On the Ramsey numbers for a combination of paths and Jahangirs

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Abstract. For given graphs G and H, the *Ramsey number* $R(G, H)$ is the least natural number n such that for every graph F of order n the following condition holds: either F contains G or the complement of F contains H. In this paper, we improve the Surahmat and Tomescu's result [9] on the Ramsey number of paths versus Jahangirs. We also determine the Ramsey number $R(\cup G, H)$, where G is a path and H is a Jahangir graph.

Keywords : Ramsey number, path, Jahangir

1 Introduction

The study of Ramsey Numbers for (general) graphs have received tremendous efforts in the last two decades, see few related papers [1–4, 6, 8] and a nice survey paper [7]. One of useful results on this is the establishment of a general lower bound by Chvátal and Harary [5], namely $R(G, H) \geq (\chi(G) - 1)(c(H) - 1) + 1$, where $\chi(G)$ is the chromatic number of G and $c(H)$ is the number of vertices in the largest component of H.

Let $G(V, E)$ be a graph with the vertex-set $V(G)$ and edge-set $E(G)$. If $(x, y) \in E(G)$ then x is called *adjacent* to y, and y is a *neighbor* of x and vice versa. For any $A \subseteq V(G)$, we use $N_A(x)$ to denote the set of all neighbors of x in A, namely $N_A(x) = \{y \in A | (x, y) \in E(G)\}$. Let P_n be a path with n vertices, C_n be a cycle with n vertices, and W_m be a wheel of $m + 1$ vertices, i.e., a graph consisting of a cycle C_m with one additional vertex adjacent to all vertices of C_m. For $m \geq 2$, the *Jahangir graph* J_{2m} is a graph consisting of a cycle C_{2m} with one additional vertex adjacent alternatively to m vertices of C_{2m}. For example, Figure 1[1] shows a Jahangir graph J_{16}.

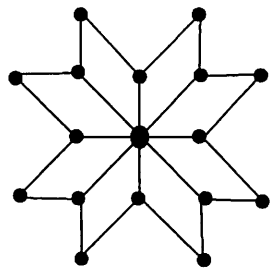

Fig. 1. Jahangir J_{16}

Recently, Surahmat and Tomescu [9] studied the Ramsey number of a combination of P_n versus a J_{2m}, and obtained the following result.

Theorem A. (Surahmat and Tomescu [9])

$$R(P_n, J_{2m}) = \begin{cases} 6 & \text{if } (n,m) = (4,2), \\ n+1 & \text{if } m = 2 \text{ and } n \geq 5, \\ n+m-1 & \text{if } m \geq 3 \text{ and } n \geq (4m-1)(m-1)+1. \end{cases}$$

In this paper, we determine the Ramsey numbers involving paths and Jahangir graphs. For particular, we improve the Surahmat and Tomescu's result for Jahangir graphs J_6, J_8 and J_{10} as follows.

Theorem 1. *$R(P_n, J_{2m}) = n + m - 1$ for $n \geq 2m + 1$ and $m = 3, 4$ or 5.*

We are also able to determine the Ramsey number $R(kP_n, J_{2m})$, for any integer $k \geq 2$, $m \geq 2$. These results are stated in the following theorems.

Theorem 2. *$R(kP_n, J_4) = kn + 1$, for $n \geq 4$, $k \geq 1$, except for $(n = 4, k = 1)$.*

Theorem 3. *$R(kP_n, J_{2m}) = kn + m - 1$, for any integer $n \geq 2m + 1$ if $m = 3, 4$ or 5; and for $n \geq (4m-1)(m-1) + 1$ if $m \geq 6$, where $k \geq 2$.*

[1] The figure J_{16} appears on Jahangir's tomb in his mausoleum, it lies in 5 km north-west of Lahore, Pakistan across the River Ravi. His tomb was built by his Queen Noor Jehan and his son Shah-Jehan (This was emperor who constructed one of the wonder of world Taj Mahal in India) around 1637 A.D. It has a majestic structure made of red sand-stone and marble.

2 The Proof of Theorems

The proof of Theorem 1.

