Total vertex irregular labelings of wheels, fans, suns and friendship graphs

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Abstract. A *total vertex irregular labeling* of a graph G with v vertices and e edges is an assignment of integer labels to both vertices and edges so that the weights calculated at vertices are distinct. The *total vertex irregularity strength* of G, denoted by $tvs(G)$, is the minimum value of the largest label over all such irregular assignments. In this paper, we consider the total vertex irregular labelings of wheels W_n, fans F_n, suns S_n and friendship graphs f_n. We show that $tvs(W_n) = \lceil \frac{n+3}{4} \rceil$ for $n \geq 3$, $tvs(F_n) = \lceil \frac{n+2}{4} \rceil$ for $n \geq 3$, $tvs(S_n) = \lceil \frac{n+1}{2} \rceil$ for $n \geq 3$, and $tvs(f_n) = \lceil \frac{2n+2}{3} \rceil$ for all n.

1 Introduction

Throughout this paper all graphs are finite, simple, undirected, and connected. A *total vertex irregular labeling* on a graph G with n vertices and m edges is an assignment of integer labels to both vertices and edges so that the weights calculated at vertices are distinct. The *weight* of a vertex v in G is defined as the sum of the label of v and the labels of all the edges incident with v, that is,

$$wt(v) = \lambda(v) + \sum_{uv \in E} \lambda(uv)$$

The notion of the total vertex irregular labeling was introduced by Bača, et al.[1]. The *total vertex irregularity strength* of G, denoted by $tvs(G)$, is the minimum value of the largest label over all such irregular assignments.

Bača et al.[1] proved that for a tree T with n pendant vertices and no vertices of degree 2, $\lceil \frac{n+1}{2} \rceil \leq tvs(T) \leq n$. In the same paper, Bača et al.[1] gave the lower bound and upper bound on total vertex irregularity strength of any graph with minimum degree δ and maximum degree Δ, that is

$$\left\lceil \frac{|V| + \delta}{\Delta + 1} \right\rceil \leq tvs(G) \leq |V| + \Delta - 2\delta - 1.$$

If G is $r-$regular, then obviously, $\left\lceil \frac{|V|+r}{r+1} \right\rceil \leq tvs(G) \leq |V| - r - 1$. For cycles C_n, the total vertex irregularity strength of cycles C_n equals to the lower bound, that is $tvs(C_n) = \lceil \frac{n+2}{3} \rceil$. Because $C_4 \simeq K_{2,2}$, we have $tvs(K_{2,2}) = 2$. Moreover, if G is a regular hamiltonian graph then $tvs(G) \leq \left\lceil \frac{|V|+2}{3} \right\rceil$. Bača et al.[1] also proved that $tvs(K_{1,n}) = \lceil \frac{n+1}{2} \rceil$, $tvs(K_n) = 2$ for all $n \geq 2$ and for prisms D_n, $tvs(D_n) = \lceil \frac{2n+3}{4} \rceil$ for $n \geq 3$.

Wijaya et al. [2] determined the total vertex irregularity strength of complete bipartite graphs, that is for $n \geq 3$, $tvs(K_{2,n}) = \lceil \frac{n+2}{3} \rceil$, $tvs(K_{n,n}) = 3$, $tvs(K_{n,n+1}) = 3$, for $n \geq 4$, $tvs(K_{n,n+2}) = 3$, and for all n, $tvs(K_{n,an}) = \lceil \frac{n(a+1)}{n+1} \rceil$ for $a > 1$. Wijaya et al. [2] also gave the lower bound on $tvs(K_{m,n})$ for $m < n$, that is $tvs(K_{m,n}) \geq \max\{\lceil \frac{m+n}{m+1} \rceil, \lceil \frac{2m+n-1}{n} \rceil\}$.

In this paper we determine the total vertex irregularity strength of wheels, fans, suns, and friendship graphs.

2 Main Result

In this section, we present the total vertex irregularity strength of wheels, fans, suns, and friendship graphs.

A wheel W_n contains a cycle on n vertices and a vertex adjacent to all vertices on the cycle. A fan F_n consists of a path on n vertices and a vertex adjacent to every vertex on the path. is a graph obtained by joining all vertices of path P_n to a further vertex called the *center*. A sun S_n is a cycle on n vertices with an edge terminating in a vertex of degree 1 attached to each vertex on the cycle. A friendship graph f_n is obtained by identifying a vertex from n K_3's. Note that if $f_1 \simeq K_3$.

