Supertough 5-Regular Graphs

Lynne L. Doty
Lynne.Doty@marist.edu
Marist College, Poughkeepsie, NY 12601

Kevin K. Ferland
kferland@bloomu.edu
Bloomsburg University, Bloomsburg, PA 17815

Abstract

The computation of the maximum toughness among graphs with n vertices and m edges is considered for $\lceil 5n/2 \rceil \leq m < 3n$. We show that there are only finitely many cases in which the toughness value 5/2 cannot be achieved. This is in stark contrast with the known result that there is a 3/2-tough graph on n vertices and $\lceil 3n/2 \rceil$ edges if and only if $n \equiv 0, 5 \mod 6$. However, constructions related to those used in the cubic case are also employed here. Our constructions additionally provide an infinite family of graphs that are supertough and not $K_{1,3}$-free.

Keywords: toughness, maximum connectivity, maximum toughness, inflations

1 Introduction

A graph $G = (V, E)$ is an (n, m)-graph if $|V| = n$ and $|E| = m$. Given a set of vertices U in a graph G, the subgraph of G induced by U shall be denoted by $\langle U \rangle$. The toughness [1] of a non-complete graph $G = (V, E)$ is

$$\tau(G) = \min\{\frac{|S|}{\omega(G \setminus S)} : S \subseteq V \text{ and } \omega(G \setminus S) > 1\},$$

where $\omega(G \setminus S)$ is the number of components in $\langle V \setminus S \rangle$. A graph G is said to be t-tough if $\tau(G) \geq t$. A τ-set for G is a separating set S for which $\tau(G) = |S|/\omega(G \setminus S)$. Among all (n, m)-graphs, the maximum toughness [1, 2, 5, 6] is denoted by $T_n(m)$. An (n, m)-graph G is said to be maximally tough if $\tau(G) = T_n(m)$ and supertough if $\tau(G) = (1/2)\lfloor 2m/n \rfloor$. All standard notation and terminology not presented here can be found in [9].

The farthest reaching known result on the values of $T_n(\lceil 5n/2 \rceil)$ is the following.

Theorem 1.1 ([1, 2]). *If n is divisible by 10 or 12, then* $T_n(\frac{5n}{2}) = \frac{5}{2}$.

It is also known that $T_n(\lceil 5n/2 \rceil)$ is not always 5/2. In [4], we show that $T_{11}(28) = T_{11}(29) = 7/3$. That $T_{11}(30) = 5/2$ is established in [5]. The contribution of this paper to the problem of computing $T_n(\lceil 5n/2 \rceil)$ is the following theorem.

Theorem 1.2. *For $n \geq 6$ with $n \notin \{11, 17, 18, 19, 21, 33\}$,*

$$T_n(\lceil \tfrac{5n}{2} \rceil) = \tfrac{5}{2}.$$

We prove Theorem 1.2 by constructing a family of 5/2-tough $(n, \lceil 5n/2 \rceil)$-graphs $G_5(n)$ that, in fact, has additional interesting properties. The search for graphs which are maximally tough or supertough has typically focused on the presence of $K_{1,3}$-centers. These are vertices with 3 non-adjacent neighbors. Graphs without $K_{1,3}$-centers are said to be $K_{1,3}$-free. Matthews and Sumner [8] show that, if a graph is $K_{1,3}$-free, then its toughness is half of its connectivity. In [4], we provided an example of a (14, 35)-graph refuting a conjecture of Goddard and Swart [6] that regular supertough graphs must be $K_{1,3}$-free. Our constructions here generalize that example and provide an infinite family of 5-regular 5/2-tough graphs that contain multiple $K_{1,3}$-centers.

