Simple Graphoidal Covers in a Graph *

S. Arumugam and I. Sahul Hamid
Department of Mathematics
Arulmigu Kalasalingam College of Engineering
Anand Nagar,Krishnankoil-626190.
INDIA
e-mail: *s_arumugam_akce@yahoo.com*

Abstract

A simple graphoidal cover of a graph G is a collection ψ of (not necessarily open) paths in G such that every path in ψ has at least two vertices, every vertex of G is an internal vertex of at most one path in ψ, every edge of G is in exactly one path in ψ and any two paths in ψ have at most one vertex in common. The minimum cardinality of a simple graphoidal cover of G is called the simple graphoidal covering number of G and is denoted by $\eta_s(G)$. In this paper we determine the value of η_s for several families of graphs. We also obtain several bounds for η_s and characterize graphs attaining the bounds.

Keywords. Graphoidal cover, Simple graphoidal cover.
2000 Mathematics Subject Classification: 05C

1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by p and q respectively. For graph theoretic terminology we refer to Harary [4]. All graphs in this paper are assumed to be connected and non-trivial.

If $P = (v_0, v_1, v_2, \ldots, v_n)$ is a path or a cycle in a graph G, then $v_1, v_2, \ldots, v_{n-1}$ are called internal vertices of P and v_0, v_n are called external vertices of P. If $P = (v_0, v_1, v_2, \ldots, v_n)$ and $Q = (v_n = w_0, w_1, w_2, \ldots, w_m)$ are two paths in G, then the walk obtained by concatenating P and Q at v_n is denoted by $P \circ Q$ and the path $(v_n, v_{n-1}, \ldots, v_2, v_1, v_0)$ is denoted by P^{-1}. For a unicyclic graph G with cycle C, if w is a vertex of degree

*Research supported by NBHM Project 48/2/2004/R&D-II/7372.

greater than 2 on C with $deg\ w = k$, let $e_1, e_2 \dots, e_{k-2}$ be the edges of $E(G) - E(C)$ incident with w. Let $T_i, 1 \leq i \leq k-2$, be the maximal subtree of G such that T_i contains the edge e_i and w is a pendant vertex of T_i. Then $T_1, T_2, \dots, T_{k-2}$ are called the *branches* of G at w. Also the maximal subtree T of G such that $V(T) \cap V(C) = \{w\}$ is called the *subtree rooted at* w.

The concept of graphoidal cover was introduced by Acharya and Sampathkumar [1].

Definition 1.1. [1] *A graphoidal cover of a graph G is a collection ψ of (not necessarily open) paths in G satisfying the following conditions.*

(i) Every path in ψ has at least two vertices.

(ii) Every vertex of G is an internal vertex of at most one path in ψ.

(iii) Every edge of G is in exactly one path in ψ.

The minimum cardinality of a graphoidal cover of G is called the graphoidal covering number of G and is denoted by $\eta(G)$.

An elaborate review of results in graphoidal covers with several interesting applications and a large collection of unsolved problems is given in Arumugam et al.[3]. Pakkiam and Arumugam [5, 6] determined the graphoidal covering number of several families of graphs.

Theorem 1.2. [5] *Let T be a tree with n pendant vertices. Then $\eta(T) = n - 1$.*

Theorem 1.3. [6] *Let G be a unicyclic graph with n pendant vertices. Let C be the unique cycle of G and let m denote the number of vertices of degree greater than 2 on C. Then*

$$\eta(G) = \begin{cases} 1 & \text{if } m = 0 \\ n+1 & \text{if } m = 1 \text{ and } deg\ v = 3, \text{ where } v \text{ is the} \\ & \text{unique vertex on } C \text{ with } deg\ v > 2. \\ n & \text{otherwise} \end{cases}$$

Theorem 1.4. [2] *For any graph G with $\delta \geq 3$, we have $\eta = q - p$.*

2. Main Results

For any graph G, $\psi = E(G)$ is trivially a graphoidal cover and has the interesting property that any two paths in ψ have at most one vertex in common. This observation motivates the following definition of simple graphoidal covers in a graph.

Definition 2.1. *A simple graphoidal cover of a graph G is a graphoidal cover ψ of G such that any two paths in ψ have at most one vertex in common. The minimum cardinality of a simple graphoidal cover of G is called the simple graphoidal covering number of G and is denoted by $\eta_s(G)$ or simply η_s.*

Remark 2.2. *We observe that every path (cycle) in a simple graphoidal cover of a graph is an induced path (cycle) so that the behaviour of simple graphoidal covers is quite different from that of graphoidal covers.*

Definition 2.3. *Let ψ be a collection of internally disjoint paths in G. A vertex of G is said to be an interior vertex of ψ if it is an internal vertex of some path in ψ, otherwise it is called an exterior vertex of ψ.*

Theorem 2.4. *For any simple graphoidal cover ψ of a (p, q)-graph G, let t_ψ denote the number of exterior vertices of ψ and let $t = \min t_\psi$, where the minimum is taken over all simple graphoidal covers ψ of G. Then $\eta_s(G) = q - p + t$.*

Proof. For any simple graphoidal cover ψ of G, we have

$$\begin{aligned} q &= \sum_{P\in\psi} |E(P)| \\ &= \sum_{P\in\psi} (t(P)+1) \quad (t(P) \text{ denotes the number of internal vertices of } P) \\ &= \sum_{P\in\psi} t(P) + |\psi| \\ &= p - t_\psi + |\psi|. \end{aligned}$$

Hence $|\psi| = q - p + t_\psi$ so that $\eta_s(G) = q - p + t$. □

Corollary 2.5. *For any graph G, $\eta_s(G) \geq q-p$. Moreover, the following are equivalent.*

(*i*) $\eta_s(G) = q - p$.