Theorem 1. *The total vertex irregularity strength of wheel W_n satisfies $tvs(W_n) = \lceil \frac{n+3}{4} \rceil$, for $n \geq 3$.*

Proof. A wheel W_n has n vertices of degree 3 and one central vertex of degree n. The smallest weight of vertices of W_n must be 4. So, the largest weight of n vertices of degree 3 is at least $(n+3)$ and the weight of central vertex is at least $(n+4)$. As a result, the value of the largest label of one of vertices or edges of W_n is at least $\max\{\lceil \frac{n+3}{4} \rceil, \lceil \frac{n+4}{n+1} \rceil\} = \lceil \frac{n+3}{4} \rceil$. Thus, $tvs(W_n) \geq \lceil \frac{n+3}{4} \rceil$.

To show that $tvs(W_n) \leq \lceil \frac{n+3}{4} \rceil$, let $V(W_n) = \{v, v_1, v_2, \cdots, v_n\}$ and $E(W_n) = \{vv_1, vv_2, \cdots, vv_n\} \cup \{v_1v_2, v_2v_3, \cdots, v_nv_1\}$. The vertex irregular labeling of W_n is as follows:

1. For $n = 3$.
   The sets of labels of vertices and edges of W_3 are $\lambda(V(W_3)) = \{2, 1, 1, 2\}$ and $\lambda(E(W_3)) = \{1, 2, 2\} \cup \{1, 1, 1\}$. The set of vertex-weights of W_3 is $wt(V(W_3)) = \{7, 4, 5, 6\}$.

2. For $n \neq 3$.
   There are 4 cases of the labeling of vertices and edges of W_n, namely:

(a) For $n \equiv 0(\mod 4)$.

$$\lambda(v) = \begin{cases} 2 & \text{for } n = 4, \\ 1 & \text{for } n \neq 4. \end{cases}$$

$$\lambda(v_i) = \begin{cases} 1 & \text{for } 1 \leq i \leq 3 \text{ and } i = n, \\ j \text{ with } j = 2, \cdots, \lfloor \frac{n+3}{4} \rfloor & \text{for } 2j \leq i \leq 2j+1 \text{ and} \\ & n+2-2j \leq i \leq n+3-2j; \end{cases}$$

$$\lambda(vv_i) = \begin{cases} 1 & \text{for } 1 \leq i \leq 2, \\ j \text{ with } j = 2, \cdots, \lceil \frac{n+3}{4} \rceil & \text{for } 2j-1 \leq i \leq 2j \text{ and} \\ & n+3-2j \leq i \leq n+4-2j; \end{cases}$$

$$\lambda(v_iv_{i+1}) = \begin{cases} 1 & \text{for } i = 1 \text{ and } i = n, \\ j \text{ with } j = 2, \cdots, \lceil \frac{n+3}{4} \rceil & \text{for } 2j-2 \leq i \leq 2j-1 \text{ and} \\ & n+2-2j \leq i \leq n+3-2j. \end{cases}$$

(b) For $n \equiv 1(\mod 4)$.

$\lambda(v) = 1$;

$$\lambda(v_i) = \begin{cases} 1 & \text{for } 1 \leq i \leq 3 \text{ and } i = n, \\ j \text{ with } j = 2, \cdots, \lceil \frac{n-1}{4} \rceil & \text{for } 2j \leq i \leq 2j+1 \text{ and} \\ & n+2-2j \leq i \leq n+3-2j, \\ \lceil \frac{n+3}{4} \rceil & \text{for } i = 2\lceil \frac{n+3}{4} \rceil; \end{cases}$$

$$\lambda(vv_i) = \begin{cases} 1 & \text{for } 1 \leq i \leq 2, \\ j \text{ with } j = 2, \cdots, \lceil \frac{n+3}{4} \rceil & \text{for } 2j-1 \leq i \leq 2j \text{ and} \\ & n+3-2j \leq i \leq n+4-2j; \end{cases}$$

$$\lambda(v_iv_{i+1}) = \begin{cases} 1 & \text{for } i = 1, \text{ and } i = n, \\ j \text{ with } j = 2, \cdots, \lceil \frac{n+3}{4} \rceil & \text{for } 2j-2 \leq i \leq 2j-1 \text{ and} \\ & n+2-2j \leq i \leq n+3-2j. \end{cases}$$

(c) For $n \equiv 2(\mod 4)$.

$\lambda(v) = 1$;

$$\lambda(v_i) = \begin{cases} 1 & \text{for } 1 \leq i \leq 3 \text{ and } i = n, \\ j \text{ with } j = 2, \cdots, \lfloor \frac{n+3}{4} \rfloor & \text{for } 2j \leq i \leq 2j+1 \text{ and} \\ & n+2-2j \leq i \leq n+3-2j; \end{cases}$$