Consider graph $G \cong K_{m-1} \cup K_{n-1}$. Clearly, G contains no P_n and $\overline{G}$ contains no J_{2m}. Thus, $R(P_n, J_{2m}) \geq n+m-1$. For $m = 3, 4$ or 5 and $n \geq 2m+1$, we will show that $R(P_n, J_{2m}) \leq n+m-1$. Let F be a graph of $n+m-1$ vertices containing no P_n. Take any longest path L in F. Let L be $(x_1, x_2, \cdots, x_k)$, and $Y = V(F)\backslash V(L)$. Since $k \leq n-1$, then $|Y| \geq m$. Obviously, yx_1, yx_k are not in $E(F)$, for any $y \in Y$. Now, consider the following two cases

Case 1. $2m \leq |L| \leq n-1$.

Let $|L| = t$ and $A = \{x_2, x_3, \cdots, x_{2m-1}\}$ be the set of first $2m-2$ vertices of L after x_1. Take the set of any m distinct vertices of Y and denote it by $B = \{y_1, \cdots, y_m\}$. By the maximality of L, every vertex of B has at most $m-1$ neighbors in A. If there are two vertices of B having $m-1$ neighbors in A then all the neighbors are intersected.

Subcase 1.1 There exists $b \in B$, $|N_A(b)| = m-1$.

Let $A_1 = A\backslash N_A(b)$ and take any vertex v_1 of A_1 whose the highest degree at B. Define $D_1 = \{x_1, x_t, b\} \cup A_1\backslash\{v_1\}$, and $D_2 = \{v_1\} \cup B\backslash\{b\}$. By the maximality of L, $d_{D_1}(w) \leq 1$ for any vertex w of D_2. In particular, $d_{D_1}(v_1) = 0$. Since v_1 has the highest degree then there are at most $m-2$ edges connecting vertices between D_1 and D_2 in F. This implies that $D_1 \cup D_2$ will induces a J_{2m} in $\overline{F}$.

Subcase 1.2 All vertices $b \in B$, $|N_A(b)| \leq m-2$.

If $m = 3$ then let $D_1 = \{$any two vertices of $A\}$. If $m = 4$ then by the Pigeon Hole principle there exists two vertices of A has neighbors at most 1 in B. In this case let $D_1 = \{$three vertices of A with two of degree at most one $\}$. If $m = 5$ then by the Pigeon Hole principle there exists three vertices of A has neighbors at most 2 in B. In this case let $D_1 = \{$four vertices of A with three of degree at most two$\}$. Therefore, $\{x_1, x_t\} \cup D_1 \cup B$ will induce a J_{2m} in $\overline{F}$.

Case 2. $1 \leq |L| \leq 2m-1$.

We breakdown the proof into several subcases.

Subcase 2.1. $1 \leq |L| \leq 3$

In this case, the component of F is either K_1, P_2, C_3 or a star. Therefore, $\overline{F}$ contains a J_{2m}, for $m = 3, 4$ or 5.

Subcase 2.2. $4 \leq |L| \leq m+1$.

Let L be $(x_1, x_2, \cdots, x_t)$, where $t \leq m+1$, and so $|Y| = |V(F)\backslash V(L)| \geq 2m-1$. Now, consider the set $N_Y(x_2)$ of vertices in Y adjacent to x_2. Note that any vertex of $N_Y(x_2)$ is nonadjacent to any other vertices of Y. If $|N_Y(x_2)| \geq m-2$ then form two sets D_1 and D_2 as follows. The set D_1 consists of x_1, x_t and any $m-2$ vertices of $N_Y(x_2)$. The set D_2 consists of the other vertices of Y not selected in D_1. Thus, $|D_1| = m$ and $|D_2| = m+1$. By the maximality of L, there is no edge connecting any vertex of D_1 to any vertex of D_2. Thus, the set $D_1 \cup D_2$ induces $K_{m,m+1} \supseteq J_{2m}$ in $\overline{F}$. If $|N_Y(x_2)| = m - 3$ then take $D_1 = \{x_1, x_t, x_2\} \cup N_Y(x_2)$, and D_2 as the set of the remaining vertices of Y. Then, $D_1 \cup D_2$ again contains $K_{m,m+1} \supseteq J_{2m}$ in $\overline{F}$. Now, if $|N_Y(x_2)| = m - 4$ (for $m = 4$ or 5) then in showing $\overline{F} \supseteq J_{2m}$ take $D_1 = \{x_1, x_t, x_2, x_{t-1}\} \cup N_Y(x_2)$, and D_2 as the set of the remaining vertices of Y not adjacent to x_{t-1}. This is true since $|N_Y(x_{t-1})| \leq 1$ (by symmetrical argument). If $|N_Y(x_2)| = m - 5$ (for $m = 5$ only), then $D_1 = \{x_1, x_2, x_{t-1}, x_t, b\}$ where b is a vertex at distance two from x_3 or b is any vertex of Y with a smallest degree, and D_2 as the set of the remaining vertices of Y. Thus, $D_1 \cup D_2$ will induce J_{10} in $\overline{F}$.