(*ii*) *There exists a simple graphoidal cover of G without exterior vertices.*

(*iii*) *There exists a set $\mathcal{P}$ of internally disjoint and edge disjoint induced paths without exterior vertices such that any two paths in $\mathcal{P}$ have at most one vertex in common. (From such a set $\mathcal{P}$ of paths the required simple graphoidal cover can be obtained by adding the edges which are not covered by the paths in $\mathcal{P}$).*

Corollary 2.6. *If there exists a simple graphoidal cover ψ of a graph G such that every vertex of G with degree at least two is interior to ψ, then ψ is a minimum simple graphoidal cover of G and $\eta_s(G) = q - p + n$, where n is the number of pendant vertices of G.*

Remark 2.7. *Since every graphoidal cover of a tree T is a simple graphoidal cover of T, it follows from Theorem 1.2 that $\eta_s(T) = \eta(T) = n - 1$, where n is the number of pendant vertices of T.*

We now proceed to determine η_s for several families of graphs.

Theorem 2.8. *Let G be a unicyclic graph with n pendant vertices. Let C be the unique cycle in G and let m be the number of vertices of degree greater than 2 on C. Then*

$$\eta_s(G) = \begin{cases} 1 & \text{if } m = 0 \\ n+1 & \text{if } m = 1 \text{ or } 2 \text{ and any vertex on } C \\ & \text{has degree at most } 3 \\ n & \text{otherwise} \end{cases}$$

Proof. **Case 1.** $m = 0$.

Then $G = C$ so that $\eta_s(G) = 1$.

Case 2. There exists a vertex v on C such that $deg\ v \geq 4$.

Let ψ_v be a minimum simple graphoidal cover of the subtree T_v of G rooted at v. Clearly v is interior to ψ_v. For any other vertex w on C with $deg\ w \geq 3$, let ψ_{w_i} be a minimum simple graphoidal cover of the branch T_i at w containing the edge ww_i, where $1 \leq i \leq (deg\ w) - 2$. Then

$$\psi = \psi_v \cup \left\{ \bigcup_{\substack{w \in V(C)-\{v\} \\ deg\ w \geq 3}} \left(\bigcup_{i=1}^{(deg\ w)-2} \psi_{w_i} \right) \right\} \cup \{C\},$$

where v is taken as the origin of C, is a simple graphoidal cover of G such that every vertex of degree at least two is interior to ψ. Hence it follows from Corollary 2.6 that $\eta_s(G) = n$.

Case 3. Every vertex on C has degree either 2 or 3.

Subcase 3.1 $m = 1$.

Let v be the unique vertex of degree 3 on C. Let T_1 be the tree rooted at v. Then T_1 has $n + 1$ pendant vertices so that $\eta_s(T_1) = n$. Let ψ_1 be a minimum simple graphoidal cover of T_1. Then $\psi = \psi_1 \cup \{C\}$, where any arbitrary vertex of C is taken as the origin of C, is a simple graphoidal cover of G and $|\psi| = n+1$. Hence $\eta_s(G) \leq n+1$. Further, for any simple graphoidal cover ψ of G, the n pendant vertices of G and at least one vertex on C are exterior to ψ so that $t \geq n+1$. Hence $\eta_s(G) = q-p+t \geq q-p+n+1 = n+1$. Thus $\eta_s(G) = n+1$.

Subcase 3.2 $m = 2$.

Let v and w be the vertices on C such that $deg\ v = deg\ w = 3$. Let P denote the (v,w)-section of C of length greater than 1. Let T_1 be the subgraph of G obtained by deleting all the internal vertices of P. Clearly T_1 is a tree with n pendant vertices and hence $\eta_s(T_1) = n-1$. Let ψ_1 be a minimum simple graphoidal cover of T_1. Let u be an internal vertex of P. Let P_1 and P_2 denote the (v,u)- section and (w,u)- section of P respectively. Then $\psi = \psi_1 \cup \{P_1, P_2\}$ is a simple graphoidal cover of G and $|\psi| = |\psi_1| + 2 = n+1$. Thus $\eta_s(G) \leq n+1$. Further, for any simple graphoidal cover ψ of G, the n pendant vertices of G and at least one vertex on C are exterior to ψ so that $t \geq n+1$. Hence $\eta_s(G) = q-p+t \geq q-p+n+1 = n+1$.

Thus $\eta_s(G) = n+1$.

Subcase 3.3 $m \geq 3$.

Let $C = (v_1, v_2, ..., v_r, v_1)$ and let $deg\ v_{i_1} = deg\ v_{i_2} = ... = deg\ v_{i_k} = 3$, where $1 \leq i_1 < i_2 < ... < i_k \leq r$. Let $\psi_j, 1 \leq j \leq k$, be a minimum simple graphoidal cover of the tree T_j rooted at v_{i_j}. Let P_1, P_2 and P_3 denote respectively the paths in ψ_1, ψ_2 and ψ_3 having v_{i_1}, v_{i_2} and v_{i_3} as its terminal vertices. Let

$$Q_1 = P_1 \circ (v_{i_1}, v_{i_1+1}, \ldots, v_{i_2}),$$
$$Q_2 = P_2 \circ (v_{i_2}, v_{i_2+1}, \ldots, v_{i_3}) \text{ and}$$
$$Q_3 = P_3 \circ (v_{i_3}, v_{i_3+1}, \ldots, v_{i_1}).$$

Then

$$\psi = \left\{ \left(\bigcup_{j=1}^{k} \psi_j \right) - \{P_1, P_2, P_3\} \right\} \cup \{Q_1, Q_2, Q_3\}$$

is a simple graphoidal cover of G in which every vertex of degree at least two is interior to ψ. Hence it follows from Corollary 2.6 that $\eta_s(G) = n$. □

Corollary 2.9. *Let G be as in Theorem 2.8. Then $\eta_s(G) \neq \eta(G)$ if and only if $m = 2$ and $deg\ v = deg\ w = 3$, where v and w are the only vertices of degree greater than 2 on C.*

Proof. The result follows from Theorem 1.3 and Theorem 2.8. □

Theorem 2.10. *For the complete graph $K_n (n \geq 3)$, we have*

$$\eta_s(K_n) = \begin{cases} 4 & \text{if } n = 4 \\ \frac{(n-1)(n-2)}{2} & \text{if } n \text{ is odd} \\ \frac{n(n-3)}{2} & \text{if } n > 4 \text{ and } n \text{ is even} \end{cases}$$

Proof. We observe that, for any simple graphoidal cover ψ of K_n, any member of ψ is either a triangle or an edge. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$.

