ for } i \geq 2$$

Let $k-1$ be the positive integer such that $x_j \geq 0$ for $1 \leq j \leq k-1$ and $x_k < 0$. Then k is the largest positive integer such that the acquisitions (*) in Claim 2 are feasible.

To make (*) in Claim 2 feasible, we need to have

$$w(v_{j+1}) \leq w(v_j) \text{ for } 2 \leq j \leq k-1 \tag{2.1}$$

$$\sum_{i=0}^{j} w(v_{k-i}) \leq w(v_{k-j-1}) \text{ for } 0 \leq j \leq k-3 \tag{2.2}$$

As $0 \leq j \leq k-3$, we have $2 \leq k-j-1 \leq k-1$. Let p be any integer such that $2 \leq p \leq k-1$. As $x_{p-1} = \min\{x_{p-2} - w(v_p), w(v_p)\}$, we have $x_{p-1} \leq w(v_p)$. Since $x_p = \min\{x_{p-1} - w(v_{p+1}), w(v_{p+1})\}$, we have $x_p \leq x_{p-1} - w(v_{p+1})$. Thus $x_p \leq w(v_p) - w(v_{p+1})$, that is,

$$x_p + w(v_{p+1}) \leq w(v_p) \tag{2.3}$$

That, together with the fact that $x_p \geq 0$ for $2 \leq p \leq k-1$, implies that $w(v_{p+1}) \leq w(v_p)$ for $2 \leq p \leq k-1$. Thus (2.1) holds. As $x_{p+1} = \min\{x_p - w(v_{p+2}), w(v_{p+2})\}$, we have

$$x_{p+1} \leq x_p - w(v_{p+2}) \Rightarrow x_{p+1} + w(v_{p+2}) \leq x_p.$$

That, together with (2.3), implies that

$$x_{p+1} + w(v_{p+2}) + w(v_{p+1}) \leq w(v_p).$$

Continuing this process and noticing that $x_{k-1} \geq 0$, we shall have

$$\sum_{i=0}^{k-1-p} w(v_{k-i}) \leq w(v_p) \text{ for } 2 \leq p \leq k-1.$$

Hence (2.2) holds. This completes the proof of Claim 3.

Claim 3 proves that the complexity of finding k with the property mentioned in Step 2.2 is linear. By the fact that $a(P_1) = a(P_2) = 1$, Step 3 is correct. The proof is complete. ∎

We conclude with two theorems regarding how to achieve the lower bound and the upper bound of $a(T)$ of order n which is mentioned in Observation 9 for caterpillars. This also means that the bounds given in Observation 9 are sharp.

Theorem 11. *Given $k \geq 2$. There exists a caterpillar T with $T' = v_1 v_2 \cdots v_k$ such that $a(T) = 1$ if and only if $n \geq n(k)$, where*

$$n(k) = \begin{cases} 2^{\frac{k}{2}+1} & \text{when } k \text{ is even} \\ 3 \cdot 2^{\frac{k-1}{2}} & \text{when } k \text{ is odd} \end{cases}.$$

Proof. ($\Leftarrow$) Let a_i be the number of end-vertices that are adjacent to v_i for $1 \leq i \leq k$. Define $a_1 = a_k = 1$, $a_i = a_{k+1-i} = 2^{i-1} - 1$ for $2 \leq i \leq \lfloor \frac{k-1}{2} \rfloor$. When k is even, define $a_{k/2} = 2^{k/2-1} - 1$ and $a_{k/2+1} = n - 2^{k/2} - 2^{k/2-1} - 1$. When k is odd, define $a_{(k+1)/2} = n - 2^{(k+1)/2} - 1$. Clearly the order of T is n. Next we need to show that $a(T) = 1$.