2 Constructing 5/2-Tough Graphs

Let $H_3(2)$ be the graph on two vertices with three parallel edges. For even $p \geq 4$, the cubic Harary graph $H_3(p)$ is constructed from the p-cycle $C_p = \langle h_1, \ldots, h_p \rangle$ by adding edges, called diameter edges, between the antipodes. The inflation of a graph G is the graph G^* whose vertices are all ordered pairs (v, e), where e is an edge of G and v is an endpoint of e, such that two vertices of G^* are adjacent if and only if they differ in exactly one coordinate. The graphs $H_3(4)$ and $H_3(4)^*$ are pictured in Figure 1. Very simply, each vertex h_i in $H_3(p)$ is

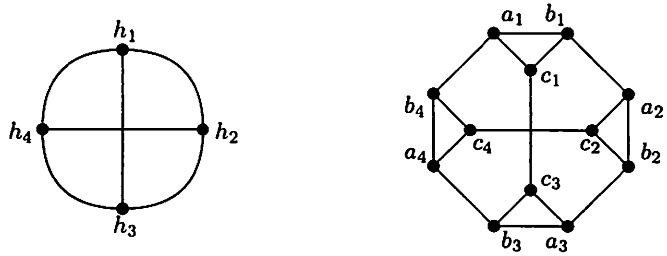

Figure 1: Harary Graph with its Inflation

inflated to a triangle $\langle a_i, b_i, c_i \rangle$ in $H_3(p)^*$. Chvátal [1] shows that the toughness of the inflation G^* of G is half of the edge-connectivity of G. Since, for even $p \geq 2$, $H_3(p)$ has edge-connectivity three [7], it follows that the cubic graph $H_3(p)^*$ has toughness 3/2. The graphs $H_3(p)^*$ play a central role in [3] and will also do so here.

Defining basic $G_5(n)$. If $n = 12d$ for some positive integer d, then we construct the $(n, 5n/2)$-graph $G_5(n)$ from $H_3(2d)^* \times K_2$ by adding more edges. First, regard each of the two copies of $H_3(2d)^*$ as a *level* of $G_5(n)$. In each level, subscripts shall be taken mod $2d$. For one level, let $\langle a_i, b_i, c_i \rangle$ be the triangle corresponding to the vertex h_i from $H_3(2d)$. We assume that $\{b_i, a_{i+1}\}$ and $\{c_i, c_{i+d}\}$ are also edges. For the other level, let $\langle x_i, y_i, z_i \rangle$ be the analogous triangle and assume that $\{y_i, x_{i+1}\}$ and $\{z_i, z_{i+d}\}$ are also edges. Moreover, the edges $\{a_i, x_i\}$, $\{b_i, y_i\}$, and $\{c_i, z_i\}$ join the levels together.

The edges that we add to $H_3(2d)^* \times K_2$ to obtain $G_5(n)$ are those of the form $\{b_i, x_{i+1}\}$, $\{y_i, a_{i+1}\}$, $\{c_i, z_{i+d}\}$, and $\{z_i, c_{i+d}\}$. In sum, we say that $G_5(n)$ is obtained from $H_3(2d)$ by inflating each vertex of $H_3(2d)$ to a prism $K_3 \times K_2$ in $G_5(n)$ and inflating each edge in $H_3(2d)$ to a K_4 subgraph in $G_5(n)$. The cycle C_{2d} in $H_3(2d)$ thus induces a cyclic ordering on the prisms in $G_5(n)$. For each diameter edge in $H_3(2d)$, the four corresponding edges joining antipodal prisms in $G_5(n)$ are also called *diameter edges*. The graph $G_5(24)$ with our vertex labeling conventions is pictured in Figure 2.