If $n = 4$, then $\psi = \{(v_1, v_2, v_3, v_1), (v_1, v_4), (v_2, v_4), (v_3, v_4)\}$ is a simple graphoidal cover of K_4 so that $\eta_s(K_4) \leq 4$. Further, for any simple graphoidal cover ψ of K_4, the number of vertices interior to ψ is at most 2 so that $t \geq 2$. Hence $\eta_s(K_4) = 4$.

Now, suppose $n = 3$ or $n \geq 5$. Let $P_i = (v_{2i-1}, v_{2i}, v_{2i+1}, v_{2i-1})$, where $i = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$ and let $Q = (v_{n-1}, v_n, v_1, v_{n-1})$.

If n is odd, then $\psi = \left\{P_1, P_2, \ldots, P_{\lfloor \frac{n}{2} \rfloor}\right\} \cup S$, where S is the set of edges of K_n not covered by the triangles $P_1, P_2, \ldots, P_{\lfloor \frac{n}{2} \rfloor}$ is a simple graphoidal cover of K_n such that v_1 is the only vertex exterior to ψ. Further, for any simple graphoidal cover ψ of K_n, at least one vertex is exterior to ψ. Hence $\eta_s(K_n) = q - p + 1 = \frac{(n-1)(n-2)}{2}$.

If n is even, then $\psi = \{P_1, P_2, \ldots, P_{\frac{n}{2}-1}, Q\} \cup S$, where S is the set of edges of K_n not covered by the triangles $P_1, P_2, \ldots, P_{\frac{n}{2}-1}$, Q is a simple graphoidal cover of K_n without exterior vertices so that $\eta_s(K_n) = q - p = \frac{n(n-3)}{2}$. □

Corollary 2.11. *$\eta_s(K_n) = \eta(K_n)$ if and only if $n = 3$ or n is even and $n \geq 6$.*

Theorem 2.12. *For the wheel $W_n = K_1 + C_{n-1}$, we have*

$$\eta_s(W_n) = \begin{cases} 4 & \text{if } n = 4, 5 \\ 5 & \text{if } n = 6 \\ n - 2 & \text{if } n \geq 7 \end{cases}$$

Proof. Let $V(W_n) = \{v_0, v_1, v_2, \ldots, v_{n-1}\}$ and $E(W_n) = \{v_0 v_i : 1 \leq i \leq n-1\} \cup \{v_i v_{i+1} : 1 \leq i \leq n-2\} \cup \{v_1 v_{n-1}\}$.

If $n = 4$, then $W_n = K_4$ so that $\eta_s(W_n) = 4$.

If $n = 5$, let $\psi = \{(v_0, v_1, v_2, v_0), (v_0, v_3, v_4, v_0), (v_1, v_4), (v_2, v_3)\}$. If $n = 6$, let $\psi = \{(v_0, v_1, v_2, v_0), (v_0, v_4, v_5, v_0), (v_2, v_3, v_4), (v_0, v_3), (v_1, v_5)\}$. Clearly ψ is a simple graphoidal cover of W_n with v_0 as its only exterior vertex. Further, for any simple graphoidal cover ψ of W_5 or W_6, at least one vertex is exterior to ψ. Hence $\eta_s(W_5) = q - p + 1 = 4$ and $\eta_s(W_6) = q - p + 1 = 5$.

Now, let $n \geq 7$. Let $P_1 = (v_{n-1}, v_1, v_2)$, $P_2 = (v_{n-3}, v_0, v_1)$, $P_3 = (v_0, v_2, v_3, v_0)$, $P_4 = (v_0, v_{n-2}, v_{n-1}, v_0)$ and $P_5 = (v_3, v_4, \ldots, v_{n-2})$. Then $\psi = \{P_1, P_2, P_3, P_4, P_5\}$ is a collection of internally disjoint and edge disjoint induced paths of W_n without exterior vertices such that any two paths in ψ have at most one vertex in common. Hence it follows from Corollary 2.5 that $\eta_s(W_n) = q - p = n - 2$. □

Corollary 2.13. *$\eta_s(W_n) = \eta(W_n)$ if and only if $n \geq 7$.*

Theorem 2.14.

(*i*) $\eta_s(K_{1,n}) = n - 1$, *for all* $n \geq 2$.

(*ii*)

$$\eta_s(K_{2,n}) = \begin{cases} 1 & \text{if } n = 2 \\ 3 & \text{if } n = 3 \\ 2(n-2) & \text{if } n \geq 4 \end{cases}$$

(*iii*)

$$\eta_s(K_{3,n}) = \begin{cases} 5 & \text{if } n = 3 \\ 6 & \text{if } n = 4 \\ 8 & \text{if } n = 5 \\ 3(n-3) & \text{if } n \geq 6 \end{cases}$$

(*iv*) *Let m and n be integers with $n \geq m \geq 4$. Then*

$$\eta_s(K_{m,n}) = \begin{cases} mn - m - n & \text{if } n \leq \binom{m}{2} + m \\ mn - m - n + r & \text{if } n = \binom{m}{2} + m + r, r > 0 \end{cases}$$

Proof. We observe that, for any simple graphoidal cover ψ of $K_{m,n}$, any member of ψ is either a cycle of length 4 or a path of length 2 or an edge.

(i) Since $K_{1,n}$ is a tree with n pendant vertices, it follows from Remark 2.7 that $\eta_s(K_{1,n}) = n - 1$.

(ii) Since $K_{2,2} = C_4$, we have $\eta_s(K_{2,2}) = 1$.

Now, let $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be the bipartition of $K_{2,n}$.

If $n = 3$, then $\psi = \{(x_1, y_1, x_2, y_2, x_1), (y_3, x_1), (y_3, x_2)\}$ is a simple graphoidal cover of $K_{2,3}$ with x_1 and y_3 as its exterior vertices. Further, for any simple graphoidal cover ψ of $K_{2,3}$, we have $t_\psi \geq 2$. Hence $\eta_s(K_{2,3}) = q - p + 2 = 3$.

Now, suppose $n \geq 4$. Let $P_1 = (x_1, y_1, x_2, y_2, x_1)$ and $P_2 = (y_3, x_1, y_4)$. Then $\psi = \{P_1, P_2\} \cup S$, where S is the set of edges of $K_{2,n}$ not covered by P_1 and P_2 is a simple graphoidal cover of $K_{2,n}$ with $n - 2$ exterior vertices.