$$\lambda(vv_i) = \begin{cases} 1 & \text{for } 1 \leq i \leq 2, \\ j \text{ with } j = 2, \cdots, \lfloor \frac{n-1}{4} \rfloor & \text{for } 2j-1 \leq i \leq 2j \text{ and} \\ & n+3-2j \leq i \leq n+4-2j, \\ \lfloor \frac{n+3}{4} \rfloor & \text{for } i = \frac{n}{2} \text{ and } \frac{n}{2}+2 \leq i \leq \frac{n}{2}+3, \\ \lceil \frac{n+3}{4} \rceil & \text{for } i = \frac{n}{2}+1; \end{cases}$$

$$\lambda(v_iv_{i+1}) = \begin{cases} 1 & \text{for } i = 1 \text{ and } i = n, \\ j \text{ with } j = 2, \cdots, \lfloor \frac{n+3}{4} \rfloor & \text{for } 2j-2 \le i \le 2j-1, \text{ and} \\ & n+2-2j \le i \le n+3-2j. \end{cases}$$

(d) For $n \equiv 3(\mod 4)$.
$\lambda(v) = 1$;

$$\lambda(v_i) = \begin{cases} 1 & \text{for } 1 \le i \le 3 \text{ and } i = n, \\ j \text{ with } j = 2, \cdots, \lfloor \frac{n+3}{4} \rfloor & \text{for } 2j \le i \le 2j+1, \text{ and} \\ & n+2-2j \le i \le n+3-2j; \end{cases}$$

$$\lambda(vv_i) = \begin{cases} 1 & \text{for } 1 \le i \le 2, \\ j \text{ with } j = 2, \cdots, \lfloor \frac{n+3}{4} \rfloor & \text{for } 2j-1 \le i \le 2j, \text{ and} \\ & n+3-2j \le i \le n+4-2j, \\ \lceil \frac{n+3}{4} \rceil & \text{for } i = \lceil \frac{n}{2} \rceil + 1; \end{cases}$$

$$\lambda(v_iv_{i+1}) = \begin{cases} 1 & \text{for } i = 1, \text{ and } i = n, \\ j \text{ with } j = 2, 3, \cdots, \lfloor \frac{n+3}{4} \rfloor & \text{for } 2j-2 \le i \le 2j-1, \text{ and} \\ & n+2-2j \le i \le n+3-2j, \\ \lceil \frac{n+3}{4} \rceil & \text{for } i = \lceil \frac{n}{2} \rceil. \end{cases}$$

Thus, the vertex-weights of W_n satisfy

$$wt(v_i) = \begin{cases} 4 & \text{for } i = 1, \\ 2i+1 & \text{for } i = 2, 3, \cdots, \lceil \frac{n+1}{2} \rceil, \\ 2(n+3-i) & \text{for } i = \lceil \frac{n+1}{2} \rceil + 1, \lceil \frac{n+1}{2} \rceil + 2, \cdots, n; \end{cases}$$

$$wt(v) = \begin{cases} \frac{1}{8}(n^2+8n+8) & \text{for } n \equiv 0(\mod 4), \\ \frac{1}{8}(n^2+8n+7) & \text{for } n \equiv 1 \text{ and } 3(\mod 4), \\ \frac{1}{8}(n^2+8n+12) & \text{for } n \equiv 2(\mod 4). \end{cases}$$

It is easy to see that the labeling is vertex irregular. Thus $tvs(W_n) = \lceil \frac{n+3}{4} \rceil$. ■

Theorem 2. *The total vertex irregularity strength of a fan F_n satisfies $tvs(F_n) = \lceil \frac{n+2}{4} \rceil$, for $n \ge 3$.*

Proof. A fan F_n has $(n-2)$ vertices of degree 3, two vertices of degree 2 and one vertex of degree n. The smallest weight of vertices of F_n must be 3. So, the largest weight of $(n-2)$ vertices of degree 3 is at least $(n+2)$ and the weight of the vertex of degree n is at least $(n+3)$. This implies that the largest label of one of vertices or edges F_n is at least $\max\{\lceil \frac{n+2}{4} \rceil, \lceil \frac{n+3}{n+1} \rceil\} = \lceil \frac{n+2}{4} \rceil$. Then $tvs(F_n) \ge \lceil \frac{n+2}{4} \rceil$.

To show that $tvs(F_n) \le \lceil \frac{n+2}{4} \rceil$, let $V(F_n) = \{v, v_1, v_2, \cdots, v_n\}$ and $E(F_n) = \{vv_1, vv_2, \cdots, vv_n\} \cup \{v_1v_2, v_2v_3, \cdots, v_{n-1}v_n\}$. The vertex irregular labeling of F_n is as follows:

1. For $3 \le n \le 5$.
   The set of labels of vertices and edges of F_3, F_4, and F_5 are as follows:
   $\lambda(V(F_3)) = \{1, 1, 1, 1\}$ and $\lambda(E(F_3)) = \{1, 2, 1\} \cup \{1, 2\}$.
   $\lambda(V(F_4)) = \{2, 1, 1, 2, 1\}$ and $\lambda(E(F_4)) = \{1, 1, 1, 1\} \cup \{1, 2, 2\}$.