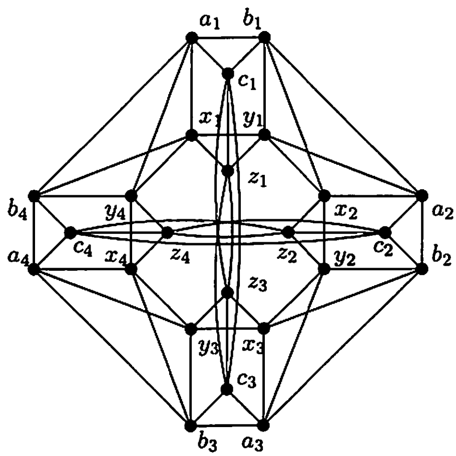

Figure 2: Maximally Tough (24, 60)-Graph $G_5(24)$

For n divisible by 12, since each level of $G_5(n)$ is the 3-connected graph $H_3(n/6)^*$, it is easy to argue that $G_5(n)$ is 5-connected. Since $G_5(n)$ is $K_{1,3}$-free, it follows that $G_5(n)$ is 5/2-tough. These graphs thus provide an alternate construction yielding part of Theorem 1.1. The utility of our construction for yielding further results here comes out of the prominent placement of prisms $K_3 \times K_2$ within it. As we shall see, each prism can be replaced by certain other special subgraphs, and the resulting graph is 5/2-tough. The three special subgraphs we employ are pictured in Figure 3 together with vertex labeling conventions that reflect how these subgraphs can be interchanged. Of course, P is the prism $K_3 \times K_2$. In the graphs W and X, note that the additional vertices w, u, v are $K_{1,3}$-centers.

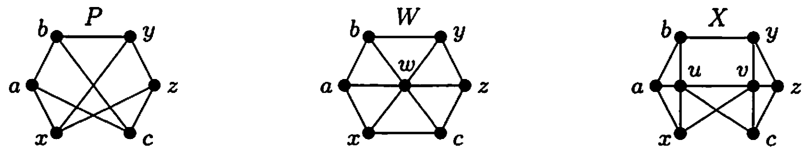

Figure 3: Key Plug-Ins P, W, and X

Definition 2.1. Given a subset $\mathcal{H} \subseteq \{P, W, X\}$, an *$\mathcal{H}$-wheel* is a graph obtained from some graph $G_5(12d)$ by replacing each subgraph of type P by one of type H for some $H \in \mathcal{H}$.

An example of a $\{P, W, X\}$-wheel obtained from $G_5(24)$ by replacing one prism by a copy of W and another by a copy of X is shown in Figure 4.

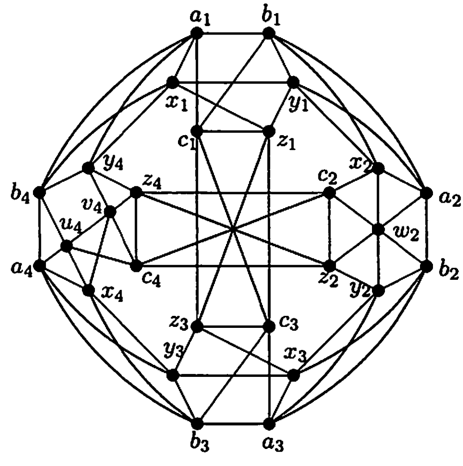

Figure 4: Maximally Tough $(27, 68)$-Graph $G_5(27)$

Theorem 2.2. *For any $\{P, W, X\}$-wheel G, we have $\tau(G) = \frac{5}{2}$.*

The proof of Theorem 2.2 is given in Section 3. Here, we first use Theorem 2.2 to extend our construction of $G_5(n)$ and prove Theorem 1.2.

Defining general $G_5(n)$. For n_0 divisible by 12, the graph $G_5(n_0)$ contains $p = n_0/6$ subgraphs of type P. By using at most one copy of W and up to p copies of X, we construct, for each $n_0 \leq n \leq \max\{n_0+2p, n_0+11\}$, a $\{P, W, X\}$-wheel $G_5(n)$ on n vertices and $\lceil 5n/2 \rceil$ edges. Theorem 2.2 tells us that $G_5(n)$ is $5/2$-tough.

Since the graphs $G_5(n)$ are not defined for all n, we additionally define a relative that achieves toughness 5/2 in some of the missing cases, namely $n = 22, 23, 34, 35$.

Filling gaps with $G_5'(n)$. For n_0 divisible by 12, assume that the prisms $P_i = \langle a_i, b_i, c_i, x_i, y_i, z_i \rangle$ in $G_5(n_0)$ are ordered cyclically. We construct $G_5'(22)$ from $G_5(24)$ by identifying c_2 with c_4 and z_2 with z_4. Then, $G_5'(23)$ is obtained from $G_5'(22)$ by replacing P_1 by a copy of W. Similarly, we construct $G_5'(34)$ from $G_5(36)$ by identifying c_2 with c_5 and z_2 with z_5, and we obtain $G_5'(35)$ by replacing P_1 with W. That the graphs $G_5'(22)$, $G_5'(23)$, $G_5'(34)$, and $G_5'(35)$ are 5/2-tough has been verified by computer.