By Proposition 8, first we can transfer the value of all end-vertices to the vertices that are adjacent to them. Let $w(v_i)$ be the value of v_i at this time, where $1 \leq i \leq k$. When k is even, $(w(v_1), w(v_2), w(v_3), \cdots, w(v_{k/2}), w(v_{k/2+1}), \cdots, w(v_{k-2}), w(v_{k-1}), w(v_k)) = (2, 2, 4, \cdots, 2^{k/2-1}, n - 2^{k/2} - 2^{k/2-1}, \cdots, 4, 2, 2)$. As $n \geq 2^{k/2+1}$, we have $w(v_{k/2+1}) = n - 2^{k/2} - 2^{k/2-1} \geq 2^{k/2-1}$. Thus we can perform the acquisitions $v_k \rightarrow v_{k-1}, v_{k-1} \rightarrow v_{k-2}, \cdots, v_{k/2+2} \rightarrow v_{k/2+1}$ and the acquisitions $v_1 \rightarrow v_2, v_2 \rightarrow v_3, \cdots, v_{k/2-1} \rightarrow v_{k/2}$, to make $v_{k/2}$ and $v_{k/2+1}$ the only two vertices with nonzero values. Since they are adjacent, $a(T) = 1$. When k is odd, by the fact that $n(k) \geq 3 \cdot 2^{\frac{k-1}{2}}$, we have $w(v_{(k+1)/2}) = n - 2^{(k+1)/2} \geq 2^{(k-1)/2} \geq w(v_{(k-1)/2})$. Thus we can perform the acquisitions $v_k \rightarrow v_{k-1}, v_{k-1} \rightarrow v_{k-2}, \cdots, v_{(k+3)/2} \rightarrow v_{(k+1)/2}$, $v_1 \rightarrow v_2, v_2 \rightarrow v_3, \cdots, v_{(k-1)/2} \rightarrow v_{(k+1)/2}$, to make $v_{(k+1)/2}$ the only vertex with nonzero values, which means $a(T) = 1$.

($\Rightarrow$) Suppose now $a(T) = 1$. Then there exists an acquisition sequence $\mathcal{S}$ such that there will be only one vertex with nonzero values left at the end of $\mathcal{S}$. Clearly this vertex must be in T'. Thus we can assume that the vertex is v_d, where $1 \leq d \leq k$.

Let u be an end-vertex that is adjacent to v_1. The existence of u is guaranteed by the definition of caterpillars. Since v_d is the only vertex with nonzero value left at the end of $\mathcal{S}$, $\mathcal{S}$ must have the following acquisitions:

$$u \rightarrow v_1, v_1 \rightarrow v_2, \cdots, v_{d-2} \rightarrow v_{d-1}, v_{d-1} \rightarrow v_d.$$

Notice that after $u \rightarrow v_1$, v_1 has a value of at least 2. After $v_1 \rightarrow v_2$, v_2 has a value of at least 4. Continuing this process, we know that v_{d-1} has a value of at least 2^{d-1}. Let w be an end-vertex that is adjacent to v_k. It is obvious that $\mathcal{S}$ needs to have the following acquisitions:

$$w \rightarrow v_k, v_k \rightarrow v_{k-1}, \cdots, v_{d+2} \rightarrow v_{d+1}, v_{d+1} \rightarrow v_d.$$

Similarly, v_{d+1} has a value of at least 2^{k-d}. To make the acquisitions $v_{d-1} \rightarrow v_d$ and $v_{d+1} \rightarrow v_d$ feasible, v_d needs to have a value of at least $\min\{2^{d-1}, 2^{k-d}\}$.

By the above discussion, we have

$$n \geq 2^{d-1} + \min\{2^{d-1}, 2^{k-d}\} + 2^{k-d}.$$

It is easy to show from this that $n \geq 2^{k/2+1}$ when n is even, and $n \geq 3 \cdot 2^{(k-1)/2}$ when n is odd.