$\lambda(V(F_5)) = \{2,1,1,1,1,1\}$ and $\lambda(E(F_5)) = \{1,1,2,1,1\} \cup \{1,2,2,2\}$. The sets of vertex-weights of F_3, F_4, and F_5 are $wt(V(F_3)) = \{5,3,6,4\}$, $wt(V(F_4)) = \{6,3,5,7,4\}$, $wt(V(F_5)) = \{8,3,5,7,6,4\}$.

2. For $n \geq 6$.
   The labeling of vertices and edges of F_n are as follows.
   (a) For $n \equiv 0 (\mod 4)$.
   $\lambda(v) = 1$;

$$\lambda(v_i) = \begin{cases} 1 & \text{for } i = 1, \\ j \text{ with } j = 1,2,\cdots,\lceil\frac{n-2}{4}\rceil & \text{for } 2j \leq i \leq 2j+1 \text{ and} \\ & n+1-2j \leq i \leq n+2-2j; \end{cases}$$

$$\lambda(vv_i) = \begin{cases} 1 & \text{for } 1 \leq i \leq 2 \text{ and} \\ & n-1 \leq i \leq n, \\ 2 & \text{for } 3 \leq i \leq 4 \text{ and } i = n-2, \\ j \text{ with } j = 3,\cdots,\lceil\frac{n-2}{4}\rceil & \text{for } 2j-1 \leq i \leq 2j \text{ and} \\ & n+2-2j \leq i \leq n+3-2j, \\ \lceil\frac{n+2}{4}\rceil & \text{for } i = 2\lceil\frac{n+2}{4}\rceil - 1; \end{cases}$$

$$\lambda(v_iv_{i+1}) = \begin{cases} 1 & \text{for } i = 1, \\ 2 & \text{for } i = n-1, \\ j \text{ with } j = 2,\cdots,\lceil\frac{n-2}{4}\rceil & \text{for } 2j-2 \leq i \leq 2j-1 \text{ and} \\ & n+1-2j \leq i \leq n+2-2j, \\ \lceil\frac{n+2}{4}\rceil & \text{for } i = 2\lceil\frac{n+2}{4}\rceil - 2. \end{cases}$$

   (b) For $n \equiv 1 (\mod 4)$.
   $\lambda(v) = 1$;

$$\lambda(v_i) = \begin{cases} 1 & \text{for } i = 1, \\ j \text{ with } j = 1,2,\cdots,\lceil\frac{n-2}{4}\rceil & \text{for } 2j \leq i \leq 2j+1 \text{ and} \\ & n+1-2j \leq i \leq n+2-2j; \end{cases}$$

$$\lambda(vv_i) = \begin{cases} 1 & \text{for } 1 \leq i \leq 2 \text{ and} \\ & n-1 \leq i \leq n, \\ 2 & \text{for } 3 \leq i \leq 4 \text{ and } i = n-2, \\ j \text{ with } j = 3,4,\cdots,\lceil\frac{n+2}{4}\rceil & \text{for } 2j-1 \leq i \leq 2j \text{ and} \\ & n+2-2j \leq i \leq n+3-2j; \end{cases}$$

$$\lambda(v_iv_{i+1}) = \begin{cases} 1 & \text{for } i = 1, \\ 2 & \text{for } i = n-1, \\ j \text{ with } j = 2,\cdots,\lceil\frac{n+2}{4}\rceil & \text{for } 2j-2 \leq i \leq 2j-1 \text{ and} \\ & n+1-2j \leq i \leq n+2-2j. \end{cases}$$

   (c) For $n \equiv 2 (\mod 4)$.
   $\lambda(v) = 1$;

$$\lambda(v_i) = \begin{cases} 1 & \text{for } i = 1, \\ j \text{ with } j = 1, 2, \cdots, \lceil \frac{n-2}{4} \rceil & \text{for } 2j \le i \le 2j+1 \quad \text{and} \\ & n+1-2j \le i \le n+2-2j \\ \lceil \frac{n+2}{4} \rceil & \text{for } i = 2\lceil \frac{n+2}{4} \rceil; \end{cases}$$

$$\lambda(vv_i) = \begin{cases} 1 & \text{for } 1 \le i \le 2 \text{ and} \\ & n-1 \le i \le n, \\ 2 & \text{for } 3 \le i \le 4 \text{ and } i = n-2, \\ j \text{ with } j = 3, 4, \cdots, \lceil \frac{n+2}{4} \rceil & \text{for } 2j-1 \le i \le 2j \text{ and} \\ & n+2-2j \le i \le n+3-2j; \end{cases}$$

$$\lambda(v_i v_{i+1}) = \begin{cases} 1 & \text{for } i = 1, \\ 2 & \text{for } i = n-1, \\ j \text{ with } j = 2, 3, \cdots, \lceil \frac{n+2}{4} \rceil & \text{for } 2j-2 \le i \le 2j-1 \text{ and} \\ & n+1-2j \le i \le n+2-2j. \end{cases}$$