Further, for any simple graphoidal cover ψ of $K_{2,n}$, at most two vertices in Y are interior to ψ so that $t \geq n-2$. Hence $\eta_s(K_{2,n}) = q-p+(n-2) = 2(n-2)$.

(iii) Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be the bipartition of $K_{3,n}$.

If $n = 3$, then $\psi = \{(x_1, y_1, x_2, y_2, x_1), (x_1, y_3, x_3), (x_2, y_3), (x_3, y_1), (x_3, y_2)\}$ is a simple graphoidal cover of $K_{3,3}$ with x_1 and x_3 as its exterior vertices. Further, $t_\psi \geq 2$ for any simple graphoidal cover ψ of $K_{3,3}$ so that $\eta_s(K_{3,3}) = q - p + 2 = 5$. By a similar argument, it can be easily proved that $\eta_s(K_{3,4}) = q - p + 1 = 6$ and $\eta_s(K_{3,5}) = q - p + 1 = 8$.

Now, suppose $n \geq 6$. Let $P_1 = (x_1, y_1, x_2, y_2, x_1)$, $P_2 = (x_2, y_3, x_3, y_4, x_2)$ and $P_3 = (x_3, y_5, x_1, y_6, x_3)$. Then $\psi = \{P_1, P_2, P_3\} \cup S$, where S is the set of edges of $K_{3,n}$ not covered by P_1, P_2 and P_3 is a simple graphoidal cover of $K_{3,n}$ with $n-6$ exterior vertices. Further, for any simple graphoidal cover ψ of $K_{3,n}$ the number of vertices in Y which are interior to ψ is at most 6 and hence $t \geq n - 6$. Hence $\eta_s(K_{3,n}) = q - p + (n - 6) = 3(n - 3)$.

(iv) Let m and n be integers with $n \geq m \geq 4$. Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be the bipartition of $K_{m,n}$.

If $m = n = 4$, then $\psi = \{(x_1, y_1, x_2, y_2, x_1), (x_2, y_3, x_4), (x_2, y_4, x_3), (y_3, x_1, y_4), (y_1, x_3, y_3), (y_1, x_4, y_4), (x_3, y_2), (x_4, y_2)\}$ is a simple graphoidal cover of $K_{4,4}$ without exterior vertices so that $\eta_s(K_{4,4}) = 8$.

Assume that $m \geq 4$ and $n \geq 5$ with $m \leq n$.
Suppose $n < 2m$. Let

$$P_i = (x_i, y_{2i-1}, x_{i+1}, y_{2i}, x_i),\ i = 1, 2, \ldots, \lceil \tfrac{n}{2} \rceil - 1,$$

$$P_{\lceil \frac{n}{2} \rceil} = \begin{cases} (x_{\frac{n}{2}},\ y_{n-1},\ x_1,\ y_n,\ x_{\frac{n}{2}}) & \text{if } n \text{ is even} \\ (x_2,\ y_n,\ x_4) & \text{if } n \text{ is odd,} \end{cases}$$

$$P = (y_3, x_1, y_5) \text{ and}$$

$$Q_i = (y_1, x_{\lceil \frac{n}{2} \rceil + i}, y_{i+2}), \text{ where } i = 1, 2, \ldots, m - \lceil \tfrac{n}{2} \rceil.$$

Now, let $\psi_1 = \{P_i : 1 \leq i \leq \lceil \frac{n}{2} \rceil\} \cup \{Q_i : 1 \leq i \leq m - \lceil \frac{n}{2} \rceil\}$ if n is even and $\psi_1 = \{P_i : 1 \leq i \leq \lceil \frac{n}{2} \rceil\} \cup \{Q_i : 1 \leq i \leq m - \lceil \frac{n}{2} \rceil\} \cup \{P\}$ if n is odd. Then ψ_1 is a collection of internally disjoint and edge disjoint induced paths in $K_{m,n}$ without exterior vertices such that any two paths in ψ_1 have at most one vertex in common and hence $\eta_s(K_{m,n}) = q - p = mn - m - n$.

Now, suppose $n = \binom{m}{2} + m$. Let

$P_i = (x_i, y_{2i-1}, x_{i+1}, y_{2i}, x_i)$, where $i = 1, 2, \ldots, m - 1$, and

$P_m = (x_m, y_{2m-1}, x_1, y_{2m}, x_m)$.

Let $S = \{(i, j) : 1 \leq i < j \leq m\}$ and $S_1 = \{(i, i+1) : 1 \leq i \leq m - 1\} \cup$

$\{(1,m)\}$ so that $|S| = \binom{m}{2}$ and $|S_1| = m$. Fix a bijection $f : S - S_1 \to \{y_{2m+1}, y_{2m+2}, \ldots, y_n\}$. Let $Q_k = (x_i, y_k, x_j)$, where $2m+1 \leq k \leq n$ and $f(i,j) = y_k$. Then $\psi_1 = \{P_i : 1 \leq i \leq m\} \cup \{Q_k : 2m+1 \leq k \leq n\}$ is a collection of internally disjoint and edge disjoint induced paths in $K_{m,n}$ without exterior vertices such that any two paths in ψ_1 have at most one vertex in common. Hence $\eta_s(K_{m,n}) = q - p = mn - m - n$.

Suppose $2m \leq n < \binom{m}{2} + m$. Let $n = 2m + r$, where $1 \leq r < \binom{m}{2} - m$. Then $\psi_1 = \{P_i : 1 \leq i \leq m\} \cup \{Q_k : 2m+1 \leq k \leq n\}$ is a collection of internally disjoint and edge disjoint induced paths in $K_{m,n}$ without exterior vertices such that any two paths in ψ_1 have at most one vertex in common. Hence $\eta_s(K_{m,n}) = q - p = mn - m - n$.