Proof of Theorem 1.2. The cases in which $n \leq 10$ are completed in [5]. The case in which $n = 20$ is handled by Theorem 1.1, and the cases in which $n = 22, 23, 34, 35$ are handled by the graphs $G_5'(n)$. The remaining cases are managed by our graphs $G_5(n)$ and Theorem 2.2. □

Note that one quarter of the vertices in an $\{X\}$-wheel are $K_{1,3}$-centers. This provides an infinite family of regular supertough graphs rich in $K_{1,3}$-centers.

3 Proof of Theorem 2.2

Since a $\{P, W, X\}$-wheel G has minimum degree 5, it follows that $\tau(G) \leq 5/2$. Hence, it suffices to establish that $\tau(G) \geq 5/2$.

Lemma 3.1. *Let G be a $\{P, W, X\}$-wheel. Then, there is a $\{P, W\}$-wheel G' such that $\tau(G) \geq \tau(G')$.*

Proof. Suppose G has X as a subgraph, and let G' be the graph obtained from G by replacing X by W. Since we can repeat this process until no copies of X remain, it suffices to show that $\tau(G) \geq \tau(G')$. Let S be a τ-set for G. We shall form a disconnecting set S' for G' such that

$$\tau(G) = \frac{|S|}{\omega(G \setminus S)} \geq \frac{|S'|}{\omega(G' \setminus S')}.$$

If $u, v \notin S$, then let $S' = S$. So suppose exactly one of u or v is in S, say $u \in S$ and $v \notin S$. Since $v \notin S$, it follows that x and c are not in different components of $G \setminus S$. Thus, let $S' = (S \setminus \{u\}) \cup \{w\}$. It now remains to assume that $u, v \in S$.

Case 1: x or c is in S.
Let $S' = (S \setminus \{u, v\}) \cup \{w\}$.

Case 2: x or c is not an isolated vertex component, say x is not.
Let $S' = (S \setminus \{u, v\}) \cup \{w, x\}$.

Case 3: x or c are both isolated vertex components.
Since a and z must each separate at least two components, $b, y \notin S$. Let $S_1 = (S \setminus \{a\}) \cup \{b\}$. So $|S_1| = |S|$ and $\omega(G \setminus S_1) = \omega(G \setminus S)$. Using the τ-set

S_1, we see that x is not an isolated vertex component. So we can apply case 2 to S_1 to get S'. □

Definition 3.2. Let G be a $\{P, W\}$-wheel.

(a) Given a subgraph of type W in G, say $W = \langle a, b, c, x, y, z, w \rangle$ with neighboring K_4 subgraphs $\langle a, x, b', y' \rangle$, $\langle b, y, a', x' \rangle$, and $\langle c, z, c', z' \rangle$, we define $R_W = \{w, x, b, z, b', y', a', x', c', z'\}$ and $R'_W = \{w, a, y, c, b', y', a', x', c', z'\}$.

(b) A disconnecting set S for a $\{P, W\}$-wheel G is said to be *W-reduced* if each vertex of S is adjacent to (separates) at least two components of $G \setminus S$ and, for each subgraph of type W in G, either $R_W \subseteq S$ or $R'_W \subseteq S$.

Note that, for the graph G in Figure 4, R_W provides a τ-set for G.

Lemma 3.3. *Let G be a $\{P, W\}$-wheel. Then, there is a $\{P, W\}$-wheel G' and a W-reduced disconnecting set S' for G' such that $\tau(G) \geq |S'|/\omega(G' \setminus S')$.*

Proof. Let S be a τ-set for G. Suppose there is a subgraph of type W in G such that $R_W \not\subseteq S$ and $R'_W \not\subseteq S$. Let G' be the graph obtained from G by replacing W by a copy of P. We shall define a disconnecting set S' for G' such that

$$|S|/\omega(G \setminus S) \geq |S'|/\omega(G' \setminus S')$$

and each vertex of S' separates at least two components of $G \setminus S'$.

