Edge antimagic total labeling on paths and unicycles

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Abstract. Let $G = (V, E)$ be a simple and undirected graph with v vertices and e edges. An (a, d)-*edge-antimagic total labeling* is a bijection f from $V(G) \cup E(G)$ to the set of consecutive integers $\{1, 2, \cdots, v + e\}$ such that the weight of edges form arithmetical progression with initial term a and common difference d. A *super* (a, d)-*edge antimagic total labeling*is an edge antimagic total labeling f which $f(V(G)) = \{1, \cdots, v\}$. In this paper we solve some problems on edge antimagic total labeling, such as on paths and unicyclic graphs.

key words: Edge antimagic total labeling, super edge antimagic total labeling

1 Introduction

In this paper we only deal with simple and connected graphs. Let $G = (V, E)$ be a simple and undirected graph with v vertices and e edges. Let $V = V(G)$ be the set of vertices of G and $E = E(G)$ be the set of edges of G.

Hartsfield and Ringel [5] introduced the concept of an antimagic labeling. For a graph G of size e, it is said to be *antimagic* if there is a labeling from $E(G)$ into $\{1, 2, \ldots, e\}$ such that all sums of the labels on the edges incident with each vertex v are distinct. Such a labeling is called an *antimagic labeling.* Paths, cycles, complete graphs, and wheels are the examples of antimagic graphs. We know that K_2 is not antimagic. In fact, Hartsfield and Ringel conjecture that every connected graph other than K_2 is antimagic. A (v, e)-graph G is defined to be *edge-antimagic* if there exists a labeling f from $V(G) \cup E(G)$ into $\{1, 2, \ldots, v + e\}$ which satisfies the condition that $f(u_1) + f(v_1) + f(u_1v_1) \neq f(u_2) + f(v_2) + f(u_2v_2)$ for any pair of distinct edges u_1v_1 and u_2v_2 of G.

The term of edge-antimagic labeling was introduced by Kotzig and Rosa [6], but it started to get more attention after Enomoto et al work [3]. Many families of graphs have been proved to have an (a,d)-edge antimagic total labeling. As examples see [2, ?]. For the complete survey, see [4]. Several works still left some open problems and some of them have been collected in [1]. We list some open problems that has relations with our work in the next section.

A *total labeling* of G is a bijection $f: V(G) \cup E(G) \to 1, 2, \ldots, v+e$ and the associated *weight* $w_f(xy)$ of an edge xy in G is $w_f(xy) = f(x) + f(y) + f(xy)$. All labelings considered in this paper are total labelings, and so from now on by a labeling we always mean a total labeling. The labeling f of G is *edge magic* if every edge has the same weight. f is called a *super edge labeling*if $f(V) = \{1, ..., n\}$. Sedláček [9] introduced magic labeling of graphs in 1963, and since then there have been many results in magic labeling, especially in edge magic labeling. For the details see [4].

A bijection $f: V(G) \cup E(G) \to \{1, 2, ..., v+e\}$ is called an (a,d)*-edge-antimagic total* ((a,d)-EAT) *labeling of* G if the set of edge-weights of all edges in G is $\{a, a+d, a+2d, \ldots, a+(v-1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed nonnegative integers. If $d = 0$ then we call f a *edge-magic total labeling* (EMT). An (a,d)-edge-antimagic total labeling f is called a *super* (a,d)*-edge-antimagic total super* (a,d)*-EAT*) labeling if $f(V)) = \{1, 2, ..., v\}$ and $f(E) = \{v+1, v+2, ..., v+e\}$. Note that if we restrict the domain of f to $V(G)$ then we have (a', d')-edge antimagic vertex labeling ((a', d')-EAV labeling).

In this paper we solve some open problems on edge antimagic total labeling that proposed on several published papers, such as on paths and unicyclic graphs.

2 Basic properties and open problems

Let $f: V(G) \cup E(G) \to \{1, 2, \ldots, v+e\}$ be an (a,d)-edge-antimagic total labeling of G and $W = \{wt_f(uv) : wt_f(uv) = f(u) + f(v) + f(uv), uv \in E(G)\} = \{a, a+d, a+2d, \ldots, a+(e-1)d\}$ is the set of edge-weights. Then the minimum possible edge-weight in (a,d)-edge-antimagic total labeling is at least 6. Consequently $a \geq 6$. On the other hand, the maximum possible edge-weight is at most $3v+3e-3$. Then

$$d \leq \frac{3v + 3e - 9}{e - 1}.$$

Bača *et al.* had survey on edge antimagic labeling in [1]. In their survey, they proposed some open problems regarding with paths and cycles, such as follows:

- Find $(a, 5)$-EAT labelings for paths P_n, for the feasible values.
- Find (a,d)-EAT labelings for even cycles with $d \in \{4, 5\}$ and for odd cycles with $d = 5$.

