Optimum sum labeling of finite union of sum graphs

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Abstract. Let $G = (V, E)$ be a simple, finite and undirected graphs. A sum labeling is a one to one mapping L from a set of vertices of G to a finite set of positive integers S such that if u and v are vertices of G then uv is an edge in G if and only if there is a vertex w in G and $L(w) = L(u) + L(v)$. A graph G that has a sum labeling is called sum graph. The minimal isolated vertex that needed to make G a sum labeling is called sum number of G, notated as $\sigma(G)$. The sum number of a sum graph G always greater or equal to $\delta(G)$, a minimum degree of G. An optimum sum graph is a sum graph that has $\sigma(G) = \delta(G)$. In this paper, we discuss sum numbers of finite union of some family of optimum sum graphs, such as cycles and friendship graphs.

Keywords: Optimal sum labeling, cycles, complete graphs and friendship graphs.

1 Introduction

In this paper, we consider finite simple undirected graphs. The set of vertices and edges of a graph G will be denoted by $V(G)$ and $E(G)$, respectively. We put $v = |V(G)|$ and $e = |E(G)|$. For simplicity, we denote $V(G)$ by V and $E(G)$ by E.

A *sum labeling* of a graph G is a one to one mapping L from a set of vertices of G to a finite set of positive integers S such that if u and v are vertices of G then uv is an edge in G if and only if there is a vertex w in G which label $L(w) = L(u) + L(v)$. A graph G that has sum labeling is called *sum graph*. The concept of sum graph labeling was introduced by Harary in 1990 [1].

A vertex with the largest label cannot be adjacent with any vertex in G. Thus every sum graph has at least one isolated vertex. The minimum number of isolated

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vertex that should be added to make G a sum graph is called *sum number of* G, denoted by $\sigma(G)$. The sum number of a sum graph G is always greater than or equal to $\delta(G)$, the minimum degree of G. An optimum sum graph is a sum graph that has $\sigma(G) = \delta(G)$.

In their paper, Miller, Ryan and Smyth [4] showed that the sum number of a disjoint union of two graphs satisfy $\sigma(G_1 \cup G_2) \leq \sigma(G_1) + \sigma(G_2) - 1$.

From an observation of sum labeling, it can be concluded that if G_1 and G_2 are sum graphs then $\sigma(G_1 \cup G_2) \geq \min(\sigma(G_1), \sigma(G_2))$. Hence, a disjoint union of two unit graphs is also a unit graph. Recall that a unit sum graph is a sum graph with $\sigma(G) = \delta(G) = 1$. This paper presents the sum labeling for union of disjoint cycle graphs and union of disjoint friendships. We chose these classes of graphs because they are optimum sum graph which is $\sigma(G) = \delta(G)$. A friendship graph is an optimum sum graph has been proved by Fernau *et al.* [2]. The preliminary results on the disjoint union of two graphs has been published in [5].

2 Sum labeling of a disjoint union of cycles

Theorem 1 : *Let $C_{m_1}, C_{m_2}, \ldots C_{m_n}$ are n disjoint cycle graphs with consecutively have $m_1, m_2, \ldots m_n$ vertices, where $m_1, m_2, \ldots m_n > 4$, then*

$$\sigma(\bigcup_{i=1}^{n} C_{m_i}) = \delta(\bigcup_{i=1}^{n} C_{m_i}) = 2.$$

Proof:
Let $U_i = \{v_1^i, v_2^i, \ldots, v_{m_i}^i\}$ be the set of vertices of graph C_{m_i}.
Define the labeling F of the disjoint union of C_{m_i}, $i = 1, 2, \ldots, n$ as follows:

$$\begin{aligned} F(v_1^1) &= 2, \qquad F(v_2^1) = 3 \\ F(v_j^1) &= F(v_{j-1}^1) + F(v_{j-2}^1), \qquad j = 3, 4, \ldots, m_1 \end{aligned} \tag{1}$$

and for $i = 1, 2, 3, \ldots, (n-1)$

$$\begin{aligned} F(v_1^{i+1}) &= F(v_{m_i}^i) + F(v_1^i), \\ F(v_2^{i+1}) &= F(v_{m_i}^i) + F(v_{m_i-1}^i), \\ F(v_j^{i+1}) &= F(v_{j-1}^{i+1}) + F(v_{j-2}^{i+1}), \qquad j = 3, 4, \ldots, m_i. \end{aligned} \tag{2}$$

Let us define the sets

$$S_i = \left\{F(v_r^i) + F(v_s^i), s = r + 1\right\}, \qquad i = 1, 2, \ldots, n$$

where $v_{m_i+1}^i = v_1^i$.