The proof of Theorem 11 is complete. ■

Theorem 12. *Given $k \geq 3$. There exists a caterpillar T with $T' = v_1 v_2 \cdots v_k$ such that $a(T) = \lceil \frac{k}{2} \rceil$ if and only if $n \geq g(k)$, where*

$$g(k) = \begin{cases} \frac{3k}{2} & \text{when } k \text{ is even} \\ \frac{3k+1}{2} & \text{when } k \text{ is odd} \end{cases}.$$

Proof. ($\Leftarrow$) Let a_i be the number of end-vertices that are adjacent to v_i for $1 \leq i \leq k$. Define $a_i = 0$ when $i \equiv 0 \pmod 2$ and $a_i = 1$ when $i \equiv 1 \pmod 2$ for $1 \leq i \leq k-2$, and $a_{k-1} = 0$, and $a_k = n - g(k) + 1$. Clearly the order of T is n.

By Proposition 8, first we can transfer the value of all end-vertices to the vertices that are adjacent to them. Let $w(v_i)$ be the value of v_i at this time, where $1 \leq i \leq k$. When k is even, $(w(v_1), w(v_2), w(v_3), \cdots, w(v_{k-3}), w(v_{k-2}), w(v_{k-1}), w(v_k)) = (2, 1, 2, \cdots, 2, 1, 1, n - g(k) + 2)$. Notice that each vertex in $X = \{v_1, v_3, \cdots, v_{k-5}, v_{k-3}, v_k\}$ has a value of at least 2 and it is adjacent to vertices having a value of one. Thus X must be a subset of any acquisition set of T and then $a(T) \geq \frac{k}{2}$. That, together with the fact that $\beta(P_k) = \lceil \frac{k}{2} \rceil$, implies that $a(T) = \frac{k}{2}$ when k is even. When k is odd, $(w(v_1), w(v_2), w(v_3), \cdots, w(v_{k-3}), w(v_{k-2}), w(v_{k-1}), w(v_k)) = (2, 1, 2, \cdots, 1, 2, 1, n - g(k) + 2)$. Notice that each vertex in $X' = \{v_1, v_3, \cdots, v_{k-2}, v_k\}$ has a value of at least 2 and it is adjacent to vertices having a value of one. By a similar argument as is given above, $a(T) = \lceil \frac{k}{2} \rceil$ when k is odd.

By the above discussion, $a(T) = \lceil \frac{k}{2} \rceil$.

($\Rightarrow$) By contradiction. Suppose that there exists a caterpillar T with $T' = v_1 v_2 \cdots v_k$ such that $a(T) = \lceil \frac{k}{2} \rceil$ and $n < g(k)$. If $n < g(k)$, then by Theorem 6,

$$a(T) \leq \frac{n+1}{3} \leq \frac{g(k)}{3} = \frac{k}{2} \leq \left\lceil \frac{k}{2} \right\rceil.$$

So the result holds unless each of the inequalities in the previous equation are equalities, and hence k is even and $n = \frac{3k}{2} - 1$. The case when $k = 4$ is trivial. We can assume that $k \geq 6$. Define $Y_i = \{v_{2i-1}, v_{2i}\}$ for $1 \leq i \leq \frac{k}{2}$. As $n = \frac{3k}{2} - 1$, the number of end-vertices in T is $\frac{k}{2} - 1$. Notice that v_1 and v_k are adjacent to at least one end-vertex. Thus there exists an integer m with $2 \leq m \leq \frac{k}{2} - 1$ such that each vertex in Y_m is not adjacent to any end-vertex.

Describe an acquisition sequence S as follows. First transfer the value of all end-vertices to the vertices that are adjacent to them. At this time, v_{2m-1} and v_{2m} have a value of one. Second perform the acquisitions $v_{2m-1} \to v_{2m-2}$ and $v_{2m} \to v_{2m+1}$. Now the vertices with nonzero values in T form two disjoint paths $P_1 = v_1v_2 \cdots v_{2m-3}v_{2m-2}$ and $P_2 = v_{2m+1}v_{2m+2} \cdots v_{2j-1}v_{2j}$. Finally perform acquisitions on P_1 and P_2. Since the acquisition set obtained at the end of S has at most $\beta(P_1) + \beta(P_2) = m - 1 + (\frac{k}{2} - m) = \frac{k}{2} - 1$, $a(T) \leq \frac{k}{2} - 1$, a contradiction. Hence such T does not exist.

The proof of Theorem 12 is complete. ■

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