(d) For $n \equiv 3(\mod 4)$.
$\lambda(v) = 1$;

$$\lambda(v_i) = \begin{cases} 1 & \text{for } i = 1, \\ j \text{ with } j = 1, 2, \cdots, \lceil \frac{n-2}{4} \rceil & \text{for } 2j \le i \le 2j+1 \text{ and} \\ & n+1-2j \le i \le n+2-2j; \end{cases}$$

$$\lambda(vv_i) = \begin{cases} 1 & \text{for } 1 \le i \le 2 \text{ and} \\ & n-1 \le i \le n, \\ 2 & \text{for } 3 \le i \le 4 \text{ and } i = n-2, \\ j \text{ with } j = 3, \cdots, \lceil \frac{n-6}{4} \rceil & \text{for } 2j-1 \le i \le 2j \text{ and} \\ & n+2-2j \le i \le n+3-2j, \\ \lceil \frac{n-2}{4} \rceil & \text{for } i = 2\lceil \frac{n+2}{4} \rceil - 3 \text{ and} \\ & 2\lceil \frac{n+2}{4} \rceil - 1 \le i \le 2\lceil \frac{n+2}{4} \rceil, \\ \lceil \frac{n+2}{4} \rceil & \text{for } i = 2\lceil \frac{n+2}{4} \rceil - 2; \end{cases}$$

$$\lambda(v_i v_{i+1}) = \begin{cases} 1 & \text{for } i = 1, \\ 2 & \text{for } i = n-1, \\ j \text{ with } j = 2, \cdots, \lceil \frac{n-2}{4} \rceil & \text{for } 2j-2 \le i \le 2j-1 \text{ and} \\ & n+1-2j \le i \le n+2-2j. \end{cases}$$

Thus, the vertex-weights of F_n are as follows.

$$wt(v_i) = \begin{cases} 3 & \text{for } i = 1, \\ 4 & \text{for } i = n, \\ 2i+1 & \text{for } i = 2, 3, \cdots, \lceil \frac{n}{2} \rceil, \\ 2(n+2-i) & \text{for } i = \lceil \frac{n}{2} \rceil + 1, \lceil \frac{n}{2} \rceil + 2, \cdots, n-1; \end{cases}$$

$$wt(v) = \begin{cases} \frac{1}{8}(n^2+6n) & \text{for } n \equiv 0 \text{ and } 2(\mod 4), \\ \frac{1}{8}(n^2+6n+1) & \text{for } n \equiv 1(\mod 4), \\ \frac{1}{8}(n^2+6n+5) & \text{for } n \equiv 3(\mod 4). \end{cases}$$

It is easy to see that the labeling is vertex irregular. Thus $tvs(F_n) = \lceil \frac{n+2}{4} \rceil$. ∎

Theorem 3. *The total vertex irregularity strength of a sun S_n satisfies $tvs(S_n) = \lceil \frac{n+1}{2} \rceil$, for $n \geq 3$.*

Proof. A sun S_n has n vertices u_i of degree 1 and n vertices v_i of degree 3. Not that the smallest weight of vertices of S_n must be 2. It follows that the largest weight of n vertices of degree 1 is at least $(n+1)$ and of n vertices of degree 3 is at least $(2n+1)$. As a consequence, at least one vertex u_i or one edge incident with u_i has label at least $\lceil \frac{n+1}{2} \rceil$. Moreover, at least one vertex v_i or one edge incident with v_i has label at least $\lceil \frac{2n+1}{4} \rceil$. Then $tvs(S_n) \geq \max\{\lceil \frac{n+1}{2} \rceil, \lceil \frac{2n+1}{4} \rceil\}$. Because of $\lceil \frac{n+1}{2} \rceil = \lceil \frac{2n+1}{4} \rceil$, then $tvs(S_n) \geq \lceil \frac{n+1}{2} \rceil$.