If $w \notin S$, then let $S' = S$. If $w \in S$, then we consider three cases requiring different modifications to S.

Case 1: x and c are in S.
Let $S' = S \setminus \{w\}$.

Case 2: x and c are not in S.
Since w must separate two components of $G \setminus S$, one of b or y is not in S. Without loss of generality, say $b \notin S$, so $a \in S$. If $y \notin S$, and hence $z \in S$, then let $S' = (S \setminus \{w, z\}) \cup \{c, y\}$. If $y \in S$, then let $S' = (S \setminus \{w\}) \cup \{c\}$.

Case 3: Exactly one of x or c is not in S, say x.
If x is not an isolated vertex component of $G \setminus S$, then let $S' = (S \setminus \{w\}) \cup \{x\}$. So we may assume that x is an isolated vertex component, and hence $a \in S$. Since a must separate two components, $b \notin S$. If b is not an isolated vertex component, then $S_1 = (S \setminus \{a\}) \cup \{b\}$ leaves x not isolated, as handled above. So we may assume that b is an isolated vertex component, and hence $y \in S$. Since y must separate two components, $z \notin S$. Of course, since $R'_W \not\subseteq S$, it follows that z is not an isolated vertex in $G \setminus S$. Thus $S_1 = (S \setminus \{y\}) \cup \{z\}$ leaves b not isolated, as handled above.

We can repeat this process of removing subgraphs of type W as necessary to obtain a $\{P, W\}$-wheel G' and a disconnecting set S' such that each remaining subgraph of type W has either $R_W \subseteq S$ or $R'_W \subseteq S$. Moreover, if necessary, we can further remove any vertices of S' outside of a subset of the form R_W or R'_W that do not separate two components. □

Definition 3.4. Let G be a $\{P, W\}$-wheel, and let S be a W-reduced disconnecting set for G.

(a) Let $W_1, \ldots, W_k$ be the subgraphs of type W, ordered cyclically around G. A *W-segment* of G is a subgraph M of G such that, for some i, the vertices of M are those of the prisms strictly between W_i and W_{i+1} (in the cyclic ordering) and the edges of M are those induced by its vertices but excluding any diameter edges. We regard $W_{k+1} = W_1$.

(b) We say that S is *W-normalized* if every component of $G \setminus S$ that intersects a W-segment in just one vertex is an isolated vertex component.

Lemma 3.5. *Let G be a $\{P, W\}$-wheel, and let S be a W-reduced disconnecting set for G. Let C be a component of $G \setminus S$ that intersects a W-segment M in at least two vertices. Then,*

(a) $|N(C) \cap S \cap M| \geq 4$.

(b) if $C \subseteq M$, *then* $|N(C) \cap S| \geq 6$.

Proof. Let d be the integer such that, if each copy of W in G is replaced by a copy of P, then the resulting graph is $G_5(12d)$. Let $P_1, \ldots, P_q$ be the prisms in M, listed according to the cyclic ordering. Say $P_i = \langle a_i, b_i, c_i, x_i, y_i, z_i \rangle$, for each i. Since M is a W-segment and S is W-reduced, we must have $a_1, x_1, b_q, y_q \in S$ and $q \geq 1$. Vertices known to be in S are circled in Figure 5. Note that there

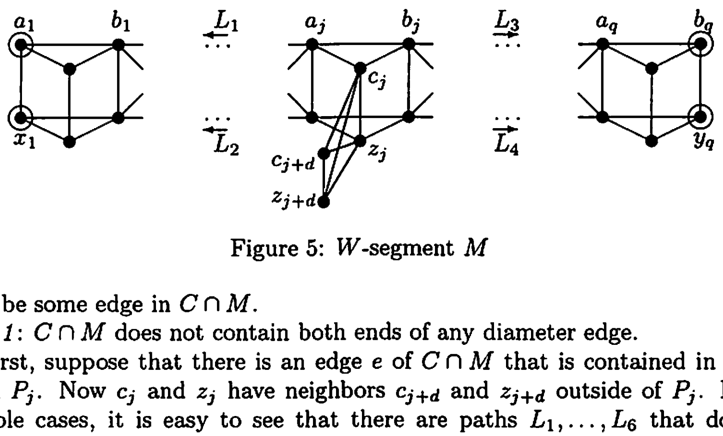

Figure 5: W-segment M

must be some edge in $C \cap M$.