Suppose $n = \binom{m}{2} + m + r$, where $r > 0$. Then in any simple graphoidal cover ψ at least r vertices are exterior to ψ and hence $t_\psi \geq r$. Further, exactly r vertices are exterior to ψ_1 and hence $\eta_s(K_{m,n}) = q - p + r$. □

Corollary 2.15. *For the complete bipartite graph $K_{m,n}(m \leq n)$, $\eta_s = \eta$ if and only if $m = 1$ and $n \geq 1$ or $m = n = 2$ or $m \geq 4$ and $n \leq \binom{m}{2} + m$.*

It is obvious that for any graph, $\eta_s \geq \eta$. The difference $\eta_s - \eta$ can be made arbitrarily large. For example, for the complete bipartite graph $K_{m,\binom{m}{2}+m+r}$, where $m > 3$ and $r > 0$, we have $\eta_s = q-p+r$ and $\eta = q-p$ so that $\eta_s - \eta = r$. Further, Remark 2.7 and Corollaries 2.9, 2.11, 2.13 and 2.15 give several families of graphs for which $\eta = \eta_s$. Also one can construct infinite families of graphs for which $\eta = \eta_s$.

Example 2.16. *Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$ be the cycle on n vertices. Let G_n be the graph obtained from C_n by adding the vertices $w_1, w_2, \ldots, w_n$ and joining w_i with v_i, v_{i+1}, suffices being taken modulo n. It can be easily verified that $\eta_s(G_n) = \eta(G_n) = q - p$.*

Example 2.17. *Let G be a connected (p,q)-graph. Then $\eta(G^+) = \eta_s(G^+) = q$, where G^+ is the graph obtained from G by attaching one pendant edge to every vertex of G. Since the p pendant vertices of G^+ are exterior to any (simple) graphoidal cover of G, it follows that $\eta_s(G^+) \geq |E(G^+)| - |V(G^+)| + p = q$. Similarly $\eta(G^+) \geq q$. Further, it can be proved by induction on q that $\eta_s(G^+) \leq q$ and $\eta(G^+) \leq q$, so that $\eta(G^+) = \eta_s(G^+) = q$.*

The above results lead to the following problems.

Problem 2.18. *Characterize the class of graphs for which*

(i) $\eta = \eta_s$.

(ii) $\eta_s = q - p$.

In the following theorems we obtain bounds for η_s and characterize graphs attaining the bounds.

Theorem 2.19. *For any graph G with girth g, we have $\eta_s(G) \leq q-g+1$. Further, equality holds if and only if G is isomorphic to one of the graphs G_i, $1 \leq i \leq 6$, given in Figure 1.*

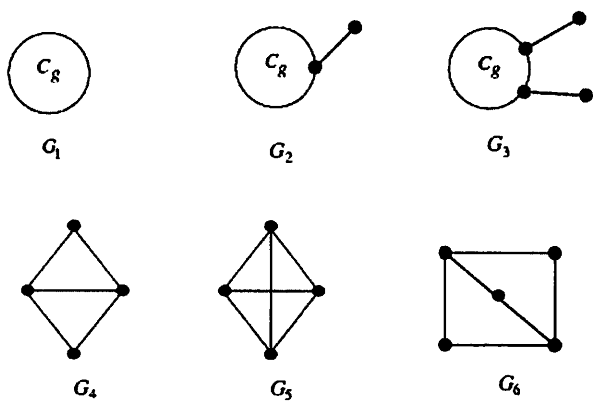

Figure 1

Proof. Let C be an induced cycle of length g in G. Then $\psi = \{C\} \cup (E(G) - E(C))$ is a simple graphoidal cover of G such that $|\psi| = q - g + 1$. Hence $\eta_s(G) \leq q - g + 1$.

Now, let G be a graph of girth g with $\eta_s(G) = q - g + 1$. Let $C = (v_1, v_2, \ldots, v_g, v_1)$ be a cycle of length g in G. If there exists an induced path P in G such that length of P is at least 2 and $|V(P) \cap V(C)| = 1$, then $\{C, P\} \cup S$, where S is the set of edges of G not covered by C and P is a simple graphoidal cover of G and $|\psi| < q - g + 1$, which is a contradiction. Thus every induced path P in G with $|V(P) \cap V(C)| = 1$ has length 1. Hence it follows that every vertex not on C is adjacent to a vertex on C, the set of all vertices not on C is independent and every vertex on C has degree at most 3.

Claim 1. $\Delta(G) \leq 3$.

Suppose there exists a vertex v of G with *deg* $v \geq 4$. Then v is not on C. Let $v_{i_1}, v_{i_2}, v_{i_3}$ and v_{i_4} be neighbours of v on C, where $1 \leq i_1 < i_2 < i_3 < i_4 \leq g$. We may assume without loss of generality that the length of the path $P' = (v_{i_1}, v_{i_1+1}, \ldots, v_{i_2})$ is at most $\frac{g}{4}$. Then $Z = P' \circ (v_{i_2}, v, v_{i_1})$

is a cycle of length at most $\frac{g}{4}+2$. Hence $g \le \frac{g}{4}+2$ so that $g < 3$, which is a contradiction. Thus $\Delta(G) \le 3$.

Claim 2. There exist at most two vertices not on C.

Suppose there exist vertices u, v and w not on C. Let v_{i_1}, v_{i_2} and v_{i_3} be vertices on C such that v_{i_1}, v_{i_2} and v_{i_3} are adjacent to u, v and w respectively. It follows from by Claim 1 that the vertices v_{i_1}, v_{i_2} and v_{i_3} are distinct so that we may assume that $1 \le i_1 < i_2 < i_3 \le g$. Let $P_1 = (u, v_{i_1}, v_{i_1+1}, \ldots, v_{i_2})$, $P_2 = (v, v_{i_2}, v_{i_2+1}, \ldots, v_{i_3})$ and $P_3 = (w, v_{i_3}, v_{i_3+1}, \ldots, v_{i_1})$. Since the girth of G is g, it follows that P_1, P_2 and P_3 are induced paths in G. Now, $\psi = \{P_1, P_2, P_3\} \cup S$, where S is the set of edges not covered by P_1, P_2 and P_3 is a simple graphoidal cover of G and $|\psi| = q - g$, which is a contradiction.

Thus there exist at most two vertices not on C.

Claim 3. If there exists a vertex v not on C with $deg\ v = 2$, then $g = 3$ or 4.