Subcase 2.3. $|L| = m + 2$.
Let L be $(x_1, x_2, \cdots, x_t)$ where $t = m + 2$, then $|Y| = |V(F) \backslash V(L)| \geq 2m - 2$. Now, consider the set $N_Y(x_2)$ of vertices in Y adjacent to x_2. Note that any vertex of $N_Y(x_2)$ is nonadjacent to any other vertices of Y. If $|N_Y(x_2)| \geq m - 2$ then form two sets D_1 and D_2 as follows. If x_3 is nonadjacent to x_{m+2} then $D_1 = \{x_1, x_{m+2}\} \cup \{\text{any } m - 2 \text{ vertices of } N_Y(x_2)\}$ and D_2 consists of x_3 together with the remaining vertices of Y. Otherwise (if $x_3 \sim x_{m+2}$), take $D_1 = \{x_1, x_{m+2}, x_4\} \cup \{\text{any } m - 2 \text{ vertices of } N_Y(x_2)\}$ and D_2 consists of any m remaining vertices of Y. By the maximality of L, there is no edge connecting any vertex of D_1 to any vertex of D_2. Thus, the set $D_1 \cup D_2$ induces $K_{m,m+1} \supseteq J_{2m}$ in $\overline{F}$.

If $|N_Y(x_2)| = m - 3$ then take $D_1 = \{x_1, x_t, x_2\} \cup N_Y(x_2)$, and D_2 as the set of the remaining vertices of Y. Then, $D_1 \cup D_2$ again contains $K_{m,m+1} \supseteq J_{2m}$ in $\overline{F}$. Now, if $|N_Y(x_2)| = m - 4$ (for $m = 4$ or 5) then in showing $\overline{F} \supseteq J_{2m}$ take $D_1 = \{x_1, x_t, x_2, x_{t-1}\} \cup N_Y(x_2)$, and D_2 as the set of the remaining vertices of Y not adjacent to x_{t-1}. This is true since $|N_Y(x_{t-1})| \leq 1$ (by symmetrical argument). If $|N_Y(x_2)| = m - 5$ (for $m = 5$ only), then $D_1 = \{x_1, x_2, x_{t-1}, x_t, b\}$ where b is a vertex at distance two from x_3 or b is any vertex of Y with a smallest degree, and D_2 as the set of the remaining vertices of Y. Thus, $D_1 \cup D_2$ will induce J_{10} in $\overline{F}$.

Subcase 2.4. $|L| = m + 3$ (or $2m - 1, 2m - 2$ if $m = 4, 5$ respectively).
Let L be $(x_1, x_2, \cdots, x_t)$ where $t = m + 3$, then $|Y| = |V(F) \backslash V(L)| \geq 2m - 3$. Now, consider the set $N_Y(x_2)$ of vertices in Y adjacent to x_2. Note that any vertex of $N_Y(x_2)$ is nonadjacent to any other vertices of Y. If $|N_Y(x_2)| \geq m - 1$ then form two sets D_1 and D_2 as follows. If x_{t-1} is adjacent to some vertex of $N_Y(x_2)$ then by the maximality of L, x_{t-2} is nonadjacent to x_1 and any vertex of $N_Y(x_2)$. In this case set $b = x_{t-2}$. If x_{t-1} is nonadjacent to any vertex of $N_Y(x_2)$, then take $b = x_{t-1}$ provided $x_{t-1} \not\sim x_1$. Otherwise (if $x_{t-1} \sim x_1$), by the maximality of L we have that x_{t-2} is nonadjacent to x_1 and to any vertex of $N_Y(x_2)$. In this case, again take $b = x_{t-2}$. Now, define $D_1 = \{x_1\} \cup \{\text{any } m - 1$ vertices of $N_Y(x_2)\}$ and $D_2 = \{x_3, x_t, b\} \cup \{$ any $m - 2$ other vertices of $Y\}$. By the maximality of L, there is no edge connecting any vertex of D_1 to any vertex of D_2. Thus, the set $D_1 \cup D_2$ induces $K_{m,m+1} \supseteq J_{2m}$ in $\overline{F}$.