Case 1: $C \cap M$ does not contain both ends of any diameter edge.

First, suppose that there is an edge e of $C \cap M$ that is contained in some prism P_j. Now c_j and z_j have neighbors c_{j+d} and z_{j+d} outside of P_j. In all possible cases, it is easy to see that there are paths $L_1, \ldots, L_6$ that do not intersect outside of e such that L_1 is a path in M from e to a_1, L_2 is a path in M from e to x_1, L_3 is a path in M from e to b_q, L_4 is a path in M from e to y_q, L_5 is a path in $\langle P_j, c_{j+d}, z_{j+d} \rangle$ from e to c_{j+d}, and L_6 is a path in $\langle P_j, c_{j+d}, z_{j+d} \rangle$ from e to z_{j+d}. The assertion in part (a) now follows, since each of the paths $L_1, \ldots, L_4$ must have some initial encounter with S. In the case of part (b), L_5 and L_6 must add two more distinct encounters with S.

Second, suppose that no edge of $C \cap M$ is contained in a prism. Hence, C must be a single edge in a K_4 subgraph. It is thus easy to see that C has 6 neighbors in $S \cap M$.

Case 2: $C \cap M$ contains both ends of some diameter edge e.

Say e has one endpoint f_j in prism P_j with $j < d$ and the other endpoint g_{j+d} in P_{j+d}. As in case 1, we encounter the asserted vertices of S along paths $L_1, \ldots, L_6$. However, in this case, we make the following adjustments, as

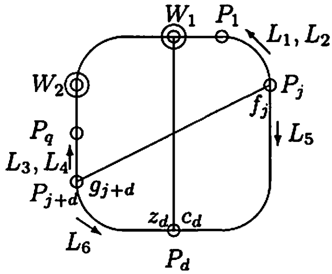

Figure 6: Viewing $\{P, W\}$-wheel G from a Distance

reflected in Figure 6: L_1 is a path in M from f_j to a_1, L_2 is a path in M from f_j to x_1, L_3 is a path in M from g_{j+d} to b_q, L_4 is a path in M from g_{j+d} to y_q, L_5 is a path in M from f_j to c_d, and L_6 is a path in M from g_{j+d} to z_d. □

Lemma 3.6. *Let G be a $\{P, W\}$-wheel, and let S be a W-reduced disconnecting set for G. Then, there is a W-normalized disconnecting set S' for G such that $|S'|/\omega(G' \setminus S') = |S|/\omega(G \setminus S)$.*

Proof. Suppose C is a component of $G \setminus S$ that is not an isolated vertex but intersects a segment in exactly one vertex. Without loss of generality, say that vertex is c_i. So $z_i \in S$. Since z_i must separate two components, without loss of generality, say that y_i is in a component of $G \setminus S$ different from C. Hence, we must have $b_i \in S$. Let $S' = (S \setminus \{b_i\}) \cup \{c_i\}$, and observe that $|S'| = |S|$ and $\omega(G \setminus S') = \omega(G \setminus S)$. By repeating this process as necessary, we can obtain a W-normalized disconnecting set. □

Lemma 3.7. *Let G be a $\{P, W\}$-wheel, and let S be a W-normalized disconnecting set for G. Then, for each W-segment, there is some component of $G \setminus S$ that intersects that W-segment in at least two vertices.*

Proof. Suppose to the contrary that some W-segment M contains only isolated vertex components of $G \setminus S$. As in Lemma 3.5 and as shown in Figure 5, say M consists of the prisms $P_1, \ldots, P_q$ and $a_1, x_1, b_q, y_q \in S$. Since each vertex of S separates two components, no triangle of any prism can be entirely contained in S. It follows that one of the vertices b_1 or y_1 must form an isolated vertex component of $G \setminus S$, forcing $a_2, x_2 \in S$. Repeating this argument along M, we get $a_q, x_q \in S$. That $b_q, y_q \in S$ contradicts the fact that one of b_q or y_q must now be in $G \setminus S$. □