Let v_{i_1} and v_{i_2} be the neighbours of v on C, where $1 \le i_1 < i_2 \le g$. Let us assume without loss of generality that the length of the path $P = (v_{i_1}, v_{i_1+1}, \ldots, v_{i_2})$ is at most $\frac{g}{2}$. Then $Z = P \circ (v_{i_2}, v, v_{i_1})$ is a cycle of length at most $\frac{g}{2}+2$. Hence $g \le \frac{g}{2}+2$ so that $g \le 4$. Thus $g = 3$ or 4.

Claim 4. If there exists a vertex v not on C with $deg\ v = 3$, then $g = 3$.

Proof of Claim 4 is similar to that of Claim 3.

Claim 5. If there exist two vertices v and w not on C, then $deg\ v = deg\ w = 1$.

It follows from Claim 1 and Claim 4 that, $deg\ v \le 2$ and $deg\ w \le 2$.

Suppose $deg\ v = deg\ w = 2$. Then by Claim 1 and Claim 3, we have $g = 4$. Hence $C = (v_1, v_2, v_3, v_4, v_1)$ and G is isomorphic to the graph given in Figure 2(a). Now, $\psi = \{(v_1, v_2, v_3, w, v_1), (v_3, v_4, v), (v, v_2), (v_1, v_4)\}$ is a simple graphoidal cover of G with $|\psi| = 4 < q - g + 1$, which is a contradiction.

Suppose $deg\ v = 1$ and $deg\ w = 2$. Then by Claim 3, we have $g = 3$ or $g = 4$. If $g = 3$, then $C = (v_1, v_2, v_3, v_1)$ and G is isomorphic to the graph given in Figure 2(b). Now, $\psi = \{(v_3, v_2, w, v_3), (v_3, v_1, v), (v_1, v_2)\}$ is a simple graphoidal cover of G with $|\psi| = 3 < q - g + 1$, which is a contradiction. If $g = 4$, then $C = (v_1, v_2, v_3, v_4, v_1)$ and G is isomorphic to the graph given in Figure 2(c). Now, $\psi = \{(v_1, v_2, v_3, w, v_1), (v_1, v_4, v), (v_4, v_3)\}$ is a simple graphoidal cover of G with $|\psi| = 3 < q - g + 1$, which is again a contradiction.

Thus $deg\ v = deg\ w = 1$.

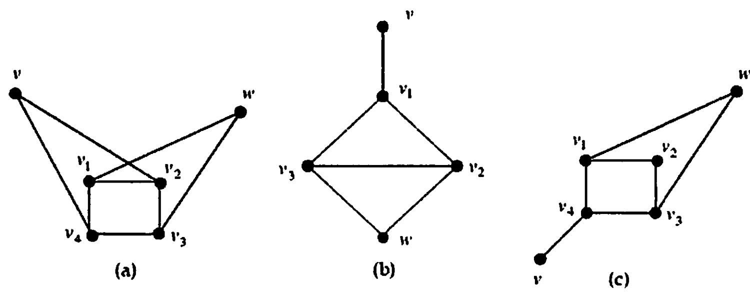

Figure 2

Now, let m denote the number of vertices not on C. By Claim 2, we have $m \leq 2$. If $m = 0$ then $G = G_1$.

Suppose $m = 1$. Let v be the unique vertex not on C. By Claim 1, we have $deg\ v \leq 3$. If $deg\ v = 1$, then $G = G_2$. Suppose $deg\ v = 2$. By Claim 3, we have $g = 3$ or 4. If $g = 3$, then $G = G_4$ and if $g = 4$, then $G = G_6$. If $deg\ v = 3$, it follows from Claim 4 that $g = 3$ and hence $G = G_5$.

Suppose $m = 2$. Let v and w be the vertices not on C. It follows from Claim 5 that $deg\ v = deg\ w = 1$ and hence $G = G_3$.

Conversely, it can be easily verified that $\eta_s(G_i) = q - g + 1$, for all $i = 1, 2, \ldots, 6$. □

Theorem 2.20. *For any graph G with diameter d, we have $\eta_s(G) \leq q - d + 1$. Further, equality holds if and only if for any diameter path $P = (u = v_1, v_2, \ldots, v_{d+1} = v)$ the following are satisfied.*

1. *$deg\ u = deg\ v = 1$.*
2. *If w is a vertex not on P, then*
   - *(i) $N(w) \subseteq V(P)$.*
   - *(ii) $deg\ w \leq 3$.*
   - *(iii) If $deg\ w = 3$ and $N(w) = \{v_i, v_j, v_k\}$, where $2 \leq i < j < k \leq d$, then $j = i + 1$ and $k = i + 2$.*
   - *(iv) If $deg\ w = 2$ and $N(w) = \{v_i, v_j\}$, where $2 \leq i < j \leq d$, then either $j = i + 1$ or $j = i + 2$. Further, if $j = i + 2$, then $deg\ v_{i+1} = 2$.*
   - *(v) If $deg\ w = 1$ and $N(w) = \{v_i\}$, where $2 \leq i \leq d$, then all the neighbours of v_i not on P are pendant vertices.*
3. *If x and y are two vertices of degree greater than 1 not on P, then $N(x) \cap N(y) = \phi$.*

Proof. Let u and v be any two vertices of G with $d(u,v) = d$. Let P be a shortest u-v path in G. Then $\psi = \{P\} \cup (E(G) - E(P))$ is a simple graphoidal cover of G and hence $\eta_s(G) \leq q - d + 1$.

Now, let G be a graph with diameter d and $\eta_s(G) = q - d + 1$. Let $P = (u = v_1, v_2, \ldots, v_d, v_{d+1} = v)$ be a diameter path in G. Then any induced path Q in G with $|V(Q) \cap V(P)| = 1$ has length 1. Hence it follows that every vertex not on P is adjacent to a vertex on P and $N(w) \subseteq V(P)$ for any vertex w not on P. This proves 2(i) of the theorem.