To show that $tvs(S_n) \leq \lceil \frac{n+1}{2} \rceil$, let $V(S_n) = \{u_1, u_2, \cdots, u_n\} \cup \{v_1, v_2, \cdots, v_n\}$ where $\deg(u_i) = 2$ and $\deg(v_i) = 3$ for all $i = 1, 2, \cdots, n$ and $E(S_n) = \{u_1v_1, u_2v_2, \cdots, u_nv_n\} \cup \{v_1v_2, v_2v_3, \cdots, v_nv_1\}$. The labeling of vertex u_i and edge u_iv_i of S_n for $i = 1, 2, \cdots, n$ is as follows:

$$\lambda(u_i) = \begin{cases} 1 & \text{for } i = 1, \\ i-1 & \text{for } i = 2, 3, \cdots, \lceil \frac{n+1}{2} \rceil, \\ n+2-i & \text{for } i = \lceil \frac{n+1}{2} \rceil + 1, \lceil \frac{n+1}{2} \rceil + 2, \cdots, n; \end{cases}$$

$$\lambda(u_iv_i) = \begin{cases} i & \text{for } i = 1, 2, \cdots, \lceil \frac{n+1}{2} \rceil, \\ n+2-i & \text{for } i = \lceil \frac{n+1}{2} \rceil + 1, \lceil \frac{n+1}{2} \rceil + 2, \cdots, n. \end{cases}$$

There are 6 cases of the labeling of vertex v_i and edge v_iv_{i+1} of S_n for $i = 1, 2, \cdots, n$, namely:

1. For $n \equiv 0 (\mod 6)$.

$$\lambda(v_i) = \begin{cases} \lfloor \frac{n+1}{3} \rfloor & \text{for } 1 \leq i \leq 3 \text{ and } i = n, \\ \lfloor \frac{n+1}{3} \rfloor + j \text{ with } j = 1, \cdots, \lfloor \frac{n-6}{6} \rfloor & \text{for } 3j+1 \leq i \leq 3j+3 \text{ and} \\ & n-3j \leq i \leq n+2-3j, \\ \lceil \frac{n}{2} \rceil & \text{for } \lceil \frac{n+1}{2} \rceil \leq i \leq \lceil \frac{n+3}{2} \rceil; \end{cases}$$

$$\lambda(v_iv_{i+1}) = \begin{cases} \lfloor \frac{n+1}{3} \rfloor & \text{for } i = 1, \\ \lfloor \frac{n+1}{3} \rfloor + j \text{ with } j = 1, \cdots, \lfloor \frac{n}{6} \rfloor & \text{for } 3j-1 \leq i \leq 3j+1 \text{ and} \\ & n+1-3j \leq i \leq n+3-3j. \end{cases}$$

2. For $n \equiv 1 (\mod 6)$.

$$\lambda(v_i) = \begin{cases} \lfloor \frac{n+1}{3} \rfloor & \text{for } 1 \leq i \leq 2, \\ \lfloor \frac{n+1}{3} \rfloor + j \text{ with } j = 1, \cdots, \lfloor \frac{n}{6} \rfloor & \text{for } 3j \leq i \leq 3j+2 \text{ and} \\ & n+1-3j \leq i \leq n+3-3j; \end{cases}$$

$$\lambda(v_iv_{i+1}) = \begin{cases} \lfloor \frac{n+1}{3} \rfloor + j \text{ with } j = 1, \cdots, \lfloor \frac{n}{6} \rfloor & \text{for } 3j-2 \leq i \leq 3j \text{ and} \\ & n+2-3j \leq i \leq n+4-3j, \\ \lceil \frac{n}{2} \rceil & \text{for } \lceil \frac{n}{2} \rceil \leq i \leq \lceil \frac{n+2}{2} \rceil. \end{cases}$$

3. For $n \equiv 2 \pmod 6$.

$$\lambda(v_i) = \begin{cases} \lfloor \frac{n+1}{3} \rfloor & \text{for } 1 \le i \le 4 \text{ and} \\ & n-1 \le i \le n, \\ \lfloor \frac{n+1}{3} \rfloor + j \text{ with } j = 1, \cdots, \lfloor \frac{n-6}{6} \rfloor & \text{for } 3j+2 \le i \le 3j+4 \text{ and} \\ & n-1-3j \le i \le n+1-3j, \\ \lceil \frac{n}{2} \rceil & \text{for } \lceil \frac{n+1}{2} \rceil \le i \le \lceil \frac{n+3}{2} \rceil; \end{cases}$$

$$\lambda(v_i v_{i+1}) = \begin{cases} \lfloor \frac{n+1}{3} \rfloor & \text{for } 1 \le i \le 2 \text{ and } i = n, \\ \lfloor \frac{n+1}{3} \rfloor + j \text{ with } j = 1, \cdots, \lfloor \frac{n}{6} \rfloor & \text{for } 3j \le i \le 3j+2 \text{ and} \\ & n-3j \le i \le n+2-3j. \end{cases}$$