Proof of Theorem 2.2. By Lemmas 3.1, 3.3, and 3.6, it suffices to consider a $\{P, W\}$-wheel G and a W-normalized disconnecting set S for G and to show that $|S|/\omega(G\setminus S) \geq 5/2$. Let k be the number of subgraphs of G of type W. Since S is W-reduced, each subgraph of type W contains three isolated vertex components of $G\setminus S$. Let j be the number of remaining isolated vertex components of $G\setminus S$. Let r be the number of components of $G\setminus S$ that are not isolated vertices and are completely contained within a W-segment. Let $C_1,\ldots,C_t$ be the remaining components of $G\setminus S$ that are not isolated vertices. For each $1\leq i\leq t$, let n_i be the number of W-segments that intersect C_i.

If, in G, we contract each component of $G\setminus S$ to a point and delete all edges with both endpoints in S, then we obtain a bipartite graph in which vertices in S are only adjacent to components of $G\setminus S$. Let e be the number of edges in this bipartite graph. Since the $K_{1,3}$-centers of G are precisely the centers of the subgraphs of type W, there are k vertices of S adjacent to three components of $G\setminus S$. Since each vertex of S must separate at least two components of $G\setminus S$, the remaining vertices of S must be adjacent to two components of $G\setminus S$. Thus,

$$2|S| + k = e. \tag{3.1}$$

Each of the $j+3k$ isolated vertex components of $G\setminus S$ is adjacent to five vertices of S. By Lemma 3.5(b), each component of $G\setminus S$ that is not an isolated vertex and is contained in some W-segment is adjacent to at least six vertices of S. By Lemma 3.5(a), for each $1\leq i\leq t$, $|N(C_i)\cap S|\geq 4n_i$. Hence,

$$e \geq 4\Big(\sum_{i=1}^{t} n_i\Big) + 6r + 5(j+3k). \tag{3.2}$$

From (3.1) and (3.2) it follows that

$$|S| \geq 2\Big(\sum_{i=1}^{t} n_i\Big) + 3r + \tfrac{5}{2}j + 7k.$$

Since each subgraph of G of type W contains three components of $G\setminus S$, we have $\omega(G\setminus S) = t + r + j + 3k$. For each $1\leq i\leq t$, by definition, $n_i\geq 2$. So,

$$\omega(G\setminus S) \leq \tfrac{1}{2}\Big(\sum_{i=1}^{t} n_i\Big) + r + j + 3k.$$

We conclude that

$$\begin{aligned}\frac{|S|}{\omega(G\setminus S)} &\geq \frac{2(\sum_{i=1}^{t} n_i) + 3r + \frac{5}{2}j + 7k}{\frac{1}{2}(\sum_{i=1}^{t} n_i) + r + j + 3k}\\ &= \frac{5}{2} + \frac{\frac{3}{4}(\sum_{i=1}^{t} n_i) + \frac{1}{2}(r-k)}{\frac{1}{2}(\sum_{i=1}^{t} n_i) + r + j + 3k}\end{aligned} \tag{3.3}$$

Since the number of W-segments in G is k, it follows from Lemma 3.7 that $r + \sum_{i=1}^{t} n_i \geq k$, and hence

$$\tfrac{3}{4}\Big(\sum_{i=1}^{t} n_i\Big) \geq \tfrac{1}{2}\Big(\sum_{i=1}^{t} n_i\Big) \geq \tfrac{1}{2}(k-r).$$

Therefore, the right-hand side of (3.3) is at least 5/2. □

4 Loose Ends

In light of Theorem 1.2 and the known results for $n = 11$, it remains to consider the cases in which $n \in \{17, 18, 19, 21, 33\}$. In this final section, we make some progress on these rogue cases. All toughness values asserted here have been verified by computer. Most of our results relate to the graphs $A_5(17)$, $B_5(18)$, $D_5(20)$, and $Q_5(33)$ defined by their pictures in Figures 7 and 8.