We now claim that $deg\ u = deg\ v = 1$. Suppose $deg\ u > 1$. Then there exists a vertex w not on P which is adjacent to u. If $deg\ w = 1$, then $(w, v_1, v_2, \ldots, v_{d+1})$ is a shortest path of length $d + 1$, which is a contradiction. Hence $deg\ w \geq 2$. Let $i > 1$ be the least positive integer such that v_i is adjacent to w. Let $C = (v_1, v_2, \ldots, v_i, w, v_1)$ and $Q = (v_i, v_{i+1}, \ldots, v_{d+1})$. Then $\psi = \{C, Q\} \cup S$, where S is the set of edges of G not covered by C and Q is a simple graphoidal cover of G such that $|\psi| < q - d + 1$, which is a contradiction. Hence $deg\ u = 1$. By a similar argument, we have $deg\ v = 1$. This proves condition (1) of the theorem.

Since P is a diameter path, condition 2(ii), 2(iii) and first part of 2(iv) follow immediately.

We now prove 2(v). Suppose $deg\ w = 1$, $N(w) = \{v_i\}$, where $2 \leq i \leq d$ and $x \in N(v_i) - V(P)$. We claim that $deg\ x = 1$. Suppose $deg\ x > 1$. Then $deg\ x = 2$ or 3.

Case 1. $deg\ x = 2$.

Without loss of generality let $N(x) = \{v_i, v_j\}$, where $j = i + 1$ or $i + 2$. Let

$$C = \begin{cases} (v_i, v_{i+1}, x, v_i) & \text{if } j = i + 1 \\ (v_i, v_{i+1}, v_{i+2}, x, v_i) & \text{if } j = i + 2, \end{cases}$$

$$P_1 = (u = v_1, v_2, \ldots, v_i, w) \text{ and}$$
$$P_2 = (v_j, v_{j+1}, \ldots, v_{d+1} = v).$$

Then $\psi = \{P_1, P_2, C\} \cup S$, where S is the set of edges of G not covered by P_1, P_2 and C is a simple graphoidal cover of G such that $|\psi| < q - d + 1$, which is a contradiction.

Case 2. $deg\ x = 3$.

Without loss of generality let $N(x) = \{v_i, v_{i+1}, v_{i+2}\}$ or $\{v_{i-1}, v_i, v_{i+1}\}$.

Let $P_1 = (u = v_1, v_2, \ldots, v_i, w)$,
$P_2 = (v_{i+1}, v_{i+2}, \ldots, v_{d+1} = v)$ and
$C = (v_i, v_{i+1}, x, v_i\}$.

Then $\psi = \{P_1, P_2, C\} \cup S$, where S is the set of edges of G not covered by P_1, P_2 and C is a simple graphoidal cover of G such that $|\psi| < q - d + 1$, which is a contradiction.

Thus $deg\ x = 1$. This proves condition 2(v) of the theorem.

We now proceed to prove (3). Let x and y be two distinct vertices of degree greater than 1 not on P. We claim that $N(x) \cap N(y) = \phi$. We have the following possibilities.

(i) $deg\ x = deg\ y = 3$

(ii) $deg\ x = 3$ and $deg\ y = 2$.

(iii) $deg\ x = deg\ y = 2$.

Suppose $deg\ x = deg\ y = 3$. Let $N(x) = \{v_i, v_{i+1}, v_{i+2}\}$ and $N(y) = \{v_j, v_{j+1}, v_{j+2}\}$ for some i, j, where $2 \leq i \leq j \leq d$. Suppose $N(x) \cap N(y) \neq \phi$.

If $j = i$, let $P_1 = (v_1, \ldots, v_i)$, $P_2 = (v_{i+2}, \ldots, v_{d+1})$, $C_1 = (v_i,\ v_{i+1},\ x,\ v_i)$ and $C_2 = (v_{i+1},\ v_{i+2},\ y,\ v_{i+1})$. If $j = i + 1$ or $j = i + 2$, let $P_1 = (v_1, \ldots, v_{j-1})$, $P_2 = (v_{j+1}, \ldots, v_{d+1})$, $C_1 = (v_{j-1}, v_j, x, v_{j-1})$ and $C_2 = (v_j,\ v_{j+1},\ y,\ v_j)$. Then $\psi = \{P_1, P_2, C_1, C_2\} \cup S$, where S is the set of edges of G not covered by P_1, P_2, C_1 and C_2 is a simple graphoidal cover of G such that $|\psi| < q - d + 1$, which is a contradiction. Hence it follows that $N(x) \cap N(y) = \phi$.

The proof for the cases when $deg\ x = 3$ and $deg\ y = 2$ or when $deg\ x = deg\ y = 2$ is similar and we omit the details.

We now prove the second part of 2(iv). Let w be a vertex not on P such that $deg\ w = 2$ and $N(w) = \{v_i, v_{i+2}\}$, where $2 \leq i \leq d$. We claim that $deg\ v_{i+1} = 2$. Suppose there exists a vertex x not on P, which is adjacent to v_{i+1}. Now, it follows from (3) that $N(x) = \{v_{i+1}\}$ or $\{v_{i+1}, v_{i+3}\}$ or $\{v_{i-1}, v_{i+1}\}$. Since the conditions $N(x) = \{v_{i+1}, v_{i+3}\}$ and $N(x) = \{v_{i-1}, v_{i+1}\}$ are similar, we assume that $N(x) = \{v_{i+1}\}$ or $\{v_{i+1}, v_{i+3}\}$. Now, let $P_1 = (v_1, v_2, \ldots, v_i, v_{i+1}, x)$ and $P_2 = (v_i, w, v_{i+2}, \ldots, v_{d+1})$. Then $\psi = \{P_1, P_2\} \cup S$, where S is the set of edges of G not covered by P_1 and P_2 is a simple graphoidal cover of G such that $|\psi| < q-d+1$, which is a contradiction. Hence $deg\ v_{i+1} = 2$. This proves the second part of condition 2(iv) of the theorem.

Thus if $\eta_s(G) = q - d + 1$, then conditions (1),(2) and (3) of the theorem are satisfied.

Conversely, suppose conditions (1),(2) and (3) of the theorem are satisfied for any diameter path $P = (u = v_1, v_2, \ldots, v_{d+1} = v)$. Let ψ be a minimum simple graphoidal cover of G

Case 1. P is a member in ψ.