The results on $(a,4)$-EAT and $(a,6)$-EAT (cited in [1] as a preprint) are given in the following section.

3 Paths

Let P_n be a path with n vertices, then according to the inequality [1], the following observation is hold.

Observation 1 *If paths P_n has an (a,d)-EAT labeling then Then*

$$d \leq 6.$$

Wallis *et al.* proved that all path is an edge magic ($(a,0)$-EAT). Bača *et al.* [2] have proved the following:

- Every path P_n has $(2n+2,1)$-EAT, $(3n,1)$-EAT, $(n+4,3)$-EAT and $(2n+2,3)$-EAT labelings.
- Every odd path $P_{2k+1}, k \geq 1$, has $(3k+4,2)$-EAT $(5k+4,3)$-EAT, $(2k+4,4)$-EAT and $(2k+6,4)$-EAT labelings.
- Every even path $P_{2k}, k \geq 1$, has $(3k+3,2)$-EAT and $(5k+1,2)$-EAT labelings.

and Ngurah [7] has proved the following theorem

Theorem 1 *Every odd path $P_{2k+1}, k \geq 1$, has $(4k+4,1)$-EAT $(6k+5,3)$-EAT, $(4k+4,2)$-EAT and $(4k+5,2)$-EAT labelings.*

In the following theorem, we want to complete the results that we mentioned above.

Theorem 2 *Every path P_n has $(n+5,4)$-EAT (for n odd), $(n+4,4)$-EAT (for n even) and $(6,6)$-EAT labelings.*

Proof.

Let P_n be the paths with n vertices. Let $v_i, i = 1,...,n$ be a set of vertices of P_n. Label the vertices and edges of P_n of G as follows.

$$f_1(v_i) = \begin{cases} i & \text{if } i \text{ is odd} \\ n+i & \text{if } i \text{ is even and } n \text{ is odd} \\ n+i-1 & \text{if } i \text{ is even and } n \text{ is even} \end{cases}$$

$$f_1(v_iv_{i+1}) = 2i, \text{for all } i$$

We can see that f_1 is a bijection and the edge weights form an arithmetic progression with difference 4. Thus f_1 is a $(n+5,4)$-EAT labeling for P_n, n odd and a $(n+4,4)$-EAT labeling for P_n, n even.
Define a new labeling f_2 as follows.

$$f_2(v_i) = 2i - 1 \quad \text{for all } i$$

$$f_2(v_i v_{i+1}) = 2i \quad \text{for all } i$$

The labeling f_2 form a bijection and the edge weights form an arithmetic progression with the difference 6. It means we obtain a (6, 6)-EAT labeling for P_n. □

4 Unicyclic graphs

In this section we presents the (a, d)-EAT labeling construction of a special unicyclyc that we called sun. The following theorem and observation will be used in the proof that a sun graph has an (a, d)-EAT labeling for $d \in \{2, 4\}$.

Theorem 3 *[2] If G has an (a, d)-EAV labeling then*

- *(i) G has a $(a + n + 1, d + 1)$-EAT labeling*
- *(ii) G has a $(a + n + e, d - 1)$-EAT labeling*

Observation 2 *If G has a super edge magic total (SEMT) labeling then G has an $(a, 1)$-EAV labeling.*

An n-crown graph $G \cong C_m \odot \bar{K}_n$ is a graph obtained by taking one copy of C_m(which has order m) and m copies of $\bar{K}_n$, and then joining the i-th vertex of C_m to every vertex in the i-th copy of $\bar{K}_n$. A sun graph $G = C_n \odot \bar{K}_1$ is obtained by joining n isolated vertices to a cycle C_n in such a way that each isolated vertex adjacent with one vertex in C_n.

Observation 3 *If a sun graph $G = C_n \odot \bar{K}_1$ has an (a, d)-EAT labeling then $d \leq 5$*

Observation 4 *[8] For every two integers $m \geq 3$ and $n \geq 1$, the n-crown $G \cong C_m \odot \bar{K}_n$ is super magic.*

Since sun graphs are special n-crown graphs then we have the following corollary:

Corollary 1 *A sun graph $G = C_n \odot \bar{K}_1$ has a super edge magic total ((a, 0)-EAT)labeling.*

Theorem 4 *Sun graphs $G = C_n \odot \bar{K}_1$ have an (a, d)-EAT labeling for $d \in \{2, 4\}$*

Proof.
According to 4 every sun graph $G = C_n \odot \bar{K}_1$ has a SEMT labeling. Obviously sun graphs also have an $(a, 1)$-EAV labeling by 4, say the labeling is f. Then the edge weight of labeling wt_f forms an arithmetic progression $a, a + 1, \ldots, a + (2n - 1)$.