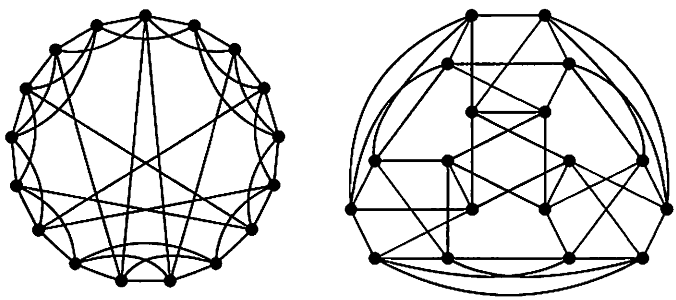

Figure 7: The (17, 43)-Graph $A_5(17)$ and the (18, 45)-Graph $B_5(18)$

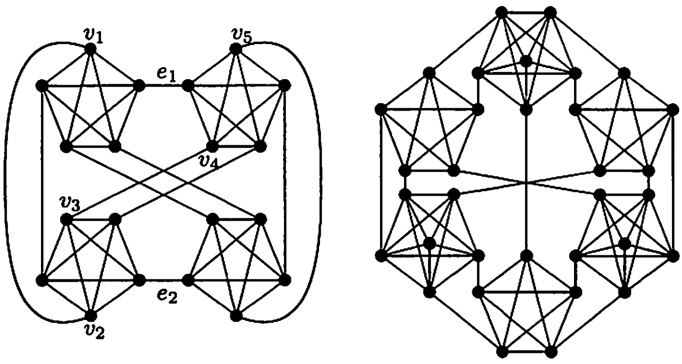

Figure 8: The (20, 50)-Graph $D_5(20)$ and the (33, 90)-Graph $Q_5(33)$

Since $\tau(A_5(17)) = 12/5$, and the (17, 46)-graph obtained from $G_5(16)$ by joining a new vertex v to the vertices in the set $\{a_1, x_1, u_1, b_2, y_2, z_2\}$ has toughness 5/2,

$$\frac{12}{5} \le T_{17}(43) \le T_{17}(46) = \frac{5}{2}.$$

Since $\tau(B_5(18)) = 12/5$, and the $(18,48)$-graph obtained from $D_5(20)$ by contracting the edges e_1 and e_2 has toughness $5/2$,

$$\frac{12}{5} \leq T_{18}(45) \leq T_{18}(48) = \frac{5}{2}.$$

The $(19,48)$-graph obtained from $B_5(18)$ by replacing one of the subgraphs of type P by one of type W has toughness $12/5$. The $(19,49)$-graph obtained from $D_5(20)$ by contracting the edge e_1 has toughness $5/2$. Hence,

$$\frac{12}{5} \leq T_{19}(48) \leq T_{19}(49) = \frac{5}{2}.$$

The $(21,53)$-graph obtained from $B_5(18)$ by replacing one of the subgraphs of type P by one of type W and another by one of type X has toughness $7/3$. The $(21,54)$-graph obtained from $D_5(20)$ by removing the edge $\{v_1, v_2\}$ and joining a new vertex v to the vertices in the set $\{v_1, v_2, v_3, v_4, v_5\}$ has toughness $12/5$. The $(21,55)$-graph obtained from $D_5(20)$ by adding a new vertex and joining it to each vertex of one of the K_5 subgraphs has toughness $5/2$. Hence,

$$\frac{7}{3} \leq T_{21}(53) \quad \text{and} \quad \frac{12}{5} \leq T_{21}(54) \leq T_{21}(55) = \frac{5}{2}.$$

Since the $(33,83)$-graph obtained from $G'_5(35)$ by identifying c_3 with c_6 and z_3 with z_6 has toughness $22/9$, and $\tau(Q_5(33)) = 5/2$,

$$\frac{22}{9} \leq T_{33}(83) \leq T_{33}(90) = \frac{5}{2}.$$

For $n \in \{17, 18, 19, 21, 33\}$ with $\lceil 5n/2 \rceil \leq m < 3n$, the inequalities above thus leave a number of open problems in the computation of $T_n(m)$.

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