Using properties of Fibonacci numbers, for $i < j$, then every element of S_i is always smaller than the elements of S_j. Consequently, there is no additional edge that connecting the vertices of C_{m_i} to the vertices of C_{m_j}.

From the definition of labeling F and the fact that $\sigma(C_{m_i}) = 2$, for $i = 1, 2, \ldots, n$ [1], then the label of the isolated vertices are $F(v_1^n) + F(v_{m_n}^n)$ and $F(v_{m_n-1}^n) + F(v_{m_n}^n)$.

Without loss of generality, let $i < j$ and let us assume that there exist an additional edge connecting a vertex v_s^i in C_i and a vertex v_t^j in C_j, then there is a vertex w_k which has label $F(w_k) = F(v_s^i) + F(v_t^j)$. Since $F(v_s^i) < F(v_t^j)$ for every i and j, and $F(w_k) = F({u^p}_{k-1}) + F({u^p}_{k-2})$ it contradicts the Fibonacci numbers property. Hence, $\sigma(\bigcup_{i=1}^{n} C_{m_i}) = \delta(\bigcup_{i=1}^{n} C_{m_i}) = 2$.

3 Sum labeling of a disjoint union of friendship graphs

In this section we show a sum labeling construction of disjoint union of friendship graphs. In [5], we proved that a disjoint union of two friendship graphs is an optimum sum graph.

Theorem 2 *[5] Let f_n and f_m be two disjoint friendship graphs and each has $2n + 1$ and $2m + 1$ vertices, respectively. Then $\sigma(f_n \cup f_m) = \delta(f_n \cup f_m) = 2$.*

In the following theorem , we generalised the result to a finite disjoint union of friendship graphs.

Theorem 3 *Let $f_{n_1}, f_{n_2}, \cdots, f_{n_m}$ be a finite disjoint friendship graphs and each has $2n_k + 1$ vertices, $k = 1, 2, ..., m$. Then $\sigma(\bigcup_{k=1}^{m} f_{n_k}) = \delta(\bigcup_{k=1}^{m} f_{n_k}) = 2$.*

Proof: Let $V_k = \{c_k, u_1^k, u_2^k, \cdots, u_{n_k}^k, v_1^k, v_2^k, \cdots, v_{n_k}^k\}$ be set of vertices of f_{n_k} with c_k be its center, $k = 1, 2, ..., m$. Note that ${u_i}^k$ is adjacent to ${v_i}^k$.
Define a labeling L of finite union of disjoint f_{n_k} for $k = 1, 2, ..., m$ as follows,
First, define a labeling L of f_{n_1} by

$$\begin{aligned} L(c_1) &= 2 \\ L(u_1^1) &= 3 \\ L(u_i^1) &= L(u_{i-1}^1) + L(c_1) \qquad i = 2, \cdots, n_1 \\ &= L(u_1^1) + (i-1)L(c_1) \end{aligned} \tag{3}$$

and

$$\begin{aligned} L(v_{n_1}^1) &= L(u_1^1) + n_1 L(c_1) \\ L(v_{n_1-i}^1) &= L(v_{n_1}^1) + iL(c_1) \qquad i = 1, 2, \cdots, (n_1 - 1) \\ &= L(u_1^1) + (n_1 + i)L(c_1) \end{aligned} \tag{4}$$

Generally, define the labeling L for f_{n_k}, $k = 2, ..., m$ by :

$$\begin{aligned}
L(c_k) &= L(u_1^{k-1}) + L(v_1^{k-1}) \qquad k = 1,2,...,m && (5)\\
L(u_1^k) &= L(v_1^{k-1}) + L(c_{k-1}) \\
L(u_i^k) &= L(u_1^k) + (i-1)L(c_k) \qquad i = 1,2,...,n_k; k = 1,2,...,m \\
&\text{and} \\
L(v_{n_k}^k) &= L(u_1^k) + n_k L(c_k) && (6)\\
L(v_{n_k-i}^k) &= L(u_1^k) + (n_k + i)L(c_k) \qquad i = 1,2,...,n_k; k = 1,2,...,m
\end{aligned}$$

Let S_k be the sum of all vertex labels in f_{n_k} and we have
$S_k = \{L(u_1^k) + jL(c_k); j = 1,2,...,n_k\} \cup \{2L(u_1^k) + (2n_k - 1)L(c_k)\}$
$\cup \{L(u_1^k) + (2n_k + 1 - j)L(c_k); j = 1,2,...,n_k\}$.