We claim that every vertex not on P is exterior to ψ. Suppose there exists a vertex w not on P which is interior to ψ. Let Q be the path (cycle) in ψ having w as an internal vertex. It follows from condition 2(i) that P and Q have two vertices in common, which is a contradiction. Hence the number of vertices interior to ψ is $d-1$ so that $t = p-(d-1) = p-d+1$. Thus $\eta_s(G) = q-p+t = q-d+1$.

Case 2. P is not a member in ψ.

We claim that if there exists a vertex w not on P which is interior to ψ, then there exists a vertex v_j on P, where $2 \le j \le d$, which is exterior to ψ. Let Q be the path (cycle) in ψ having w as an internal vertex. Now, by conditions 2(i) to 2(iv), we have $N(w) = \{v_i, v_{i+1}\}$ or $\{v_i, v_{i+2}\}$ or $\{v_i, v_{i+1}, v_{i+2}\}$ for some i, where $2 \le i \le d$.

Suppose $N(w) = \{v_i, v_{i+1}\}$. Then we may assume without loss of generality that $Q = (v_i, w, v_{i+1}, v_i)$. Now, by conditions 2(v) and (3) we have $deg\ v_i = 3$ and hence v_i is exterior to ψ.

Suppose $N(w) = \{v_i, v_{i+2}\}$. If Q is a cycle, then we may assume that $Q = (v_i, w, v_{i+2}, v_{i+1}, v_i)$. Now, by conditions 2(v) and (3) we have $deg\ v_i = 3$ and hence v_i is exterior to ψ. If Q is a path, then (v_i, w, v_{i+2}) is a section of Q and hence by condition 2(iv), $deg\ v_{i+1} = 2$ so that v_{i+1} is exterior to ψ.

Suppose $N(w) = \{v_i, v_{i+1}, v_{i+2}\}$. If Q is a cycle, then we may assume that $Q = (v_i, w, v_{i+1}, v_i)$ and hence by conditions 2(v) and (3) the vertex v_i is exterior to ψ. If Q is a path, then (v_i, w, v_{i+2}) is a section of Q and hence by conditions 2(v) and (3), the vertex v_{i+1} is exterior to ψ.

Thus for every vertex w not on P which is interior to ψ there exists a vertex v_j on P, where $2 \le j \le d$, which is exterior to ψ. Also it is clear that for any two distinct vertices not on P which are interior to ψ the corresponding vertices on P which are exterior to ψ are also distinct. Hence number of vertices interior to ψ is at most $d-1$ so that $t \ge p-(d-1) = p-d+1$. Hence $\eta_s(G) = q-p+t \ge q-d+1$.

Thus $\eta_s(G) = q-d+1$. □

Theorem 2.21. *For any graph G, $\eta_s(G) \ge \lceil \frac{\Delta}{2} \rceil$. Moreover, the following are equivalent.*

(i) $\eta_s(G) = \lceil \frac{\Delta}{2} \rceil$.

(ii) $\eta(G) = \lceil \frac{\Delta}{2} \rceil$.

(iii) G is homeomorphic to one of the graphs given in Figure 3.

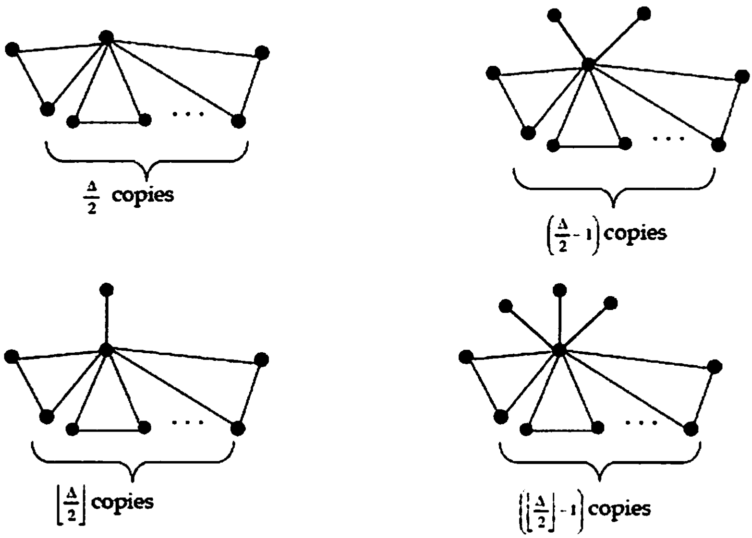

Figure 3

Proof. The inequality is obvious.

Suppose $\eta_s(G) = \left\lceil\frac{\Delta}{2}\right\rceil$. Let ψ be a minimum simple graphoidal cover of G. Let v be a vertex of G with *deg* $v = \Delta$. Then v lies on every member of ψ and except possibly for at most one member, all other members of ψ cover two edges incident with v. Also, among the members of ψ which cover two edges incident with v except possibly for at most one member all other members are cycles. Thus G is homeomorphic to one of the graphs given in Figure 3. Hence (i) and (iii) are equivalent. Equivalence of (ii) and (iii) can be proved by a similar argument. □

References

[1] B. D. Acharya and E. Sampathkumar, Graphoidal covers and graphoidal covering number of a graph, *Indian J. pure appl. Math.*, **18**(10)(1987), 882 - 890.

[2] B.D. Acharya, Further results on the graphoidal covering number of a graph, *Graph Theory Newsletter,* **17**(4)(1988), 1.

[3] S. Arumugam, B. D. Acharya and E. Sampathkumar, Graphoidal covers of a graph - A creative review, *Proceedings of the National workshop on Graph Theory and its Applications*, Manonmaniam Sundaranar University, Tirunelveli, Eds. S. Arumugam, B. D. Acharya and E. Sampathkumar, Tata McGraw Hill, (1996), 1 - 28.

[4] F. Harary, Graph Theory, Addison-Wesley, Reading, Mass, 1972.

[5] C. Pakkiam and S. Arumugam, On the graphoidal covering number of a graph, *Indian J. pure appl. Math.*, **20**(4) (1989), 330 - 333.

[6] C. Pakkiam and S. Arumugam, The graphoidal covering number of unicyclic graphs, *Indian J. pure appl. Math.*, **23**(2)(1992), 141 - 143.