If $|N_Y(x_2)| = m-2$ then take $D_1 = \{x_1, x_2\} \cup N_Y(x_2)$, and $D_2 = \{x_3, x_t\} \cup \{$ any $m-1$ other vertices of Y. Then, $D_1 \cup D_2$ contains $K_{m,m+1}$ minus at most two edges (x_2, x_3) and (x_2, x_t) in $\overline{F}$. Therefore, $\overline{F} \supseteq J_{2m}$. Now, if $|N_Y(x_2)| = m-3$ then in showing $\overline{F} \supseteq J_{2m}$ take $D_1 = \{x_1, x_2, x_t\} \cup N_Y(x_2)$, and $D_2 = \{x_3\} \cup \{$ any m other vertices of Y. This is true since $D_1 \cup D_2$ contains $K_{m,m+1}$ minus at most two edges (x_2, x_3) and (x_2, x_t) in $\overline{F}$. If $|N_Y(x_2)| = m-4$, then $D_1 = \{x_1, x_2, x_{t-1}, x_t\} \cup N_Y(x_2) \cup N_Y(x_{t-1})$ and D_2 as the set of the remaining vertices of Y. Thus, $D_1 \cup D_2$ will induce $K_{m,m+1}$ in $\overline{F}$. If $|N_Y(x_2)| = m-5$ (only for $m=5$), then $D_1 = \{x_1, x_2, x_{t-1}, x_t, b\}$, where b is either x_3, a neighbor of x_3 in Y or a vertex of Y at distance two from x_3 and D_2 as the set of the remaining vertices of Y. Thus, $D_1 \cup D_2$ will induce $K_{m,m+1}$ minus at most one edge in $\overline{F}$.

Subcase 2.5. $|L| = m+4 = 2m-1$ (only for $m=5$).
Let L be $(x_1, x_2, \cdots, x_t)$ where $t = 2m-1$, then $|Y| = |V(F) \backslash V(L)| \geq 2m-4$. Now, consider the set $N_Y(x_2)$ of vertices in Y adjacent to x_2. Note that any vertex of $N_Y(x_2)$ is nonadjacent to any other vertices of Y. If $|N_Y(x_2)| \geq m-2$ then form two sets D_1 and D_2 as follows. By the maximality of L, one element in each pair $\{x_4, x_5\}$ and $\{x_6, x_7\}$ is nonadjacent to all vertices of $N_Y(x_2)$. Call these two vertices by b and c. Therefore, there are at most four edges connecting from $\{x_1, x_t\}$ to $\{x_3, b, c\}$ in F. Now, define $D_1 = \{x_1, x_t\} \cup \{$any $m-2$ vertices of $N_Y(x_2)\}$ and $D_2 = \{x_3, b, c\} \cup$ {any $m-2$ other vertices of $Y\}$. Thus, the set $D_1 \cup D_2$ induces $K_{5,6}$ minus four edges in $\overline{F}$, and so $\overline{F} \supseteq J_{10}$.

If $|N_Y(x_2)| = m-3$ then By the maximality of L, one vertex in $\{x_4, x_5\}$ is nonadjacent to all vertices of $N_Y(x_2)$. Call this vertex by b. Therefore, there are at most four edges connecting from $\{x_1, x_2, x_t\}$ to $\{x_3, b\}$ in F. Now, take $D_1 = \{x_1, x_2, x_t\} \cup N_Y(x_2)$, and $D_2 = \{x_3, b\} \cup \{$any $m-1$ other vertices of $Y\}$. Then, $D_1 \cup D_2$ contains $K_{5,6}$ minus at most four edges in $\overline{F}$. Therefore, $\overline{F} \supseteq J_{10}$.

if $|N_Y(x_2)| = m-4$ then take $D_1 = \{x_1, x_2, x_{t-1}, x_t\} \cup N_Y(x_2) \cup N_Y(x_{t-1})$, and $D_2 = \{x_3\} \cup \{$all the remaining vertices of $Y\}$. Then, $D_1 \cup D_2$ contains $K_{5,6}$ minus possibly two edges (x_3, x_{t-1}) and (x_3, x_t) in $\overline{F}$. Therefore, $\overline{F} \supseteq J_{10}$.