Then from (5) and (6) we have 2 isolated vertices of $\bigcup_{k=1}^m f_{n_k}$, those are w_1^m and w_2^m.

$$\begin{aligned}
L(w_1^m) &= L(u_1^m) + L(v_1^m) && (7)\\
L(w_2^m) &= L(v_1^m) + L(c_m) && (8)
\end{aligned}$$

To convince that there is no additional edge that make the graph failed to be a sum graph with this labeling, we consider 2 cases as follows.

CASE 1 : Labeling in f_{n_k}, for $k = 1, ..., m$

$$\begin{aligned}
L(u_s^k) + L(u_t^k) &= L(u_1^k) + (s-1)L(c_k) + L(u_1^k) + (t-1)L(c_k) \\
&< L(c_{k+1}), \ k = 1,2,...,m-1 && (9)
\end{aligned}$$

$$\begin{aligned}
L(u_s^m) + L(u_t^m) &= L(u_1^m) + (s-1)L(c_m) + L(u_1^m) + (t-1)L(c_m) \\
&< L(w_1^m) && (10)
\end{aligned}$$

$$\begin{aligned}
L(v_s^k) + L(v_t^k) &= L(u_1^k) + (2n_k - s)L(c_k) + L(u_1^k) + (2n_k - t)L(c_k) \\
&= 2L(u_1^k) + (2n_k + 2n_k - s - t)L(c_k) \\
&\leq 2L(u_1^k) + 2(2n_k - 1)L(c_k)
\end{aligned}$$

$$\begin{aligned}
L(u_s^k) + L(v_t^k) &= L(u_1^k) + (s-1)L(c_k) + L(u_1^k) + (2n_k - t)L(c_k) \\
&\leq 2L(u_1^k) + (n_k - 1 + 2n_k - 1)L(c_k) \\
&= 2L(u_1^k) + (2n_k - 1)L(c_k) + (n_k - 1)L(c_k)
\end{aligned}$$

CASE 2 : Labeling between vertices in $f_{n_{m-1}}$ and f_{n_m}

$$\begin{aligned} L(u_s^{m-1}) + L(u_t^m) &= L(u_1^{m-1}) + (s-1)L(c_{m-1}) + L(u_1^m) + (t-1)L(c_m) \\ &\leq L(u_1^{m-1}) + (n_{m-1}-1)L(c_{m-1}) + L(u_1^m) + (n_m-1)L(c_m) \\ &< L(u_1^{m-1}) + (2n_{m-1}-1)L(c_{m-1}) + L(u_1^m) + (2n_m-1)L(c_m) \\ &= L(v_1^{m-1}) + L(v_1^m) \\ &< L(u_1^m) + L(v_1^m) \\ &= L(w_1^m) \end{aligned} \tag{11}$$

$$\begin{aligned} L(v_s^{m-1}) + L(v_t^m) &= L(u_1^{m-1}) + (2n_{m-1}-s)L(c_{m-1}) + L(u_1^m) + (2n_m-t)L(c_m) \\ &\leq L(u_1^{m-1}) + (2n_{m-1}-1)L(c_{m-1}) + L(u_1^m) + (2n_m-1)L(c_m) \\ &= L(v_1^{m-1}) + L(v_1^m) \\ &< L(u_1^m) + L(v_1^m) \\ &= L(w_1^m) \end{aligned} \tag{12}$$

$$\begin{aligned} L(u_s^{m-1}) + L(v_t^m) &= L(u_1^{m-1}) + (n_{m-1}-s)L(c_{m-1}) + L(u_1^m) + (2n_m-t)L(c_m) \\ &\leq L(u_1^{m-1}) + (n_{m-1}-1)L(c_{m-1}) + L(u_1^m) + (2n_m-1)L(c_m) \\ &< L(u_1^{m-1}) + (2n_{m-1}-1)L(c_{m-1}) + L(u_1^m) + (2n_m-1)L(c_m) \\ &= L(v_1^{m-1}) + L(v_1^m) \\ &< L(u_1^m) + L(v_1^m) \\ &= L(w_1^m) \end{aligned} \tag{13}$$

We can see from each case the result is always an even number and different from numbers in S_k and always smaller than labels of isolated vertices of f_{n_k}. Then we can conclude that there is no additional edge in $\bigcup_{k=1}^m f_{n_k}$. Thus, we have $\sigma(\bigcup_{k=1}^m f_{n_k}) = \delta(\bigcup_{k=1}^m f_{n_k}) = 2$

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