Case 1: $d = 2$. Define a new labeling g as follows. Let $\{v_1, ..., v_n\}$ be internal

vertices of sun graph G and $\{w_1, ..., w_n\}$ be external vertices of G. Note that internal vertices means all the vertices of G that has degree 3 and external vertices means all vertices of G that has degree 1.

$$g(v) = \begin{cases} f(v_i) & \text{if } v = v_i \\ f(w_i) & \text{if } v = w_i \end{cases}$$

$$g(e_i) = 2n + i, \text{ if } wt_f(e_i) = a + i - 1$$

Then the edge weight of labeling g are $a + n + 1, a + n + 3, \ldots, a + (6n - 1)$. This sequence forms an arithmetic progression with difference 2. Thus g is an $(a + n + 1, 2)$-EAT labeling for sun graphs.

Case 2: $d = 4$. The integers that we can use for labels vertices and edges of sun graphs are 1,2,..., $4n$. Define a new labeling g as follows.

$$g(v) = \begin{cases} 2f(v_i) & \text{if } v = v_i \\ 2f(w_i) & \text{if } v = w_i \end{cases}$$

$$g(e_i) = 2i + 1, \text{ if } wt_f(e_i) = a + i - 1$$

Then the edge weight of labeling g are $2a + 3, 2a7, \ldots, 2a + (6n - 2)$. This sequence forms an arithmetic progression with difference 4. Thus g is an $(2a + 3, 4)$-EAT labeling for sun graphs. □.

Theorem 5 *If n and d are odds, there is no (a, d)-EAT labeling for $G = C_n \odot \bar{K}_1$.*

Proof. Let $\{v_1, \ldots, v_n\}$ are the internal vertices (the vertex that has degree 3).

Suppose that $G = C_n \odot \bar{K}_1$ has an (a, d)-EAT labeling for d odd. Then the sum of the edge-weights.

$$w = a + (a + d) + (a + 2d) + \ldots + (a + (2n - 1)d = 2na + dn(2n - 1)$$

Since d and n are odd then the value of w is always odd. Observe that the sum of all labels is as follows.

$$S_L = 1 + \ldots + 4n + 2(x_1 + \ldots + x_n) = 2n(4n + 1) + 2(x_1 + \ldots + x_n),$$

where $(x_1 + \ldots + x_n)$ are labels for $(v_1, \ldots, v_n)$ respectively. The value of S_L is always even. Contradict with the fact that if $G = C_n \odot \bar{K}_1$ has an (a, d)-EAT labeling for n and d odds then $w = S_L$. □

5 Open Problems

In the purpose to solve open problems of (a,d)-EAT labeling of path, we only can find an $(a,5)$-EAT labeling for path $P_n, n \leq 9$. Unfortunately, we cannot find the pattern of the labeling. Thus the problem in [1] is still open for $d = 5$. We list here two problems for further investigation.

Conjecture 1 *: P_n has $(a,5)$-EAT labelings for the feasible values of a.*

Open Problems 1 *Find if there is an (a,d)-EAT labeling, $d \in \{1,3,5\}$, for sun graphs $G = C_n \odot \bar{K}_1$ for n even.*

References

1. M. Bača, E.T. Baskoro, M. Miller, J. Ryan, R. Simanjuntak and K.A. Sugeng, Survey of edge antimagic labelings of graphs, *J. Indones. Math. Soc(MIHMI)* **12** No. 1 (2006) 113-130.
2. M. Bača, Y. Lin, M. Miller and R. Simanjuntak, New construction of magic and antimagic graph labelings, *Utilitas Math* **60** (2001) 229-239.
3. H. Enomoto, A.S. Llado, T. Nakamigawa and G. Ringel, Super edge-magic graphs, *SUT J. Math.* **34** (1998) 105-109.
4. J. Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics* **6**(2003), DS6 .
5. N. Hartsfield and G. Ringel, Supermagic and antimagic graphs, *J. Rec. Math.*, 21 (1989), 107–115
6. A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.* **13** (1970) 451-461.
7. A.A. Ngurah, On (a,b)-edge antimagic total labeling of odd cycle, *MIHMI* **9** (2003) 9-12.
8. R.M. Figueroa-Centeno, R. Ichishima and F.A. Muntaner-Batle, Magical coronation of graphs, *Austral. J. Combin.* 26(2002) 199-208.
9. J. Sedlacek, Problem 27, *Theory of Graphs and its Applications*, Proc. Symposium Smolenice (1963) 163-167.
10. K.A. Sugeng and M. Miller, Relationships between adjacency matrices and super (a,d)-edge-antimagic total graphs, *JCMCC* **55** (2005) 71-82.
11. W.D. Wallis, E.T. Baskoro, M. Miller and Slamin, Edge magic total labelings, *Austral. J. Combin.* **22** (2000) 177-190.