if $|N_Y(x_2)| = m-5$ then take $D_1 = \{x_1, x_2, b, x_{t-1}, x_t\}$ where b is either x_3 or x_4 whose the smallest number of neighbors in Y, and $D_2 = Y$. Then, $D_1 \cup D_2$ contains $K_{5,6}$ minus at most three edges in $\overline{F}$. Therefore, $\overline{F} \supseteq J_{10}$. □

The proof of Theorem 2.
For $n=4$ and $k=2$, consider graph $G = K_1 \cup K_7$. Clearly G contains no $2P_n$ and $\overline{G}$ contains no J_4. Hence $R(2P_4, J_4) \geq 9$. To prove the upper bound, consider now graph F of order 9 containing no $2P_4$. Take a longest path in F and call it L. Let L be $x_1, x_2, \cdots, x_k$. Clearly, $k \leq 7$, since $F \not\supseteq 2P_4$. If $A = V(F) \backslash V(L)$, then $|A| \geq 2$. Any vertex of A is nonadjacent to x_1 and x_k. Thus, the number vertices in A must be exactly 2 and so $k=7$, since otherwise A together with $\{x_1, x_k\}$ will form a $K_{2,3} = J_4$ in $\overline{F}$. Let $A = \{y, z\}$, and consider the following two cases:

Case 1. Vertices y and z has a common neighbor in L.
Let x_i be the common neighbor of y and z in L, for some $i \in \{2, 3, \cdots, 6\}$. Then, y, z are nonadjacent to x_{i-1} and x_{i+1}, since otherwise the maximality of L will suffer. At least one of the last two vertices must differ with x_1 and x_7, call it w. So, we have a J_4 in $\overline{F}$ formed by $\{x_1, x_7, y, z, w\}$.

Case 2. Vertices y and z has no common neighbor in L.
If there exists a vertex x_i, $2 \leq i \leq 6$, is nonadjacent to y and z, then $\{x_i, x_1, x_7, y, z\}$ forms a J_4 in $\overline{F}$. Thus, every x_i is adjacent to at least one of $\{y, z\}$. Now, since y and z has no common neighbor in L, without loss of generality we can assume that $x_2y \in E(F)$, and so $x_2z \notin E(F), x_3y \notin E(F), x_3z \in E(F), x_4z \notin E(F), x_4y \in E(F), x_5y \notin E(F)$ and $x_5z \in E(F)$. Therefore, the path $x_1, x_2, y, x_4, x_3, z, x_5, x_6, x_7$ is Hamiltonian, which contradicts the maximality of path L in F.

Now, let $n \geq 5$. Consider graph $G = K_1 \cup K_{kn-1}$. Clearly G contains no kP_n and $\overline{G}$ contains no J_4. Hence $R(kP_n, J_4) \geq kn - 1 + 1 + 1 = kn + 1$. For the upper bound, let F be a graph of order $kn + 1$ such that $\overline{F}$ does not contain J_4. By induction on k, we will show that F contains kP_n. By Theorem A gives a verification of the result for $k = 1$. Assume the theorem is true for any $s \leq k - 1$, namely $R(sP_n, J_4) = sn + 1$, for $n \geq 5$. Now consider graph F of $kn + 1$ vertices such that $\overline{F} \not\supseteq J_4$. By the induction hypothesis, F will contain $(k - 1)P_n$. Let $Y = V(F) \backslash V((k-1)P_n)$. Then, $|Y| = n + 1 = R(P_n, J_4)$ and hence $F[Y]$ contains a P_n. This implies that F contains kP_n. □

The proof of Theorem 3.
Since graph $G = K_{m-1} \cup K_{kn-1}$ contains no kP_n and $\overline{G}$ contains no J_m, then $R(kP_n, J_{2m}) \geq kn + m - 1$. For proving the upper bound, let F be a graph of order $kn + m - 1$ such that $\overline{F}$ contains no a J_4. We will show that F contains kP_n. We use an induction on k. For $k = 1$ it is true from Theorem A. Now, let assume that the theorem is true for all $s \leq k - 1$. Take any graph F of $kn + m - 1$ vertices such that its complement contains no J_{2m}. By the hypothesis, F must contain $(k - 1)$ disjoint copies of P_n. Remove these copies from F, then the remaining vertices will induce another P_n in F since $\overline{F} \not\supseteq J_{2m}$. Therefore $F \supseteq kP_n$. The proof is complete. □

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