4. For $n \equiv 3 \pmod 6$.

$$\lambda(v_i) = \begin{cases} \lfloor \frac{n+1}{3} \rfloor & \text{for } 1 \le i \le 3 \text{ and } i = n, \\ \lfloor \frac{n+1}{3} \rfloor + j \text{ with } j = 1, \cdots, \lfloor \frac{n}{6} \rfloor & \text{for } 3j+1 \le i \le 3j+3 \text{ and} \\ & n-3j \le i \le n+2-3j; \end{cases}$$

$$\lambda(v_i v_{i+1}) = \begin{cases} \lfloor \frac{n+1}{3} \rfloor & \text{for } i = 1, \\ \lfloor \frac{n+1}{3} \rfloor + j \text{ with } j = 1, \cdots, \lfloor \frac{n}{6} \rfloor & \text{for } 3j-1 \le i \le 3j+1 \text{ and} \\ & n+1-3j \le i \le n+3-3j, \\ \lceil \frac{n}{2} \rceil & \text{for } \lceil \frac{n}{2} \rceil \le i \le \lceil \frac{n+2}{2} \rceil. \end{cases}$$

5. For $n \equiv 4 \pmod 6$.

$$\lambda(v_i) = \begin{cases} \lfloor \frac{n+1}{3} \rfloor & \text{for } 1 \le i \le 2, \\ \lfloor \frac{n+1}{3} \rfloor + j \text{ with } j = 1, \cdots, \lfloor \frac{n}{6} \rfloor & \text{for } 3j \le i \le 3j+2 \text{ and} \\ & n+1-3j \le i \le n+3-3j, \\ \lfloor \frac{n}{2} \rfloor & \text{for } \lceil \frac{n+1}{2} \rceil \le i \le \lceil \frac{n+3}{2} \rceil; \end{cases}$$

$$\lambda(v_i v_{i+1}) = \begin{cases} \lfloor \frac{n+1}{3} \rfloor + j \text{ with } j = 1, \cdots, \lceil \frac{n}{6} \rceil & \text{for } 3j-2 \le i \le 3j \text{ and} \\ & n+2-3j \le i \le n+4-3j. \end{cases}$$

6. For $n \equiv 5 \pmod 6$.

$$\lambda(v_i) = \begin{cases} \lfloor \frac{n+1}{3} \rfloor & \text{for } 1 \le i \le 4 \text{ and} \\ & n-1 \le i \le n, \\ \lfloor \frac{n+1}{3} \rfloor + j \text{ with } j = 1, \cdots, \lfloor \frac{n}{6} \rfloor & \text{for } 3j+2 \le i \le 3j+4 \text{ and} \\ & n-1-3j \le i \le n+1-3j; \end{cases}$$

$$\lambda(v_i v_{i+1}) = \begin{cases} \lfloor \frac{n+1}{3} \rfloor & \text{for } 1 \le i \le 2 \text{ and } i = n, \\ \lfloor \frac{n+1}{3} \rfloor + j \text{ with } j = 1, \cdots, \lfloor \frac{n}{6} \rfloor & \text{for } 3j \le i \le 3j+2 \text{ and} \\ & n-3j \le i \le n+2-3j, \\ \lceil \frac{n}{2} \rceil & \text{for } \lceil \frac{n}{2} \rceil \le i \le \lceil \frac{n+2}{2} \rceil. \end{cases}$$

Thus, the vertex-weights of S_n satisfy

$$wt(u_i) = \begin{cases} 2 & \text{for } i = 1, \\ 2i - 1 & \text{for } i = 2, 3, \cdots, \lceil \frac{n+1}{2} \rceil, \\ 2(n + 2 - i) & \text{for } i = \lceil \frac{n+1}{2} \rceil + 1, \lceil \frac{n+1}{2} \rceil + 2, \cdots, n; \end{cases}$$

$$wt(v_i) = \begin{cases} n + 2 & \text{for } i = 1, \\ n - 1 + 2i & \text{for } i = 2, 3, \cdots, \lceil \frac{n+1}{2} \rceil, \\ 3n + 4 - 2i & \text{for } i = \lceil \frac{n+1}{2} \rceil + 1, \lceil \frac{n+1}{2} \rceil + 2, \cdots, n. \end{cases}$$

It is easy to see that the labeling is vertex irregular. Thus $tvs(S_n) = \lceil \frac{n+1}{2} \rceil$. ∎

Theorem 4. *For all n, the total vertex irregularity strength of a friendship graph f_n satisfy $tvs(f_n) = \lceil \frac{2n+2}{3} \rceil$.*

Proof. A friendship graph f_n has $2n$ vertices v_i of degree 2 and one vertex v of degree $2n$. The smallest weight of vertices of f_n must be 3. So, the largest weight of $2n$ vertices of degree 2 is at least $(2n + 2)$. Hence, the largest label of one vertex v_i or one edge incident with v_i is at least $\lceil \frac{2n+2}{3} \rceil$. On the other hand, the weight of vertex v is at least $(2n+3)$. This means that the label of vertex v or one edge incident with v is at least $\lceil \frac{2n+3}{2n+1} \rceil$. Then $tvs(f_n) \geq \max\{\lceil \frac{2n+2}{3} \rceil, \lceil \frac{2n+3}{2n+1} \rceil\} = \lceil \frac{2n+2}{3} \rceil$.

To show that $tvs(f_n) \leq \lceil \frac{2n+2}{3} \rceil$, let $V(f_n) = \{v, v_1, v_2, \cdots, v_{2n}\}$ and $E(f_n) = \{vv_1, vv_2, \cdots, vv_{2n}\} \cup \{v_1v_2, v_3v_4, \cdots, v_{2i-1}v_{2i}, \cdots, v_{2n-1}v_{2n}\}$. Let the labeling of vertices of f_n be as follows:

$$\lambda(v) = \begin{cases} 2 & \text{for } n = 1, \\ 1 & \text{for } n \geq 2; \end{cases}$$

$$\lambda(v_i) = j \quad \text{with} \quad j = 1, 2, \cdots, \lfloor \tfrac{2n+2}{3} \rfloor \quad \text{for} \quad 3j - 2 \leq i \leq 3j.$$

There are 3 cases of the labeling of edges of f_n, namely:

1. For $n \equiv 0(\mod 3)$.

$$\lambda(vv_i) = \begin{cases} 1 \text{ for } i = 1, \\ j \text{ with} \begin{cases} j = 2, 4, \cdots, \lfloor \frac{2n+2}{3} \rfloor \text{ for } 3j - 4 \leq i \leq 3j - 1, \\ j = 3, 5, \cdots, \lceil \frac{2n+2}{3} \rceil \text{ for } 3j - 3 \leq i \leq 3j - 2; \end{cases} \end{cases}$$

$$\lambda(v_iv_{i+1}) = j \quad \text{with} \begin{cases} j = 1, 2 & \text{for } i = 2j - 1, \\ j = 3, 5, \cdots, \lceil \frac{2n+2}{3} \rceil & \text{for } i = 3j - 4 \text{ and} \\ & i = 3j - 2, \\ j = 4, 6, \cdots, \lfloor \frac{2n+2}{3} \rfloor & \text{for } i = 3(j - 1). \end{cases}$$

2. For $n \equiv 1(\mod 3)$.

$$\lambda(vv_i) = \begin{cases} 1 \text{ for } i = 1, \\ j \text{ with} \begin{cases} j = 2, 4, \cdots, \lceil \frac{2n+2}{3} \rceil \text{ for } 3j - 4 \leq i \leq 3j - 1, \\ j = 3, 5, \cdots, \lfloor \frac{2n+2}{3} \rfloor \text{ for } 3j - 3 \leq i \leq 3j - 2; \end{cases} \end{cases}$$

$$\lambda(v_iv_{i+1}) = j \quad \text{with} \begin{cases} j = 1,2 & \text{for } i = 2j-1, \\ j = 3,5,\cdots,\lfloor\frac{2n+2}{3}\rfloor & \text{for } i = 3j-4 \text{ and} \\ & i = 3j-2, \\ j = 4,6,\cdots,\lfloor\frac{2n-1}{3}\rfloor & \text{for } i = 3(j-1). \end{cases}$$

3. For $n \equiv 2(\mod 3)$.

$$\lambda(vv_i) = \begin{cases} 1 \text{ for } & i = 1, \\ j \text{ with} \begin{cases} j = 2,4,\cdots,\lceil\frac{2n+2}{3}\rceil & \text{for } 3j-4 \leq i \leq 3j-1, \\ j = 3,5,\cdots,\lceil\frac{2n-1}{3}\rceil & \text{for } 3j-3 \leq i \leq 3j-2; \end{cases} \end{cases}$$

$$\lambda(v_iv_{i+1}) = j \quad \text{with} \begin{cases} j = 1,2 & \text{for } i = 2j-1, \\ j = 3,5,\cdots,\lceil\frac{2n-1}{3}\rceil & \text{for } i = 3j-4 \text{ and} \\ & i = 3j-2, \\ j = 4,6,\cdots,\lceil\frac{2n+2}{3}\rceil & \text{for } i = 3(j-1). \end{cases}$$

Thus, the vertex-weights of f_n are as follows.

$$wt(v_i) = i+2 \quad \text{for} \quad i = 1,2,\cdots,2n;$$

$$wt(v_i) = \begin{cases} 5 & \text{for } n = 1, \\ \frac{1}{3}(2n^2+6n+3) & \text{for } n \equiv 0(\mod 3), \\ \frac{1}{3}(2n^2+6n+4) & \text{for } n \equiv 1 \text{ and } 2(\mod 3) \text{ with } n \neq 1. \end{cases}$$

It is easy to see that the labeling is vertex irregular. Thus $tvs(f_n) = \lceil\frac{2n+2}{3}\rceil$.

■

References

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